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ON DECAY AND BLOW-UP OF SOLUTIONS FOR A SINGULAR NONLOCAL VISCOELASTIC PROBLEM WITH A NONLINEAR SOURCE TERM

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Abstract. We consider a singular nonlocal viscoelastic problem with a nonlinear source term and a possible damping term. We prove that if the initial data enter into the stable set, the solution exists globally and decays to zero with a more general rate, and if the initial data enter into the unstable set, the solution with nonpositive initial energy as well as positive initial energy blows up in finite time. These are achieved by using the potential well theory, the modified convexity method and the perturbed energy method.

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1. Introduction

In this paper, we investigate the following one-dimensional viscoelastic problem with a nonlocal boundary condition:

$$
(1.1)\qquad \begin{cases} u_{tt} - \dfrac{1}{x}\,(xu_x)_x + \displaystyle\int_0^t g(t-s)\,\frac{1}{x}\,(xu_x(x,s))_x\,ds + au_t = |u|^{p-2}u \\ \hfill \text{for } x \in (0,\ell),\ t \in (0,\infty), \\ u(\ell,t) = 0, \quad \displaystyle\int_0^\ell xu(x,t)\,dx = 0 \qquad \text{for } t \in [0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{for } x \in [0,\ell], \end{cases}
$$

where $a \geq 0$, $\ell < \infty$, $p > 2$ and $g\colon \mathbb{R}^+ \to \mathbb{R}^+$.

This type of evolution problems, with nonlocal constraints, are generally encountered in heart transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physics. The nonlocal boundary conditions arise mainly when the data on the boundary cannot be measured directly, but their average values are known. We can refer to the works of Cahlon and Shi [4], Cannon [5], Choi and Chan [8], Ewing and Lin [9], Ionkin [10], Kamynin [11], Samarskii [33], Shi and Shilor [34], Wang et al. [36], and Wu et al. [37]. The first paper discussed second order partial differential equations with nonlocal integral conditions going back to Cannon [5]. In fact, most of the works on this topic were dedicated to classical solutions. Later, mixed problems with classical and nonlocal (integral) boundary conditions related to parabolic and hyperbolic equations received attention and have been extensively studied. Existence and uniqueness questions have been considered by Bouziani [3], Ionkin [10], Kamynin [11], Mesloub [25], Pulkina [32].

In the absence of the viscoelastic term (i.e., $g = 0$), Mesloub and Bouziani [23] studied the following equation:

$$
v_{tt} - \frac{1}{x}\,v_x - v_{xx} = f(x,t), \quad x \in (0,\ell),\ t \in (0,T),
$$

and obtained the existence and uniqueness of a strong solution. Later, Mesloub and Messaoudi [25] solved a three-point boundary-value problem for a hyperbolic equation with a Bessel operator and an integral condition based on an energy method. Then in [26] they considered a nonlinear one-dimensional hyperbolic problem with a linear damping term and established a blow-up result for large initial data and a decay result for small initial data.

In the presence of the viscoelastic term (i.e. $g \neq 0$), Mecheri et al. [22] studied the following equation:

$$
u_{tt} - \frac{1}{x}\,(xu_x)_x + \int_0^t g(t-s)\,\frac{1}{x}\,(xu_x(x,s))_x\,ds + au_t = f(x,t), \quad 0 < x < 1,\ t > 0,
$$

for $a > 0$ and proved the existence and uniqueness of the strong solution. Then, Mesloub et al. [24] considered a nonlinear mixed problem for a viscoelastic equation with a dissipation term under a nonlocal boundary condition and obtained the existence and uniqueness of the weak solution based on the iteration processes. Later, the global existence, decay and blow-up of solutions of problem (1.1) (when $a = 0$) were established by Mesloub and Messaoudi in [27], where the authors studied the blow-up result with only negative initial energy. Recently, Wu [38] improved [27] by establishing the blow-up result with nonpositive initial energy as well as positive initial energy.

For the case of initial and boundary value problems for linear and nonlinear viscoelastic equations with classical conditions, many results have also been extensively studied. Cavalcanti et al. [6] studied

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + a(x)u_t + |u|^m u = 0, \quad (x,t) \in \Omega \times (0,\infty),$$

for $a\colon \Omega \to \mathbb{R}^+$, a function which may be null on a part of the domain Ω. Under the conditions that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0,$$

the authors established an exponential rate of decay. Berrimi and Messaoudi [2] improved Cavalcanti's result by introducing a different functional which allowed to weak the conditions on both a and g. In particular, the function a can vanish on the whole domain Ω and consequently the geometry condition has disappeared. In [7], Cavalcanti et al. considered

$$u_{tt} - k_0\Delta u + \int_0^t \operatorname{div}[a(x)g(t-\tau)\nabla u(\tau)]\,d\tau + b(x)h(u_t) + f(u) = 0,$$

under similar conditions on the relaxation function g and $a(x)+b(x) \geq \rho > 0$, for all $x \in \Omega$. They improved the result of [6] by establishing exponential stability for g decaying exponentially and h linear and polynomial stability for g decaying polynomially and h nonlinear. In [1], Berrimi and Messaoudi considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau = |u|^{p-2}u$$

in a bounded domain and $p > 2$. They established a local existence result and showed that, under weaker condition $g'(t) \leq \xi g^r(t)$, the solution is global and decay in a polynomial or exponential fashion when the initial data is small enough. Then Messaoudi [30] improved this result by establishing a general decay of energy which is similar to the relaxation function under weaker condition that $g'(t) \leq \xi(t)g(t)$. In regard of nonexistence, Messaoudi [28] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + a|u_t|^{m-2}u_t = |u|^{p-2}u$$

and established a blow-up result for solutions with negative energy if $p > m$ and a global existence result for $p \le m$. Then Messaoudi [29] improved this result by accommodating certain solutions with positive initial energy. Liu [14] obtained the similar blow-up result for the viscoelastic problem with strong damping and nonlinear source by using the potential well theory and convexity technique. For other related works, we refer the readers to [12], [13], [15]–[21], [31], [35], [39]–[41] and the references therein.

Inspired by [1], [14], [20], [27], [30], we intend to study the blow-up and decay properties of problem (1.1) in this paper. Our goal is to establish a decay result with a more general rate and a blow-up result with nonpositive initial energy as well as positive initial energy. The main difficulties we encounter here arise from the simultaneous appearance of the singular nonlocal viscoelastic term, the possible damping term, as well as the nonlinear source term. We first show that if the initial data enter into the unstable set, the source term is enough to obtain blow-up result no matter $a = 0$ or $a > 0$. This is achieved by using the potential well theory and the modified convexity method. We then establish the decay result under the condition that $g'(t) \le -\xi(t)g^r(t)$, which is more general than that of [1], [30], by constructing some functionals and using the perturbed energy method.

The paper is organized as follows. In Section 2 we present some assumptions and known results and state the main results. Section 3 is devoted to the proof of the blow-up result. The decay result is proved in Sections 4.

2. Preliminaries and main results

In this section we first introduce some functional spaces and present some assumptions and known results which will be used throughout this work.

Let $L^p_x = L^p_x(0,\ell)$ be the weighted Banach space equipped with the norm

$$\|u\|_p = \bigg(\int_0^\ell x|u|^p\,dx\bigg)^{1/p}.$$

In particular, when $p = 2$, we denote $H = L^2_x(0,\ell)$ to be the weighted Hilbert space of square integrable functions having the finite norm

$$\|u\|_H = \bigg(\int_0^\ell xu^2\,dx\bigg)^{1/2}.$$

We take $V = V^{1,1}_x(0,\ell)$ to be the weighted Hilbert space equipped with the norm

$$\|u\|_V = (\|u\|_H^2 + \|u_x\|_H^2)^{1/2},$$

and $V_0 = \{u \in V : u(\ell) = 0\}$.

