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THE BOLZANO PROPERTY AND THE CUBE-LIKE COMPLEXES

Przemysław Tkacz

Abstract. Introducing the *Bolzano property*, we present a topological version of the Poincaré–Miranda theorem. One simple, and one algorithmic proof that n-cube-like complexes have this property are given. Moreover, we investigate under what conditions the inverse limit preserves the Bolzano property. Finally, we give a characterization of the Bolzano property for locally connected spaces.

1. Introduction

Bolzano proved that if a continuous function f in a closed interval $[a,b]$ changes sign at the endpoints, i.e. $f(a) \cdot f(b) \leq 0$, then this function equals zero at least at one point of the interval. Nearly a hundred years later Poincaré stated without a proof the following claim [10], [11]:

Let $f_1, \ldots, f_n$ be n continuous functions of n variables $x_1, \ldots, x_n$; the variable x_i varies between the limits a_i and $-a_i$. Suppose that for every $x_i = a_i$ the function f_i is constantly positive and that for every $x_i = -a_i$ the function f_i is constantly negative; I say there will exist a collection of values of x_i at which all f_i vanish

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The above result provides a solution to the following system of nonlinear equations:

$$\begin{cases} f_1(x_1,\dots,x_n)=0, \\ \dots\dots\dots\dots\dots\dots \\ f_n(x_1,\dots,x_n)=0, \end{cases}$$

where $f_1,\dots,f_n$ are continuous and satisfy Poincaré's boundary conditions.

In 1940, Carlo Miranda [9] rediscovered the Poincaré theorem and showed that it is equivalent to the Brouwer fixed point theorem. More information about the history, proofs, and consequences of the above mentioned theorems, the reader may find in Kulpa's paper [7].

In the papers [5], [8], the Poincaré–Miranda theorem was extended from n-cubes to *n-cube-like polyhedrons*. This gives a generalization of some classic results due to Bolzano, Poincaré, Bohl, and Brouwer [1], [2].

Kulpa described the Bolzano–Poincaré–Miranda property for topological spaces in the following way [6, p. 91]:

A family $\{(A_i,B_i) : i=1,\dots,n\}$ of pairs of non-empty disjoint closed subsets of a topological space X is said to be an n-dimensional boundary system whenever for each continuous map $f\colon X\to R^n$, $f=(f_1,\dots,f_n)$, satisfying for each $i\le n$ the Bolzano condition

$$f_i(A_i)\subset(-\infty,0], \qquad f_i(B_i)\subset[0,\infty),$$

there exists a point $c\in X$ such that $f(c)=0$.

While Kulpa's definition of the Bolzano property is external, here we present an internal one. Additionally, we compare these two definitions. Next, we show that the Bolzano property holds for n-cube-like-polyhedrons. We provide two different proofs for the latter statement: an existential one, and an algorithmic one. In the first one, we apply very simple arguments inspired by Kulpa's proof of the Poincaré theorem [7]. In the second one, we use concepts introduced in the papers [5], [8]. However, instead of copying and gluing all boundary of the n-cube-like-complex (as in [8]) or making a product (as in [5]), we copy and glue only one face.

In the next part of this paper, we investigate the conditions under which the inverse limit preserves the Bolzano property. In that way we obtain a generalization of Kulpa's theorem [6, p. 90].

Finally, in Theorem 6.1, we provide a characterization of the Bolzano property for the locally connected spaces.

2. The Bolzano property and the Poincaré–Miranda theorem

Definition 2.1. A topological space X is said to have the *n-dimensional Bolzano property* if there exists a family $\{(A_i,B_i) : i=1,\dots,n\}$ of pairs of disjoint closed subsets of X such that for every family $\{(H_i^-, H_i^+) : i = 1, \dots, n\}$ of closed sets such that for each $0 < i \le n$

$$A_i \subset H_i^-, \quad B_i \subset H_i^+, \quad \text{and} \quad H_i^- \cup H_i^+ = X,$$

we have $\bigcap\{H_i^- \cap H_i^+ : i = 1, \dots, n\} \neq \emptyset$. The family $\{(A_i, B_i) : i = 1, \dots, n\}$ is called an *n-dimensional boundary system.*

Let $n \in \mathbb{N}$ and $\{(A_i, B_i) : i = 1, \dots, n\}$ be an n-dimensional boundary system for a topological space X. For each $k \le n$, the family $\{(A_i, B_i) : i = 1, \dots, k\}$ is a k-dimensional boundary system. For $l \in \mathbb{N}$, the family $\{(A_i, B_i) : i = 1, \dots, n + l\}$, where $A_i = A_n$, $B_i = B_n$ for $i > n$, is not an $(n + l)$-dimensional boundary system. To see this put $H_n^- = A_n$, $H_n^+ = X$, $H_{n+1}^- = X$, $H_{n+1}^+ = B_n$. We obtain $(H_n^- \cap H_n^+) \cap (H_{n+1}^- \cap H_{n+1}^+) = A_n \cap B_n = \emptyset$.

Theorem 2.2. *Let $\{(A_i, B_i) : i = 1, \dots, n\}$ be an n-dimensional boundary system in the space X and $f\colon X \to R^n$, $f = (f_1, \dots, f_n)$, be a continuous map such that for each $i \le n$, $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$. Then there exists $c \in X$ such that $f(c) = 0$.*

Proof. Put $H_i^- = f_i^{-1}((-\infty, 0])$ and $H_i^+ = f_i^{-1}([0, \infty))$ for $i \le n$. □

Lemma 2.3. *Let $F_1, \dots, F_n$ be a family of closed subsets of the normal space X such that $\bigcap\limits_{i=1}^{n} F_i = \emptyset$. Then there are closed G_δ-subsets $F_1', \dots, F_n'$ of X such that for each $i \le n$, $F_i \subset F_i'$ and $\bigcap\limits_{i=1}^{n} F_i' = \emptyset$.*

Proof. The sets F_1, $\bigcap\limits_{i=2}^{n} F_i$ are closed and disjoint. Since X is normal, there exists a closed G_δ-set F_1' such that $F_1 \subset F_1'$ and $F_1' \cap \bigcap\limits_{i=2}^{n} F_i = \emptyset$. Let us consider the family $F_1', F_2, \dots, F_n$ and apply the same argument for the set F_2. The rest of the proof runs as before. □

Theorem 2.4. *Let $\{(A_i, B_i) : i = 1, \dots, n\}$ be a family of pairs of non-empty disjoint closed subsets of a normal space X such that for each continuous map $f\colon X \to R^n$ satisfying $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$ for each $i \le n$, there exists $c \in X$ such that $f(c) = 0$. Then $\{(A_i, B_i) : i = 1, \dots, n\}$ is an n-dimensional boundary system.*

Proof. Let $\{(H_i^-, H_i^+) : i = 1, \dots, n\}$ be a family of closed sets such that $A_i \subset H_i^-$, $B_i \subset H_i^+$ and $H_i^- \cup H_i^+ = X$ for $i \le n$. Suppose that $\bigcap\{H_i^- \cap H_i^+ : i = 1, \dots, n\} = \emptyset$. By Lemma 2.3, we can assume that $\{(H_i^-, H_i^+) : i = 1, \dots, n\}$ is a family of closed G_δ-sets. Hence, for each $i \le n$ there exist continuous maps $g_i, h_i\colon X \to [0, 1]$ such that $g_i^{-1}(0) = H_i^-$ and $h_i^{-1}(0) = H_i^+$. We have

$$\bigcap\{g_i^{-1}(0) \cap h_i^{-1}(0) : i = 1, \dots, n\} = \bigcap\{H_i^- \cap H_i^+ : i = 1, \dots, n\} = \emptyset.$$

For each $i \le n$ let us define a map $f_i(x) := g_i(x) - h_i(x)$. Since the map $f = (f_1, \dots, f_n)$ satisfies the assumptions, there is $c \in X$ such that $f(c) = 0$. It means that for each $i \le n$, we have $g_i(c) = h_i(c)$. Since $\{H_i^-, H_i^+\}$ is a cover of X, we infer that $g_i(c) = h_i(c) = 0$. Thus, $c \in \bigcap\{H_i^- \cap H_i^+ : i = 1, \dots, n\}$, a contradiction. □

Remark 2.5. In the metric space (X, d), Theorem 2.4 can be proved by putting $f_i(x) = d(x, H_i^-) - d(x, H_i^+)$ for $i \le n$.

The following example shows that Theorem 2.4 does not hold for all regular spaces.