On the relaxation function g we put the following assumptions:

(G1) $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing C^2 function such that

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)\,ds = l > 0.$$

(G2) There exists a positive differentiable function ξ such that

$$g'(t) \le -\xi(t) g^r(t), \quad t \ge 0,\ 1 \le r < \frac{3}{2}, \tag{2.1}$$

and ξ satisfies, for some positive constant L,

$$\left|\frac{\xi'(t)}{\xi(t)}\right| \le L, \quad \xi'(t) \le 0, \quad \text{for all } t > 0, \quad \int_0^{+\infty} \xi(s)\,ds = +\infty.$$

Furthermore, when $1 < r < 3/2$, for any fixed $t_0 > 0$, there exists a positive constant C_r depending only on r, such that

$$\frac{t}{\left(1 + \int_{t_0}^t \xi(s)\,ds\right)^{1/(2(r-1))}} \le C_r, \qquad \int_0^\infty \frac{1}{\left(1 + \int_{t_0}^t \xi(s)\,ds\right)^{\nu}}\,dt < +\infty, \tag{2.2}$$

for all $t \ge t_0$, $\nu > 1$,

Remark 2.1. The condition $r < 3/2$ is made to ensure that

$$\int_0^\infty g^{2-r}(s)\,ds < \infty.$$

Remark 2.2. If $\xi(t) \equiv \xi$ =contant, (G2) recaptures that of [1], [14], [27]. If $r \equiv 1$, (G2) recaptures that of [30], [31]. Therefore, (G2) is a generalization of [1], [14], [27], [30], [31]. In particular, when $\xi(t) \equiv \xi$ and $1 < r < 3/2$, (2.2) holds naturally.

Lemma 2.3 ([27], Poincaré-type inequality). *For any v in V_0, we have*

$$\int_0^\ell x v^2(x)\,dx \le C_p \int_0^\ell x v_x^2(x)\,dx,$$

where C_p is some positive constant.

Lemma 2.4 ([27]). *For any v in V_0, $2 < p < 4$, we have*

$$\int_0^\ell x|v|^p\,dx \le C_* \|v_x\|_2^p,$$

where C_* is a constant depending on ℓ and p only.

We have the following local existence result for problem (1.1).

Theorem 2.5. *Suppose that* (G1) *holds and $2 < p < 3$. Then, for any u_0 in V_0 and u_1 in H, problem* (1.1) *has a unique local solution*

$$u \in C(0, T_{\max}; V_0) \cap C^1(0, T_{\max}; H)$$

such that

$$\langle u_{tt}(t),\,\phi\rangle - \left(\frac{1}{x}\,(xu_x)_x,\phi\right)_H + \left(\int_0^t g(t-s)\,\frac{1}{x}\,(xu_x(x,s))_x\,ds,\phi\right)_H + (au_t,\phi)_H = (|u|^{p-2}u,\phi)_H$$

for all test functions $\phi\in V_0$ *and for almost all* $t\in[0,T_{\max})$ *with* $T_{\max}>0$ *small enough.*

Proof. The proof can be easily established by adopting the arguments of [1], [24] and [26]. That is, we consider, first, a related linear problem. Then, we use the well-known contraction mapping theorem to prove the existence of solutions to the nonlinear problem. These are quite standard so we omit it here. □

Remark 2.6. The condition $2<p<3$ is needed so that the embedding of V_0 in L_x^2 is Lipschitz (see [26, Lemma 5.2]).

Next we introduce the functionals for I,J and E:

$$\begin{aligned}
I(t) := I(u(t)) &= \left(1-\int_0^t g(s)\,ds\right)\int_0^\ell xu_x^2\,dx \\
&\quad + (g\circ u_x)(t) - \int_0^\ell x|u(t)|^p\,dx, \\
J(t) := J(u(t)) &= \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\int_0^\ell xu_x^2\,dx \\
&\quad + \frac{1}{2}\,(g\circ u_x)(t) - \frac{1}{p}\int_0^\ell x|u(t)|^p\,dx, \\
E(t) := E(u(t)) &= J(t) + \frac{1}{2}\int_0^\ell xu_t^2\,dx,
\end{aligned}$$

where

$$(g\circ u_x)(t) = \int_0^\ell\int_0^t xg(t-s)|u_x(x,t)-u_x(x,s)|^2\,ds\,dx.$$

Remark 2.7. Multiplication of equation (1.1) by xu_t and integration over $(0,\ell)$ easily yields, for all $t\ge 0$

$$\begin{aligned}
E'(t) &= \frac{1}{2}\,(g'\circ u_x)(t) - \frac{1}{2}\,g(t)\int_0^\ell xu_x^2\,dx - a\int_0^\ell xu_t^2\,dx \\
&\le -a\int_0^\ell xu_t^2\,dx \le 0.
\end{aligned}\tag{2.3}$$

We are now in position to state our main results.

Theorem 2.8. *Assume that* (G1) *holds and* $2<p<3$*, let* u *be the unique local solution to problem* (1.1) *and denote*

$$d_1 = \frac{p-2}{2p}\left(\frac{l}{C_*^{2/p}}\right)^{p/(p-2)}.$$

For any fixed $\delta < 1$, assume that u_0, u_1 satisfy

$$E(0) < \delta d_1, \qquad I(0) < 0. \tag{2.4}$$

Suppose that

$$\int_0^\infty g(s)\,ds \le \frac{p-2}{p-2+1/[(1-\widehat{\delta})^2 p + 2\delta(1-\widehat{\delta})]} \tag{2.5}$$

where $\widehat{\delta} = \max\{0,\delta\}$. Then the solution of problem (1.1) blows up in a finite time T^* in the sense that

$$\lim_{t\to T^{*-}} \|u(t)\|_H^2 = +\infty.$$

REMARK 2.9. For $a = 0$, Wu [38] established blow-up results under some restrictions on $\int_0^\ell x u_0 u_1\,dx$, which are no more needed in this paper. In fact, we use the potential well theory and the modified convexity method, which is different from that in Wu [38].

THEOREM 2.10. *Assume that (G1) holds and $2 < p < 3$, let u be the unique local solution to problem (1.1). In addition, assume that u_0, u_1 satisfy*

$$E(0) < d_1, \qquad I(0) > 0. \tag{2.6}$$

Then the solution u is global and satisfies

$$\int_0^\ell x u_x^2\,dx \le \frac{2p}{l(p-2)}\,E(t) \le \frac{2p}{l(p-2)}\,E(0), \quad \text{for all } t > 0. \tag{2.7}$$

THEOREM 2.11. *Under the assumptions of Theorem 2.10, suppose further that (G2) holds. Then for each $t_0 > 0$, there exist positive constants K and κ such that*

$$E(t) \le \begin{cases} K e^{-\kappa \int_{t_0}^t \xi(s)\,ds} & \text{if } r = 1, \\ K\left(1 + \displaystyle\int_{t_0}^t \xi(s)\,ds\right)^{-1/(r-1)} & \text{if } 1 < r < \dfrac{3}{2}. \end{cases} \tag{2.8}$$

REMARK 2.12. Note that when $1 < r < 3/2$, we obtain more general type of decays.

If we choose $\xi(t) \equiv \xi$, (2.8) gives the polynomial rate decay as

$$E(t) \le K(1+t)^{-1/(r-1)},$$

which coincides with the results of [1], [14], [27].

If we choose $\xi(t) = (1+t)^{-m}$ for $0 < m < 3 - 2r < 1$, which satisfies (2.2), we have

$$g(t) \le \frac{C_0}{(1+t)^q} \quad \text{with } q = \frac{1-m}{r-1}$$

and (2.8) also gives the polynomial rate of decay as $E(t) \le C_1/(1+t)^q$.

If we choose $\xi(t) = 2a(r-1)t^{-(3-2r)} + b$ for $a, b > 0$, then we have

$$g(t) \leq \frac{C}{[1 + at^{2(r-1)} + bt]^{1/(r-1)}},$$

which gives the polynomial rate of decay as

$$E(t) \leq \frac{K}{[1 + at^{2(r-1)} + bt]^{1/(r-1)}}.$$

If we choose $\xi(t) = 2(r-1)(1+t)^{-(3-2r)} + (1+t)^{-1}$, which satisfies (G2), then we have

$$g(t) \leq \frac{C}{[(1+t)^{2(r-1)} + \ln(1+t) - 1]^{1/(r-1)}}$$

and a new type of decay as

$$E(t) \leq \frac{K}{[(1+t)^{2(r-1)} + \ln(1+t) - 1]^{1/(r-1)}}$$

is established.