Example 2.6 ([4, Example 1.5.9]). Let M_0 be the subset of the plane defined by the condition $y \ge 0$, i.e. the closed upper half-plane, let z_0 be the point $(0, -1)$ and let $M = M_0 \cup \{z_0\}$. Denote by L the line $y = 0$ and by L_i the segment consisting of all points $(x, 0) \in L$ with $i - 1 \le x \le i$, $i = 1, 2, \dots$. For each point $z = (x, 0) \in L$ denote by $C_1(z)$ the set of all points $(x, y) \in M_0$, where $0 \le y \le 2$, by $C_2(z)$ the set of all points $(x + y, y) \in M_0$, where $0 \le y \le 2$, and let $\mathcal{B}(z)$ be the family of all sets of the form $(C_1(z) \cup C_2(z)) \setminus D$, where D is a finite set such that $z \notin D$. Furthermore, for each point $z \in M_0 \setminus L$ let $\mathcal{B}(z) = \{\{z\}\}$ and, finally, let $\mathcal{B}(z_0) = \{U_i(z_0)\}_{i=1}^{\infty}$, where $U_i(z_0)$ consists of z_0 and all points $(x, y) \in M_0$ with $x \ge i$. The topology of the space M is generated by the neighbourhood system $\{\mathcal{B}(z)\}_{z \in M}$.

Let n be a natural number, $n > 1$. Put $A_i = \{z_0\}$, $B_i = L_1$ for $i \le n$. First, we show that $f(z_0) = (0, \dots, 0)$ for each continuous map $f\colon M \to R^n$, $f = (f_1, \dots, f_n)$, such that $f_i(A_i) \subset (-\infty, 0]$, $f_i(B_i) \subset [0, \infty)$ for $i \le n$. We claim that for each $i \le n$ and each $j \ge 1$ we have $f_i^{-1}([0, \infty)) \cap L_j \ne \emptyset$. The proof of this fact is analogous to the one presented in [4, Examples 1.4.6, 1.5.9]. Since $f_i(z_0) \le 0$ for $i \le n$ and each neighbourhood of z_0 contains some segment L_{j_0}, we conclude that $f(z_0) = (0, \dots, 0)$. But for the sets $H_1^- = A_1$, $H_1^+ = M$ and $H_2^- = M$, $H_2^+ = \{z_0\}$ we have $\bigcap\{H_i^- \cap H_i^+ : i = 1, 2\} = \emptyset$.

Question 2.7. Does Theorem 2.4 hold for $T_{3\frac{1}{2}}$ spaces?

Remark 2.8. The topological space X is T_5 if and only if for each pair of closed sets $A, B \subset X$ there exist closed sets $F, G \subset X$ such that $F \cap (A \cup B) = A$, $G \cap (A \cup B) = B$ and $F \cup G = X$.

Theorem 2.9. *Let $\{(A_i, B_i) : i = 1, \dots, n\}$ be an n-dimensional boundary system in a T_5 space X. Then for each $i_0 \le n$ the subspaces A_{i_0}, B_{i_0} have the $(n-1)$-dimensional Bolzano property. Moreover, the families*

$$\{(A_{i_0} \cap A_i, A_{i_0} \cap B_i) : i \ne i_0\}, \qquad \{(B_{i_0} \cap A_i, B_{i_0} \cap B_i) : i \ne i_0\}$$

are $(n-1)$-dimensional boundary systems in A_{i_0}, B_{i_0}, respectively.

PROOF. For an arbitrary $i_0 \in \{1,\dots,n\}$ take the set A_{i_0}.

Let $\{(F_i^-, F_i^+) : i \neq i_0\}$ be a family of closed sets such that $A_{i_0} \cap A_i \subset F_i^-$, $A_{i_0} \cap B_i \subset F_i^+$ and $F_i^- \cup F_i^+ = A_{i_0}$. By Remark 2.8, for $i \neq i_0$ there exist closed sets $F_i'^-, F_i'^+$ such that

$$F_i'^- \cap (F_i^- \cup F_i^+) = F_i^-, \quad F_i'^+ \cap (F_i^- \cup F_i^+) = F_i^+, \quad \text{and} \quad F_i'^- \cup F_i'^+ = X.$$