3. Blow-up of solutions

In this section, we prove a finite time blow-up result for initial data in the unstable set. For $t \geq 0$, we define $d(t) = \inf_{u \in V_0 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u)$ and

$$\mathcal{N} = \{u \in V_0 \setminus \{0\} : I(u) = 0\}. \tag{3.1}$$

Then we can prove the following lemma.

Lemma 3.1. *For $t \geq 0$, we have $0 < d_1 \leq d(t) \leq d_2(u) = \sup_{\lambda \geq 0} J(\lambda u)$ and*

$$d(t) = \inf_{u \in \mathcal{N}} J(u). \tag{3.2}$$

Proof. Obviously, $d(t) \leq d_2(u) = \sup_{\lambda \geq 0} J(\lambda u)$. Since

$$J(\lambda u) = \frac{\lambda^2}{2}\left[\left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell x u_x^2\,dx + (g \circ u_x)(t)\right] - \frac{\lambda^p}{p}\int_0^\ell x|u|^p\,dx.$$

We get

$$\frac{d}{d\lambda}J(\lambda u) = \lambda\left[\left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell x u_x^2\,dx + (g \circ u_x)(t)\right] - \lambda^{p-1}\int_0^\ell x|u|^p\,dx.$$

Let

$$\frac{d}{d\lambda}J(\lambda u) = 0,$$

which implies

$$\overline{\lambda_1} = 0, \qquad \overline{\lambda_2} = \left[\frac{\left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell x u_x^2\,dx + (g \circ u_x)(t)}{\int_0^\ell x|u|^p\,dx}\right]^{1/(p-2)}.$$

An elementary calculation shows

$$\frac{d^2}{d\lambda^2}\, J(\overline{\lambda_1} u) > 0 \quad \text{and} \quad \frac{d^2}{d\lambda^2}\, J(\overline{\lambda_2} u) < 0.$$

Using (G1) and Lemma 2.4, we get

$$\begin{aligned}
\sup_{\lambda \geq 0} J(\lambda u) &= J(\overline{\lambda_2} u) \\
&= \frac{p-2}{2p} \left[\frac{\left(1 - \int_0^t g(s)\, ds\right) \int_0^\ell x u_x^2\, dx + (g \circ u_x)(t)}{\left(\int_0^\ell x|u|^p\, dx\right)^{2/p}} \right]^{p/(p-2)} \\
&\geq \frac{p-2}{2p} \left[\frac{l \int_0^\ell x u_x^2\, dx}{\left(\int_0^\ell x|u|^p\, dx\right)^{2/p}} \right]^{p/(p-2)} \geq \frac{p-2}{2p} \left(\frac{l}{C_*^{2/p}} \right)^{p/(p-2)} = d_1 > 0,
\end{aligned}$$

which implies that $d(t) \geq d_1$.

To get (3.2), straightforward computations lead to

$$\begin{aligned}
I(\overline{\lambda_2} u) &= \left(1 - \int_0^t g(s)\, ds\right) \int_0^\ell x (\overline{\lambda_2} u)_x^2\, dx + (g \circ (\overline{\lambda_2} u)_x)(t) - \int_0^\ell x |\overline{\lambda_2} u|^p\, dx \\
&= \left[\frac{\left(1 - \int_0^t g(s)\, ds\right) \int_0^\ell x u_x^2\, dx + (g \circ u_x)(t)}{\int_0^\ell x|u|^p\, dx} \right]^{2/(p-2)} \\
&\quad \times \left[\left(1 - \int_0^t g(s)\, ds\right) \int_0^\ell x u_x^2\, dx + (g \circ u_x)(t) \right] \\
&\quad - \left[\frac{\left(1 - \int_0^t g(s)\, ds\right) \int_0^\ell x u_x^2\, dx + (g \circ u_x)(t)}{\int_0^\ell x|u|^p\, dx} \right]^{p/(p-2)} \int_0^\ell x|u|^p\, dx \\
&= \frac{\left[\left(1 - \int_0^t g(s)\, ds\right) \int_0^\ell x u_x^2\, dx + (g \circ u_x)(t) \right]^{p/(p-2)}}{\left(\int_0^\ell x|u|^p\, dx\right)^{2/(p-2)}} \\
&\quad \times \left\{ \frac{\left(1 - \int_0^t g(s)\, ds\right) \int_0^\ell x u_x^2\, dx + (g \circ u_x)(t)}{\left(1 - \int_0^t g(s)\, ds\right) \int_0^\ell x u_x^2\, dx + (g \circ u_x)(t)} - 1 \right\} = 0,
\end{aligned}$$

which implies that $\overline{\lambda_2}u \in \mathcal{N}$. Also, for any $u \in \mathcal{N}$, we note that

$$\overline{\lambda_2}(u) = \left[\frac{\left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell xu_x^2\,dx + (g \circ u_x)(t)}{\int_0^\ell x|u|^p\,dx}\right]^{1/(p-2)} = 1.$$

Therefore we have $\overline{\lambda_2}(u)u = u$ for all $u \in \mathcal{N}$. □

Lemma 3.2. *Under the same assumptions as in Theorem* 2.8*, one has* $I(u(t)) < 0$ *and, for all* $t \in [0, T_{\max})$,

$$d_1 < \frac{p-2}{2p}\left[\left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell xu_x^2\,dx + (g \circ u_x)(t)\right] < \frac{p-2}{2p}\int_0^\ell x|u|^p\,dx.$$

Proof. Using (2.3) and (2.4), we have $E(t) \le \delta d_1$ for all $t \in [0, T_{\max})$. Furthermore, we can obtain $I(u(t)) < 0$ for all $t \in [0, T_{\max})$.

In fact, if it is not true, then there exists some $t_0 \in [0, T_{\max})$ such that $I(t_0) \ge 0$. Since $I(0) < 0$, it follows that there exists some $\widetilde{t} \in (0, t_0]$ such that $I(u(\widetilde{t})) = 0$. Define

$$\begin{aligned}(3.3)\quad t^* = \inf\bigg\{&\widetilde{t} \in (0, t_0] :\\ &\left(1 - \int_0^{\widetilde{t}} g(s)\,ds\right)\int_0^\ell xu_x^2(\widetilde{t})\,dx + (g \circ u_x)(\widetilde{t}) = \int_0^\ell x|u(\widetilde{t})|^p\,dx\bigg\}.\end{aligned}$$

Then, we have $I(u(t^*)) = 0$ and

$$(3.4)\quad \left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell xu_x^2\,dx + (g \circ u_x)(t) < \int_0^\ell x|u|^p\,dx, \quad 0 \le t < t^*.$$

Next, we consider two cases:

Case 1. Suppose that $\|u(t^*)\|_H^2 = 0$, using the regularity of u, we have

$$(3.5)\quad \lim_{t \to t^{*-}} \|u(t)\|_H^2 = 0.$$

On the other hand, from (3.4) and Lemma 2.4, we obtain

$$(3.6)\quad \left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell xu_x^2\,dx + (g \circ u_x)(t) < \int_0^\ell x|u|^p\,dx \le C_*\|u_x\|_2^p,$$

and $\|u(t)\|_H^2 \ne 0$, for all $t \in [0, t^*)$. Therefore we have

$$\lim_{t \to t^{*-}} \|u(t)\|_H^2 > \left(\frac{l}{C_*}\right)^{1/(p-2)},$$

which contradicts to (3.5).

Case 2. Suppose that $\|u(t^*)\|_H^2 \ne 0$. Applying Lemma 3.1, we see that $d(t)$ is the infimum of $J(u(t))$ over all functions u in $\mathcal{N}$ and $J(u(t^*)) \ge d(t) \ge d_1$, which contradicts to $J(u(t^*)) \le E(t^*) < d_1$. Thus, we conclude that $I(t) < 0$ for all $t \in [0, T_{\max})$.

To get (3.2), we use (3.4), Lemma 3.1 and the conclusion that $I(t) < 0$ for all $t \in [0, T_{\max})$ and get

$$
\begin{aligned}
(3.7)\quad d_1 &\le \frac{p-2}{2p} \frac{\left[\left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell x u_x^2\,dx + (g \circ u_x)(t)\right]^{2/(p-2)}}{\left(\int_0^\ell x|u|^p\,dx\right)^{p/(p-2)}} \\
&< \frac{p-2}{2p}\left[\left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell x u_x^2\,dx + (g \circ u_x)(t)\right], \quad 0 \le t < T_{\max}.
\end{aligned}
$$

It follows from (3.4) and (3.7) that

$$
\begin{aligned}
0 < d_1 &< \frac{p-2}{2p}\left[\left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell x u_x^2\,dx + (g \circ u_x)(t)\right] \\
&\qquad < \frac{p-2}{2p}\int_0^\ell x|u|^p\,dx,
\end{aligned}
$$

for $0 \le t < T_{\max}$. □

Lemma 3.3 ([12]). *Let L be a positive C^2 function, which satisfies, for $t > 0$, the inequality*

$$
L(t)L''(t) - (1+\zeta)L'(t)^2 \ge 0
$$

with some $\zeta > 0$. If $L(0) > 0$ and $L'(0) > 0$, then there exists time $T^* \le L(0)/\zeta L'(0)$ such that

$$
\lim_{t \to T^{*-}} L(t) = \infty.
$$

Proof of Theorem 2.8. Assume by contradiction that the solution u is global. Then, we consider $L\colon [0,T] \to \mathbb{R}_+$ defined by

$$
(3.8)\quad L(t) = \int_0^\ell x u^2\,dx + a\int_0^t \int_0^\ell x u^2\,dx\,ds + a(T-t)\int_0^\ell x u_0^2\,dx + b(t+T_0)^2,
$$

where T, b and T_0 are positive constants to be chosen later. Then $L(0) > 0$. Furthermore,

$$
\begin{aligned}
L'(t) &= 2\int_0^\ell x u u_t\,dx + a\int_0^\ell x(u^2 - u_0^2)\,dx + 2b(t+T_0) \\
&= 2\int_0^\ell x u u_t\,dx + 2a\int_0^t \int_0^\ell x u u_s\,dx\,ds + 2b(t+T_0),
\end{aligned}
$$

and, consequently,

$$
L''(t) = 2\int_0^\ell x u u_{tt}\,dx + 2\int_0^\ell x u_t^2\,dx + 2a\int_0^\ell x u u_t\,dx + 2b
$$

for almost every $t \in [0,T]$. Testing equation (1.1) with xu and plugging the result into the expression of $L''(t)$, we obtain

$$\begin{aligned}
L''(t) = &-2\int_0^\ell xu_x^2\,dx + 2\int_0^\ell\int_0^t g(t-s)xu_x(x,t)u_x(x,s)\,ds\,dx \\
&-2a\int_0^\ell xuu_t\,dx + 2\int_0^\ell x|u|^p\,dx + 2\int_0^\ell xu_t^2\,dx + 2a\int_0^\ell xuu_t\,dx + 2b \\
=&\,2\bigg[\int_0^\ell xu_t^2\,dx - \bigg(1-\int_0^t g(s)\,ds\bigg)\int_0^\ell xu_x^2\,dx \\
&-\int_0^\ell\int_0^t g(t-s)xu_x(x,t)(u_x(x,t)-u_x(x,s))\,ds\,dx + \int_0^\ell x|u|^p\,dx + b\bigg]
\end{aligned}$$

for almost every $t \in [0,T]$. Therefore, we get

$$\begin{aligned}
L(t)L''(t) - \frac{p+2}{4}\,L'(t)^2 = &\,2L(t)\bigg[\int_0^\ell xu_t^2\,dx - \bigg(1-\int_0^t g(s)\;ds\bigg)\int_0^\ell xu_x^2\,dx \\
&-\int_0^\ell\int_0^t g(t-s)xu_x(x,t)(u_x(x,t)-u_x(x,s))\,ds\,dx + \int_0^\ell x|u|^p\,dx + b\bigg] \\
&+(p+2)\bigg[\eta(t) - \bigg(L(t) - a(T-t)\int_0^\ell xu_0^2\,dx\bigg) \\
&\times\bigg(\int_0^\ell xu_t^2\,dx + a\int_0^t\int_0^\ell xu_s^2\,dx\,ds + b\bigg)\bigg],
\end{aligned}$$

where

$$\begin{aligned}
\eta(t) = &\bigg(\int_0^\ell xu^2\,dx + a\int_0^t\int_0^\ell xu^2\,dx\,ds + b(t+T_0)^2\bigg) \\
&\times\bigg(\int_0^\ell xu_t^2\,dx + a\int_0^t\int_0^\ell xu_s^2\,dx\,ds + b\bigg) \\
&-\bigg[\int_0^\ell xuu_t\,dx + a\int_0^t\int_0^\ell xuu_s\,dx\,ds + b(t+T_0)\bigg]^2.
\end{aligned}$$

Using Schwarz's inequality, we can easily get $\eta(t) \ge 0$ for every $t \in [0,T]$. As a consequence, we reach the following differential inequality:

$$L(t)L''(t) - \frac{p+2}{4}\,L'(t)^2 \ge L(t)\Phi(t), \quad \text{for a.e. } t \in [0,T], \tag{3.9}$$

where $\Phi\colon [0,T] \mapsto \mathbb{R}_+$ is the map defined by

$$\begin{aligned}
\Phi(t) = &-p\int_0^\ell xu_t^2\,dx - 2\bigg(1-\int_0^t g(s)\,ds\bigg)\int_0^\ell xu_x^2\,dx - a(p+2)\int_0^t\int_0^\ell xu_s^2\,dx\,ds \\
&-2\int_0^\ell\int_0^t g(t-s)xu_x(x,t)(u_x(x,t)-u_x(x,s))\,ds\,dx + 2\int_0^\ell x|u|^p\,dx - pb
\end{aligned}$$

$$= -2pE(t) + p(g \circ u_x)(t) + (p-2)\left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell xu_x^2\,dx$$
$$- pb - 2\int_0^\ell \int_0^t g(t-s)xu_x(x,t)(u_x(x,t) - u_x(x,s))\,ds\,dx$$
$$- a(p+2)\int_0^t \int_0^\ell xu_s^2\,dx\,ds.$$

By (2.3), for all $t \in [0,T]$ we may also write

$$\Phi(t) \geq -2pE(0) + p(g \circ u_x)(t) + (p-2)\left(1 - \int_0^t g(s)\,ds\right)\int_0^\ell xu_x^2\,dx \tag{3.10}$$
$$-pb-2\int_0^\ell\int_0^t g(t-s)xu_x(x,t)(u_x(x,t)-u_x(x,s))\,ds\,dx+a(p-2)\int_0^t\int_0^\ell xu_s^2\,dx\,ds.$$

By using Young's inequality, we have

$$2\int_0^\ell \int_0^t g(t-s)xu_x(x,t)(u_x(x,t) - u_x(x,s))\,ds\,dx \tag{3.11}$$
$$\leq \frac{1}{\varepsilon}\int_0^t g(s)\int_0^\ell xu_x^2\,ds\,dx + \varepsilon(g \circ u_x)(t),$$

for any $\varepsilon > 0$. Substituting (3.11) for the fifth term of the right hand side of (3.10), we obtain

$$\Phi(t) \geq -2pE(0) + \left[(p-2) - \left(p - 2 + \frac{1}{\varepsilon}\right)\int_0^t g(s)\,ds\right]\int_0^\ell xu_x^2\,dx \tag{3.12}$$
$$+ (p-\varepsilon)(g \circ u_x)(t) + a(p-2)\int_0^t \int_0^\ell xu_s^2\,dx\,ds - pb.$$