Let $H_i^- = F_i'^- \cup A_i$, $H_i^+ = F_i'^+ \cup B_i$ for $i \neq i_0$, and $H_{i_0}^- = A_{i_0}$, $H_{i_0}^+ = X$. Since the space X has the n-dimensional Bolzano property, we have $\bigcap\{H_i^- \cap H_i^+ : i = 1,\dots,n\} \neq \emptyset$. We leave it to the reader to verify that $\bigcap\{H_i^- \cap H_i^+ : i = 1,\dots,n\} = \bigcap\{F_i^- \cap F_i^+ : i \neq i_0\} \neq \emptyset$. The proof for B_{i_0} is similar. $\square$

By induction we get:

COROLLARY 2.10. *Let $I_1, I_2 \subset \{1,\dots,n\}$, $I_1 \cap I_2 = \emptyset$. Then the subspace $\bigcap_{i\in I_1} A_i \cap \bigcap_{i\in I_2} B_i$ has the $(n-(\mathrm{card}(I_1)+\mathrm{card}(I_2)))$-dimensional Bolzano property.*

EXAMPLE 2.11. Let $X = [0,1]\times[0,1]$ be a subspace of the half-disk topology (see [12, p. 96]). We will show that the thesis of Theorem 2.9 is not valid for the space X. Let $A_1 = \{0\}\times[0,1]$, $B_1 = \{1\}\times[0,1]$, $A_2 = [0,1]\times\{0\}$, $B_2 = [0,1]\times\{1\}$. First, we prove that the family $\{(A_i,B_i) : i = 1,2\}$ forms a 2-dimensional boundary system. Suppose not. There exists a family $\{(H_i^-, H_i^+) : i = 1,2\}$ satisfying the conditions from Definition 2.1 such that $\bigcap\{H_i^- \cap H_i^+ : i = 1,2\} = \emptyset$. Observe that for each $x \in A_2$, there is an open neighborhood $U(x)$, contained in one of the sets $H_1^- \setminus H_1^+, H_1^+ \setminus H_1^-$, or $H_2^- \setminus H_2^+$. Let $U = \bigcup\{U(x) : x \in A_2\}$.

There is $n_0 \in N$ such that $[0,1]\times\{1/n_0\} \subset U$: If not, then there is a sequence $\{x_n\}$ such that $x_n \in ([0,1]\times\{1/n\}) \cap (X \setminus U)$ for all $n \in N$. Since $[0,1]\times[0,1]$ with Euclidean metric is a compact space, we may assume that $x_n \to x$. Each neighbourhood (in the half-disk topology) of x meets a closed set $X \setminus U$, and thus $x \in X \setminus U$. We have $x \in A_2$, a contradiction.

Since the subspace $Y = [0,1]\times[1/n_0, 1]$ has the Euclidean topology its opposite faces form a 2-dimensional boundary system, consequently the sets $H_1^- \cap Y$, $H_1^+ \cap Y$, $(H_2^- \cap Y) \cup ([0,1]\times\{1/n_0\})$, $H_2^+ \cap Y$ have nonempty intersection. This intersection is contained in $X \setminus U$. It follows that $\bigcap\{H_i^- \cap H_i^+ : i = 1,2\} \neq \emptyset$, which is the desired conclusion. Furthermore, the subspace A_2 has discrete topology, hence it does not have the 1-dimensional Bolzano property.

QUESTION 2.12. Does Theorem 2.9 hold for T_4 spaces?

3. Combinatorial techniques

3.1. Notation. We use terminology of Dugundji and Granas [3]. Let A be a finite set. Denote by $P(A)$ the family of all subsets of A, and by $P_{n+1}(A)$ the family of all subsets of A of the cardinality $n+1$. The elements of $P_{n+1}(A)$ are called *n-simplexes* defined on the set A. Let $S \in P_{n+1}(A)$. Then $T \in P_{k+1}(S)$ is called *a k-face* of the n-simplex S.

Definition 3.1. The family $\mathcal{K} \subset P(A)$ is called an *abstract complex* if for each $V \in \mathcal{K}$, we have $P(V) \subset \mathcal{K}$. The *support* of the abstract complex $\mathcal{K}$ is defined by the formula:

$$|\mathcal{K}| := \bigcup \{V : V \in \mathcal{K}\}.$$

The elements of $|\mathcal{K}|$ are called *vertices*.

Definition 3.2. Let $\mathcal{S} \subset P(A)$. Then $\mathcal{K}(\mathcal{S}) := \bigcup_{S \in \mathcal{S}} P(S)$ is called a *complex generated by the family $\mathcal{S}$*.

Definition 3.3. If $\mathcal{S} \subset P_{n+1}(A)$, then the *boundary of the complex $\mathcal{K}(\mathcal{S})$* is the subcomplex $\partial\mathcal{K}(\mathcal{S})$ generated by the family

$$\mathcal{B} = \{T \in P_n(A) : \exists! S \in \mathcal{S} \text{ such that } T \subset S\}.$$

3.2. Cube-like complex. Before we introduce the main definition, we present some intuition. Consider an n-dimensional cube $I^n = [0,1]^n$ in R^n. Observe that the boundary ∂I^n is the union of n pairs of opposite faces, $(n-1)$-dimensional cubes, i.e.

$$\partial I^n = \bigcup_{i=1}^{n} I_i^- \cup I_i^+,$$

where $I_i^- = \{x \in I^n : x(i) = 0\}$, $I_i^+ = \{x \in I^n : x(i) = 1\}$ for all $i \leq n$. Moreover, for all $i_0 \in \{1, \dots, n\}$ and $\varepsilon \in \{-,+\}$ the opposite faces of an $(n-1)$-dimensional cube $I_{i_0}^\varepsilon$ have the following form: $I_{i_0}^\varepsilon \cap I_i^-, I_{i_0}^\varepsilon \cap I_i^+$ for $i \neq i_0$. The above observation, Theorem 2.9, and Corollary 2.10 underlie the definition of an n-cube-like complex.

Definition 3.4. Let A be a non-empty finite set. Every complex consisting of a single vertex (an element of the set A) is called a *0-cube-like complex*, denoted by $\mathcal{K}^0$. The complex $\mathcal{K}^n$ generated by the family $\mathcal{S} \subset P_{n+1}(A)$ is said to be an *n-cube-like complex* if:

(a) For every $(n-1)$-face $T \in \mathcal{K}^n \setminus \partial\mathcal{K}^n$, there exist exactly two n-simplexes $S, S' \in \mathcal{K}^n$ such that $S \cap S' = T$.

(b) There exists a sequence of n pairs of subcomplexes $\mathcal{F}_i^-, \mathcal{F}_i^+$ called *i-th opposite faces* such that:

  (b$_1$) $\partial\mathcal{K}^n = \bigcup_{i=1}^{n} \mathcal{F}_i^- \cup \mathcal{F}_i^+$,

  (b$_2$) $\mathcal{F}_i^- \cap \mathcal{F}_i^+ = \emptyset$ for $i \in \{1, \dots, n\}$,

  (b$_3$) for each $i_0 \in \{1, \dots, n\}$, and each $\varepsilon \in \{-,+\}$, the subcomplex $\mathcal{F}_{i_0}^\varepsilon$ is an $(n-1)$-cube-like complex such that its opposite faces have a form $\mathcal{F}_{i_0}^\varepsilon \cap \mathcal{F}_i^-, \mathcal{F}_{i_0}^\varepsilon \cap \mathcal{F}_i^+$ for $i \neq i_0$.

Let $(\overline{K}, \overline{\mathcal{K}})$ be a polyhedron, where $\overline{\mathcal{K}}$ is a simplicial complex and $\overline{K}$ is the support of $\overline{\mathcal{K}}$. Each polyhedron determines an abstract complex $\mathcal{K}$ called its vertex-scheme: $\mathcal{K}$ consists of subsets of vertices that span the simplexes of $\overline{\mathcal{K}}$. $|\mathcal{K}|$ is the set of vertices of $(\overline{K}, \overline{\mathcal{K}})$ (see [3]).

Definition 3.5. The polyhedron $(\overline{K}, \overline{\mathcal{K}})$ in R^m is said to be an *n-cube-like polyhedron* if its vertex-scheme abstract complex $\mathcal{K}$ is n-cube-like. The opposite faces of $\overline{\mathcal{K}}$ correspond to faces of $\mathcal{K}$ and are denoted by $\overline{\mathcal{F}_i^-}$ and $\overline{\mathcal{F}_i^+}$ (supports by $\overline{F_i^-}$ and $\overline{F_i^+}$) for $i \leq n$.