If $\delta \leq 0$, i.e. $E(0) < 0$, we choose $\varepsilon = p$ in (3.12) and b small enough such that $b \leq -2E(0)$. Together with (2.5), we obtain

$$\Phi(t) \geq \left[(p-2) - \left(p - 2 + \frac{1}{p}\right)\int_0^t g(s)\,ds\right]\int_0^\ell xu_x^2\,dx \tag{3.13}$$
$$+ a(p-2)\int_0^t \int_0^\ell xu_s^2\,dx\,ds \geq a(p-2)\int_0^t \int_0^\ell xu_s^2\,dx\,ds \geq 0.$$

If $0 < \delta < 1$, i.e. $E(0) < \delta d_1$, we choose $\varepsilon = (1-\delta)p+2\delta$ and $b = 2(\delta d_1 - E(0)) > 0$ in (3.12). Then we get

$$\Phi(t) \geq -2p\delta d_1 + \left[(p-2) - \left(p - 2 + \frac{1}{(1-\delta)p+2\delta}\right)\int_0^t g(s)\,ds\right]\int_0^\ell xu_x^2\,dx$$
$$+ \delta(p-2)(g \circ u_x)(t) + a(p-2)\int_0^t \int_0^\ell xu_s^2\,dx\,ds.$$

By (2.5), we have

$$(p-2) - \left(p - 2 + \frac{1}{(1-\delta)p+2\delta}\right)\int_0^t g(s)\,ds \geq \delta(p-2)\left(1 - \int_0^t g(s)\,ds\right)$$

and therefore, by (3.2) and (2.4), we get

$$\begin{aligned}(3.14)\quad \Phi(t) \geq& -2p\delta d_1 + \delta(p-2)\bigg[\bigg(1-\int_0^t g(s)\,ds\bigg)\int_0^\ell xu_x^2\,dx + (g\circ u_x)(t)\bigg] \\ &+ a(p-2)\int_0^t\int_0^\ell xu_s^2\,dx\,ds \\ \geq& 2p(\delta d_1 - \delta d_1) + a(p-2)\int_0^t\int_0^\ell xu_s^2\,dx\,ds \geq 0.\end{aligned}$$

Therefore, combining (3.9), (3.13), and (3.14), we arrive at

$$L(t)L''(t) - \frac{p+2}{4}L'(t)^2 \geq 0, \quad \text{for a.e. } t\in[0,T].$$

Let T_0 be any number which depends only on $p, b, \int_0^\ell xu_0^2\,dx$ and $\int_0^\ell xu_1^2\,dx$ as

$$T_0 > \frac{(p-2+4a)\displaystyle\int_0^\ell xu_0^2\,dx + (p-2)\int_0^\ell xu_1^2\,dx}{2(p-2)b},$$

which fulfills the requirement of

$$L'(0) = 2\int_0^\ell xu_0u_1\,dx + 2bT_0 > 0.$$

Then using Lemma 3.3, we obtain that $L(t)$ goes to ∞ as t tends to some T^* satisfying

$$(3.15)\qquad T^* \leq \frac{4L(0)}{(p-2)L'(0)} = \frac{2(1+aT)\displaystyle\int_0^\ell xu_0^2\,dx + 2bT_0^2}{(p-2)\displaystyle\int_0^\ell xu_0u_1\,dx + (p-2)bT_0}.$$

Finally, for fixed T_0, we choose T as

$$(3.16)\qquad T > \frac{4\bigg(\displaystyle\int_0^\ell xu_0^2\,dx + bT_0^2\bigg)}{2(p-2)bT_0 - (p-2+4a)\displaystyle\int_0^\ell xu_0^2\,dx - (p-2)\int_0^\ell xu_1^2\,dx}.$$

Combing (3.15) and (3.16), we get $T > T^*$ and this contradicts to our assumption, which finishes our proof. □

4. Decay of solutions

In this section we prove our decay result. For this purpose, we need the following lemmas.

Lemma 4.1 ([27, Lemma 4.1]). *Under the same assumption as in Theorem 2.11, one has $I(u(t)) > 0$ for all $t \in [0, T_{\max})$.*

Proof of Theorem 2.10. We can refer to [27, Lemma 4.2]. □

Next, we use the following "modified" functional

$$F(t) := E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t), \tag{4.1}$$

where ε_1 and ε_2 are positive constants and

$$\Psi(t) = \xi(t) \int_0^\ell x u_t u \, dx, \tag{4.2}$$

$$\chi(t) = -\xi(t) \int_0^\ell x u_t \int_0^t g(t-s)(u(t)-u(s)) \, ds \, dx. \tag{4.3}$$

Lemma 4.2. *For ε_1 and ε_2 small enough, we have*

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t) \tag{4.4}$$

holds for two positive constants α_1 and α_2.

Proof. Straightforward computations lead to

$$\begin{aligned}
F(t) =& E(t) + \varepsilon_1 \xi(t) \int_0^\ell x u_t u \, dx - \varepsilon_2 \xi(t) \int_0^\ell x u_t \int_0^t g(t-s)(u(t)-u(s)) \, ds \, dx \\
\leq& E(t) + \frac{\varepsilon_1}{2} \xi(t) \int_0^\ell x u_t^2 \, dx + \frac{\varepsilon_1}{2} \xi(t) \int_0^\ell x u^2 \, dx + \frac{\varepsilon_2}{2} \xi(t) \int_0^\ell x u_t^2 \, dx \\
&+ \frac{\varepsilon_2}{2} \xi(t) \int_0^\ell x \Big(\int_0^t g(t-s)(u(t)-u(s)) \, ds \Big)^2 dx \\
\leq& E(t) + \frac{\varepsilon_1}{2} \xi(t) \int_0^\ell x u_t^2 \, dx + \frac{\varepsilon_1}{2} \xi(t) \int_0^\ell x u^2 \, dx + \frac{\varepsilon_2}{2} \xi(t) \int_0^\ell x u_t^2 \, dx \\
&+ \frac{\varepsilon_2}{2} \xi(t) \int_0^\ell x \int_0^t g(s) \, ds \int_0^t g(t-s)(u(t)-u(s))^2 \, ds \, dx \\
\leq& E(t) + \frac{(\varepsilon_1+\varepsilon_2)\xi(t)}{2} \int_0^\ell x u_t^2 \, dx + \frac{C_p \varepsilon_1}{2} \xi(t) \int_0^\ell x u_x^2 \, dx \\
&+ \frac{\varepsilon_2}{2} (1-l) \xi(t) \int_0^\ell \int_0^t x g(t-s)(u(t)-u(s))^2 \, ds \, dx \\
\leq& E(t) + \frac{(\varepsilon_1+\varepsilon_2)\xi(t)}{2} \int_0^\ell x u_t^2 \, dx \\
&+ \frac{C_p \varepsilon_1}{2} \xi(t) \int_0^\ell x u_x^2 \, dx + \frac{\varepsilon_2}{2} (1-l) C_p \xi(t) (g \circ u_x)(t) \leq \frac{1}{\alpha_1} E(t),
\end{aligned}$$

and in the same way, we get

$$\begin{aligned}
F(t) \geq& E(t) - \frac{(\varepsilon_1+\varepsilon_2)\xi(t)}{2} \int_0^\ell x u_t^2 \, dx - \frac{C_p \varepsilon_1}{2} \xi(t) \int_0^\ell x u_x^2 \, dx \\
&- \frac{\varepsilon_2}{2} (1-l) C_p \xi(t) (g \circ u_x)(t)
\end{aligned}$$

$$\geq \left[\frac{1}{2} - \frac{(\varepsilon_1+\varepsilon_2)\xi(t)}{2}\right]\int_0^\ell xu_t^2\,dx + \left(\frac{1}{2}\,l - \frac{C_p\varepsilon_1}{2}\,\xi(t)\right)\int_0^\ell xu_x^2\,dx$$
$$+ \left[\frac{1}{2} - \frac{C_p}{2}\,\varepsilon_2(1-l)\xi(t)\right](g\circ u_x)(t) - \frac{1}{p}\int_0^\ell x|u|^p\,dx \geq \frac{1}{\alpha_2}\,E(t),$$

for ε_1 and ε_2 small enough. □

Lemma 4.3 ([27, Lemma 4.5]). *Let $v \in L^\infty((0,T);H), v_x \in L^\infty((0,T);H)$ and g be a continuous function on $[0,T]$ and suppose that $0<\tau<1$ and $r>0$. Then there exists a constant $C>0$ such that*

$$\begin{aligned}\int_0^t g(t-s)\|v_x(\,\cdot\,,t)-v_x(\,\cdot\,,s)\|_H^2\,ds \\ \leq C\bigg(\sup_{0<s<T}\|v(\,\cdot\,,s)\|_H^2\int_0^t g^{1-\tau}(s)\,ds\bigg)^{(r-1)/(r-1+\tau)} \\ \times\bigg(\int_0^t g^r(t-s)\|v_x(\,\cdot\,,t)-v_x(\,\cdot\,,s)\|_H^2\,ds\bigg)^{\tau/(r-1+\tau)}.\end{aligned}$$

Lemma 4.4 ([27, Lemma 4.6]). *Let $v \in L^\infty((0,T);H), v_x \in L^\infty((0,T);H)$ and g be a continuous function on $[0,T]$ and suppose that $r>0$. Then there exists a constant $C>0$ such that*

$$\begin{aligned}\int_0^t g(t-s)\|v_x(\,\cdot\,,t)-v_x(\,\cdot\,,s)\|_H^2\,ds \\ \leq C\bigg(t\|v_x(\,\cdot\,,t)\|_H^2+\int_0^t\|v_x(\,\cdot\,,s)\|_H^2\,ds\bigg)^{(r-1)/r} \\ \times\bigg(\int_0^t g^r(t-s)\|v_x(\,\cdot\,,t)-v_x(\,\cdot\,,s)\|_H^2\,ds\bigg)^{1/r}.\end{aligned}$$

Lemma 4.5. *Assume that $2<p<3$ and that (G1), (G2) and (2.10) hold. Then the functional Ψ, defined by (4.2), satisfies, for all $\alpha,\beta>0$,*

$$\begin{aligned}\Psi'(t) \leq \bigg(1+\frac{a}{2\beta}+\frac{L}{2\alpha}\bigg)\xi(t)\int_0^\ell xu_t^2\,dx \\ -\bigg(\frac{l-a\beta C_p-\alpha C_pL}{2}\bigg)\xi(t)\int_0^\ell xu_x^2\,dx \\ +\frac{\xi(t)}{2l}\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t)+\xi(t)\|u\|_{L_x^p}^p.\end{aligned}\tag{4.5}$$

Proof. By using the differential equation in (1.1), we easily see that

$$\begin{aligned}\Psi'(t) &= \xi(t)\int_0^\ell xu_t^2\,dx+\xi(t)\int_0^\ell xuu_{tt}\,dx+\xi'(t)\int_0^\ell xuu_t\,dx \\ &= \xi(t)\int_0^\ell xu_t^2\,dx-\xi(t)\int_0^\ell xu_x^2\,dx\end{aligned}\tag{4.6}$$

$$+\,\xi(t)\int_0^\ell x|u|^p\,dx - a\xi(t)\int_0^\ell xuu_t\,dx$$
$$+\,\xi(t)\int_0^\ell xu_x\int_0^t g(t-s)u_x(x,s)\,ds\,dx + \xi'(t)\int_0^\ell xuu_t\,dx.$$

By Young's inequality, (G1), (G2), Lemma 2.3 and direct calculations, we arrive at (see [27])

$$\begin{aligned}(4.7)\quad \xi(t)\int_0^\ell xu_x\int_0^t g(t-s)u_x(x,s)\,ds\,dx &\le \frac{\xi(t)}{2}\int_0^\ell xu_x^2\,dx\\ &+\frac{\xi(t)}{2}\int_0^\ell x\bigg[\int_0^t g(t-s)(|u_x(s)-u_x(t)|+|u_x(t)|)\,ds\bigg]^2\,dx\\ &\le \frac{\xi(t)}{2}\int_0^\ell xu_x^2\,dx+\frac{\xi(t)}{2}\,(1+\eta)(1-l)^2\int_0^\ell xu_x^2\,dx\\ &+\frac{\xi(t)}{2}\left(1+\frac{1}{\eta}\right)\int_0^t g^{2-r}(s)\,ds\int_0^\ell\int_0^t xg^r(t-s)|u_x(s)-u_x(t)|^2\,ds\,dx\end{aligned}$$

for any $\eta>0$. We also have

$$(4.8)\qquad \xi'(t)\int_0^\ell xuu_t\,dx \le \frac{\xi(t)}{2}\left|\frac{\xi'(t)}{\xi(t)}\right|\bigg(C_p\alpha\int_0^\ell xu_x^2\,dx+\frac{1}{\alpha}\int_0^\ell xu_t^2\,dx\bigg),$$

for all $\alpha>0$, and

$$(4.9)\qquad -a\xi(t)\int_0^\ell xuu_t\,dx \le \frac{a\beta C_p}{2}\,\xi(t)\int_0^\ell xu_x^2\,dx+\frac{a}{2\beta}\,\xi(t)\int_0^\ell xu_t^2\,dx.$$

Combining (4.6)–(4.9), we arrive at

$$\begin{aligned}\Psi'(t)\le&\bigg(1+\frac{L}{2\alpha}+\frac{a}{2\beta}\bigg)\xi(t)\int_0^\ell xu_t^2\,dx\\ &-\frac{\xi(t)}{2}\,\big[1-(1+\eta)(1-l)^2-aC_p\beta-\alpha C_pL\big]\int_0^\ell xu_x^2\,dx\\ &+\frac{\xi(t)}{2}\left(1+\frac{1}{\eta}\right)\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t)+\xi(t)\|u\|_{L_x^p}^p.\end{aligned}$$

By choosing $\eta = l/(1-l)$, (4.5) is established. $\square$

Lemma 4.6. *Assume $2<p<3$ and that* (G1), (G2) *and* (2.10) *hold. Then the functional χ, defined by* (4.3), *satisfies, for all $\theta>0$,*

$$\begin{aligned}(4.10)\quad \chi'(t)\le&\,\xi(t)\theta[1+C^*+2(1-l)^2]\int_0^\ell xu_x^2\,dx\\ &+\xi(t)\bigg[\theta-\int_0^t g(s)\,ds+a\theta+\theta L\bigg]\int_0^\ell xu_t^2\,dx\\ &+\bigg[\frac{1}{2\theta}+2\theta+\frac{C_p+(a+L)C_p}{4\theta}\bigg]\xi(t)\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t)\\ &-\frac{C_p}{4\theta}\,\xi(t)g(0)(g'\circ u_x)(t).\end{aligned}$$

Proof. Direct calculations give

$$
\begin{aligned}
\chi'(t) = {}&\xi(t)\int_0^\ell x u_x(t)\bigg(\int_0^t g(t-s)(u_x(t)-u_x(s))\,ds\bigg)\,dx \\
&-\xi(t)\int_0^\ell x\bigg(\int_0^t g(t-s)(u_x(t)-u_x(s))\,ds\bigg) \\
&\times\bigg(\int_0^t g(t-s)u_x(s)\bigg)\,dx \\
&-\xi(t)\int_0^\ell x|u|^{p-2}u\bigg(\int_0^t g(t-s)(u(t)-u(s))\,ds\bigg)\,dx \\
&-\xi(t)\int_0^\ell x u_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx \\
&+a\xi(t)\int_0^\ell x u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \\
&-\xi(t)\int_0^\ell x u_t^2\int_0^t g(t-s)\,ds\,dx \\
&-\xi'(t)\int_0^\ell x u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx.
\end{aligned}
\tag{4.11}
$$

We now estimate the right hand side of (4.11). For $\theta > 0$, similar as in [27], we have the estimates of the first to the fourth terms. The first term

$$
\begin{aligned}
&\xi(t)\int_0^\ell x u_x(t)\bigg(\int_0^t g(t-s)(u_x(t)-u_x(s))\,ds\bigg)\,dx \\
&\qquad\le \theta\xi(t)\int_0^\ell x u_x^2\,dx + \frac{1}{4\theta}\,\xi(t)\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t).
\end{aligned}
\tag{4.12}
$$

The second term

$$
\begin{aligned}
&\xi(t)\int_0^\ell x\bigg(\int_0^t g(t-s)(u_x(t)-u_x(s))\,ds\bigg)\bigg(\int_0^t g(t-s)u_x(s)\bigg)\,dx \\
&\quad\le 2\theta(1-l)^2\xi(t)\int_0^\ell x u_x^2\,dx + \bigg(2\theta+\frac{1}{4\theta}\bigg)\xi(t)\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t).
\end{aligned}
\tag{4.13}
$$

The third term

$$
\begin{aligned}
&\xi(t)\int_0^\ell x|u|^{p-2}u\bigg(\int_0^t g(t-s)(u(t)-u(s))\,ds)\bigg)\,dx \\
&\qquad\le \theta C^*\xi(t)\int_0^\ell x u_x^2\,dx + \xi(t)\,\frac{C_p}{4\theta}\,\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t),
\end{aligned}
\tag{4.14}
$$

where

$$
C^* = \frac{C_*}{3-p}\left(\frac{2p}{l(p-2)}\,E(0)\right)^{p-2}.
$$

The fourth term

$$(4.15)\quad -\xi(t)\int_0^\ell xu_t\int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx \le \theta\xi(t)\int_0^\ell xu_t^2\,dx - \frac{g(0)}{4\theta}\,C_p\xi(t)(g'\circ u_x)(t).$$

For the fifth term, by Young's inequality and Lemma 2.3, we have

$$(4.16)\quad a\xi(t)\int_0^\ell xu_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx \le a\theta\xi(t)\int_0^\ell xu_t^2\,dx + \frac{aC_p}{4\theta}\,\xi(t)\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t).$$

For the sixth term

$$\begin{aligned}(4.17)\quad -\xi'(t)\int_0^\ell xu_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx &\le \xi(t)\left|\frac{\xi'(t)}{\xi(t)}\right|\left[\theta\int_0^\ell xu_t^2\,dx + \frac{C_p}{4\theta}\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t)\right]\\ &\le \theta L\xi(t)\int_0^\ell xu_t^2\,dx + \frac{C_pL}{4\theta}\,\xi(t)\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t).\end{aligned}$$

A combination of (4.11)–(4.17) yields (4.10). □

Proof of Theorem 2.11. Since g is continuous and $g(0)>0$, then for any $t_0>0$, we have

$$(4.18)\qquad \int_0^t g(s)\,ds \ge \int_0^{t_0} g(s)\,ds := g_0, \quad \text{for all } t\ge t_0.$$

By using (2.3), (4.5), (4.10) and (4.18), we obtain

$$\begin{aligned}F'(t) =& E'(t)+\varepsilon_1\Psi'(t)+\varepsilon_2\chi'(t)\\ =& \frac{1}{2}\,(g'\circ u_x)(t) - \frac{1}{2}\,g(t)\int_0^\ell xu_x^2\,dx - a\int_0^\ell xu_t^2\,dx + \varepsilon_1\Psi'(t)+\varepsilon_2\chi'(t)\\ \le& -\left[a-\varepsilon_1\bigg(1+\frac{a}{2\beta}+\frac{L}{2\alpha}\bigg)\xi(t)+\varepsilon_2\xi(t)(g_0-\theta(1+L)-a\theta)\right]\int_0^\ell xu_t^2\,dx\\ &+\varepsilon_1\xi(t)\int_0^\ell x|u|^p\,dx + \left[\frac{1}{2}-\frac{\varepsilon_2\xi(0)}{4\theta}\,C_pg(0)\right](g'\circ u_x)(t)\\ &-\left\{\frac{\varepsilon_1}{2}\,(l-a\beta C_p-\alpha C_pL)-\varepsilon_2\theta\Big[(1+C^*+2(1-l)^2\Big]\right\}\xi(t)\int_0^\ell xu_x^2\,dx\\ &+\left\{\frac{\varepsilon_1}{2l}+\varepsilon_2\left[\frac{1}{2\theta}+2\theta+\frac{C_p+(a+L)C_p}{4\theta}\right]\right\}\\ &\times\xi(t)\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t)\end{aligned}$$

$$\begin{aligned}
&\leq -\bigg[\frac{a}{\xi(0)} - \varepsilon_1\bigg(1+\frac{a}{2\beta}+\frac{L}{2\alpha}\bigg) + \varepsilon_2(g_0-\theta(1+L)-a\theta)\bigg]\xi(t)\int_0^\ell x u_t^2\,dx \\
&\quad + \varepsilon_1\xi(t)\int_0^\ell x|u|^p\,dx + \bigg[\frac{1}{2} - \frac{\varepsilon_2\xi(0)}{4\theta}\,C_p g(0)\bigg](g'\circ u_x)(t) \\
&\quad - \bigg\{\frac{\varepsilon_1}{2}\,(l-a\beta C_p-\alpha C_p L) - \varepsilon_2\theta\big[(1+C^*+2(1-l)^2\big]\bigg\}\xi(t)\int_0^\ell x u_x^2\,dx \\
&\quad + \bigg\{\frac{\varepsilon_1}{2l} + \varepsilon_2\bigg[\frac{1}{2\theta}+2\theta+\frac{C_p+(a+L)C_p}{4\theta}\bigg]\bigg\} \\
&\quad \times \xi(t)\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)(g^r\circ u_x)(t),
\end{aligned}$$

since $0<\xi(t)\leq\xi(0)$. When $a>0$, we choose α and β so small that $l-a\beta C_p-\alpha C_p L>l/2$ and then choose θ small enough satisfying

$$k_2 = \frac{\varepsilon_1 l}{4} - \varepsilon_2\theta\big[(1+C^*+2(1-l)^2\big] > 0. \tag{4.19}$$

As far as α,β and θ are fixed, we then pick ε_1 and ε_2 so small that (4.4) and (4.19) remain valid and

$$\begin{aligned}
k_1 &= \frac{a}{\xi(0)} - \varepsilon_1\bigg(1+\frac{a}{2\beta}+\frac{L}{2\alpha}\bigg) + \varepsilon_2(g_0-\theta(1+L)-a\theta) > 0, \\
k_3 &= \frac{1}{2} - \frac{\varepsilon_2 C_p g(0)}{4\theta}\,\xi(0) \\
&\quad - \bigg\{\frac{\varepsilon_1}{2l} + \varepsilon_2\bigg[\frac{1}{2\theta}+2\theta+\frac{C_p+(a+L)C_p}{4\theta}\bigg]\bigg(\int_0^t g^{2-r}(s)\,ds\bigg)\bigg\} > 0.
\end{aligned}$$

Therefore, using the assumption $g'(t)\leq-\xi(t)g^r(t)$ in (G2), we have, for some $\sigma>0$ and for all $t\geq t_0$,

$$F'(t) \leq -\sigma\xi(t)\bigg[\int_0^\ell x u_t^2\,dx - \int_0^\ell x|u|^p\,dx + \int_0^\ell x u_x^2\,dx + (g^r\circ u_x)(t)\bigg]. \tag{4.20}$$

When $a=0$, we choose θ,α so small that $g_0-(1+L)\theta>g_0/2$, $l-\alpha C_p L>l/2$, and

$$\frac{4\theta[1+C^*+2(1-l)^2]}{l} < \frac{g_0}{2+L/\alpha}.$$

Whence θ and α are fixed, the choice of ε_1 and ε_2 satisfying

$$\frac{4\theta[1+C^*+2(1-l)^2]}{l}\,\varepsilon_2 < \varepsilon_1 < \frac{g_0\varepsilon_2}{2+L/\alpha}$$

will make

$$k_1 = -\varepsilon_1\bigg(1+\frac{L}{2\alpha}\bigg)\xi(0) + \varepsilon_2\xi(0)(g_0-\theta(1+L)) > 0, \tag{4.21}$$

$$k_2 = \frac{\varepsilon_1}{2}\,(l-\alpha C_p L) - \varepsilon_2\theta[(1+C^*+2(1-l)^2] > 0. \tag{4.22}$$

We then pick ε_1 and ε_2 so small that (4.4), (4.21) and (4.22) remain valid and

$$\begin{aligned} k_3 = \frac{1}{2} - \frac{\varepsilon_2 C_p g(0)}{4\theta}\,\xi(0) \\ - \left\{ \frac{\varepsilon_1}{2l} + \varepsilon_2 \left[\frac{1}{2\theta} + 2\theta + \frac{C_p + LC_p}{4\theta} \right] \left(\int_0^t g^{2-r}(s)\,ds \right) \right\} > 0. \end{aligned}$$

We can still get (4.20). Next, as (4.20) is proved, we will give the following two cases according to the different ranges of r:

Case 1. $r = 1$.

By virtue of the choice of $\varepsilon_1, \varepsilon_2$ and θ, we estimate (4.20) and obtain, for some constant $\alpha > 0$,

$$F'(t) \le -\alpha \xi(t) E(t), \quad \text{for all } t \ge t_0. \tag{4.23}$$

Hence, with the help of the left hand side inequality in (4.4) and (4.23), we find

$$F'(t) \le -\alpha\alpha_1 \xi(t) F(t), \quad \text{for all } t \ge t_0. \tag{4.24}$$

A simple integration of (4.24) over (t_0, t) leads to

$$F(t) \le F(t_0) e^{-(\alpha\alpha_1)\int_{t_0}^t \xi(s)\,ds}, \quad \text{for all } t \ge t_0. \tag{4.25}$$

Therefore, (2.8) is established by virtue of (4.4) again.

Case 2. $1 < r < 3/2$.

By using (2.1), we get

$$g(t)^{1-r} \ge (r-1) \int_{t_0}^t \xi(s)\,ds + g(t_0)^{1-r}.$$

For all $0 < \tau < 1$, we further have

$$\int_0^\infty g^{1-\tau}(s)\,ds \le \int_0^\infty \frac{1}{[(r-1)\int_{t_0}^t \xi(s)\,ds + g(t_0)^{1-r}]^{(1-\tau)/(r-1)}}\,dt.$$

For $0 < \tau < 2 - r < 1$, we have $(1-\tau)/(r-1) > 1$. And using (2.2), we obtain

$$\int_0^\infty g^{1-\tau}(s)\,ds < \infty, \quad \text{for all } 0 < \tau < 2 - r.$$

So Lemma 4.3 and (2.7) yield

$$\begin{aligned} (g \circ u_x)(t) &\le C\left(E(0) \int_0^\infty g^{1-\tau}(s)\,ds \right)^{(r-1)/(r-1+\tau)} (g^r \circ u_x)^{\tau/(r-1+\tau)} \\ &\le C (g^r \circ u_x)^{\tau/(r-1+\tau)} \end{aligned}$$

for some positive constant C. Therefore, for any $r_1 > 1$, we arrive at

$$\begin{aligned}E^{r_1}(t) \le& \, CE^{r_1-1}(0)\bigg(\int_0^\ell xu_t^2\,dx - \int_0^\ell x|u|^p\,dx + \int_0^\ell xu_x^2\,dx\bigg) \\ &+ C(g\circ u_x)^{r_1} \\ \le& \, CE^{r_1-1}(0)\bigg(\int_0^\ell xu_t^2\,dx - \int_0^\ell x|u|^p\,dx + \int_0^\ell xu_x^2\,dx\bigg) \\ &+ C(g^r\circ u_x)^{\tau r_1/(r-1+\tau)}.\end{aligned} \tag{4.26}$$

By choosing $\tau = 1/2$ and $r_1 = 2r-1$ (hence $\tau r_1/(r-1+\tau) = 1$), estimate (4.26) gives, for some $\Gamma > 0$,

$$E^{r_1}(t) \le \Gamma\bigg[\int_0^\ell xu_t^2\,dx - \int_0^\ell x|u|^p\,dx + \int_0^\ell xu_x^2\,dx + (g^r\circ u_x)(t)\bigg] \tag{4.27}$$

By combining (4.4), (4.20) and (4.27), we obtain

$$F'(t) \le -\frac{\sigma}{\Gamma}\,\xi(t)E^{r_1}(t) \le -\frac{\sigma}{\Gamma}\,\alpha_1^{r_1}F^{r_1}(t)\xi(t), \quad \text{for all } t \ge t_0. \tag{4.28}$$

A simple integration of (4.28) leads to

$$F(t) \le C_1\bigg(1 + \int_{t_0}^t \xi(s)\,ds\bigg)^{-1/(r_1-1)}, \quad \text{for all } t \ge t_0. \tag{4.29}$$

Therefore,

$$\int_{t_0}^\infty F(t)\,dt \le C_1\int_{t_0}^\infty \frac{1}{\Big(1 + \int_{t_0}^t \xi(s)\,ds\Big)^{1/(r_1-1)}}\,dt.$$

Since $1/(r_1-1) > 1$ and $1 + \int_{t_0}^t \xi(s)\,ds \to +\infty$ as $t \to +\infty$, we get from (2.2) that

$$\int_{t_0}^\infty F(t)\,dt < \infty. \tag{4.30}$$

In addition, by using (2.2), we have

$$tF(t) \le \frac{C_1 t}{\Big(1 + \int_{t_0}^t \xi(s)\,ds\Big)^{1/(r_1-1)}} \le C_r.$$

Therefore, we obtain

$$\sup_{t\ge t_0} tF(t) < +\infty. \tag{4.31}$$

Since E is bounded, we use (4.4), (4.30) and (4.31) to get

$$\int_0^\infty F(t)\,dt + \sup_{t\ge 0} tF(t) < \infty.$$

Then, by using (2.7) and Lemma 4.4, we have

$$\begin{aligned}(g\circ u_x)(t) \leq{}& C_2\bigg(t\|u_x(\,\cdot\,,t)\|_H^2+\int_0^t\|u_x(\,\cdot\,,s)\|_H^2\,ds\bigg)^{(r-1)/r}\\ &\times\bigg(\int_0^t g^r(t-s)\|u_x(\,\cdot\,,t)-u_x(\,\cdot\,,s)\|_H^2\,ds\bigg)^{1/r}\\ \leq{}& C_2\bigg(tF(t)+\int_0^t F(s)\,ds\bigg)^{(r-1)/r}(g^r\circ u_x)^{1/r}\leq C_3(g^r\circ u_x)^{1/r},\end{aligned}$$

which implies that

$$(g^r\circ u_x)(t)\geq C_4(g\circ u_x)^r, \tag{4.32}$$

for some constant $C_4>0$. Consequently, a combination of (4.20) and (4.32) yields, for all $t\geq t_0$,

$$F'(t)\leq -C_5\xi(t)\bigg[\int_0^\ell xu_t^2\,dx-\int_0^\ell x|u|^p\,dx+\int_0^\ell xu_x^2\,dx+(g\circ u_x)^r\bigg],$$

for some constant $C_5>0$. On the other hand, as in [1], we can get

$$E^r(t)\leq C_6\bigg[\int_0^\ell xu_t^2\,dx-\int_0^\ell x|u|^p\,dx+\int_0^\ell xu_x^2\,dx+(g\circ u_x)^r\bigg]$$

for all $t\geq 0$ and some constant $C_6>0$. Combining the last two inequalities and (4.4), we obtain

$$F'(t)\leq -C_7\xi(t)F^r(t),\quad\text{for all } t\geq t_0 \tag{4.33}$$

for some constant $C_7>0$. A simple integration of (4.33) over (t_0,t) gives

$$F(t)\leq C_8\bigg(1+\int_{t_0}^t\xi(s)\,ds\bigg)^{-1/(r-1)},\quad\text{for all } t\geq t_0.$$

Therefore, (2.8) is obtained by virtue of (4.4) again. □

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