Obviously, n-dimensional cubes (triangulated) are n-cube-like polyhedrons, but not the only ones. An n-cube-like polyhedron can be not connected. The example is a disjoint sum of an n-cube and a number of closed simplicial n-manifolds. Moreover a Möbius strip, a solid torus, a cube with holes are also examples of n-cube-like polyhedra (see [5], [8]).

Note that for a given n-cube-like complex, we can find more than one sequence of opposite faces. In the further part of the paper, by an n-cube like complex, we mean an n-cube like complex equipped with a fixed sequence of opposite faces.

Observe that if $(\overline{K}, \overline{\mathcal{K}})$ is an n-cube-like polyhedron and $(\overline{F_i^-}, \overline{\mathcal{F}_i^-}), (\overline{F_i^+}, \overline{\mathcal{F}_i^+})$ are its i-th opposite faces, then the polyhedron $(\overline{K}, \overline{\mathcal{K}'})$ where $\overline{\mathcal{K}'}$ is the barycentric subdivision of $\overline{\mathcal{K}}$ is an n-cube-like polyhedron equipped with faces $(\overline{F_i^-}, \overline{\mathcal{F}'^-_i})$, $(\overline{F_i^+}, \overline{\mathcal{F}'^+_i})$, where $\overline{\mathcal{F}'^-_i}, \overline{\mathcal{F}'^+_i}$ are barycentric subdivision of $\overline{\mathcal{F}_i^-}$ and $\overline{\mathcal{F}_i^+}$, respectively . The fact that a triangulation of an arbitrary k-simplex $T \in \overline{\mathcal{K}}$ agrees with the triangulation of simplexes containing T as a face, allows the reader to proceed with the proof of this observation by induction on n.

3.3. Combinatorial lemma.

Definition 3.6. Let $\mathcal{K}^n$ be an n-cube-like complex. A map $\phi\colon |\mathcal{K}^n| \to \{0, \ldots, n\}$ is said to be a *coloring function.* A subset $C \subset |\mathcal{K}^n|$ is called *k-colored,* if $\phi(C) = \{0, \ldots, k\}$.

Definition 3.7. Let $\phi\colon |\mathcal{K}^n| \to \{0, \ldots, n\}$ be a coloring function of the n-cube-like complex. A sequence of different n-simplexes $S_1, \ldots, S_m \in \mathcal{K}^n$ is called a *chain,* if $\phi(S_i \cap S_{i+1}) = \{0, \ldots, n-1\}$ for $i < m$.

The chain $S_1, \ldots, S_m$ is called *maximal,* if for each chain $T_1, \ldots, T_{m'}$ such that $\{S_1, \ldots, S_m\} \subset \{T_1, \ldots, T_{m'}\}$ we have $m = m'$.

The maximal chains $S_1, \ldots, S_m$ and $T_1, \ldots, T_m$ are called *equivalent* if

$$\{S_1, \ldots, S_m\} = \{T_1, \ldots, T_m\}.$$

Observation 3.8. Let $\phi\colon |\mathcal{K}^n| \to \{0, \ldots, n\}$ be a coloring function of an n-cube-like complex. Each $(n-1)$-colored $(n-1)$-face $T \in \mathcal{K}^n$ uniquely (up to equivalence) determines a maximal chain $S_1, \dots, S_m$ such that $T \subset S_i$ for some $i \leq m$.

Proof. The observation follows from the fact that any $(n-1)$-face is a face of exactly one, or two n-simplexes, depending whether it lies in the boundary of $\mathcal{K}^n$, or not. Moreover, each $(n-1)$-colored n-simplex has exactly two $(n-1)$-colored $(n-1)$-faces, and each n-colored n-simplex has exactly one $(n-1)$-colored $(n-1)$-face. $\square$

Lemma 3.9. *Let $\mathcal{K}^n$ be an n-cube-like complex. Let $\{H_i^-, H_i^+ : i = 1, \dots, n\}$ be a family of subsets of $|\mathcal{K}^n|$ such that $|\mathcal{F}_i^-| \subset H_i^-$, $|\mathcal{F}_i^+| \subset H_i^+$, and $H_i^- \cup H_i^+ = |\mathcal{K}^n|$ for $i \leq n$. Then there exists an n-simplex $S \in \mathcal{K}^n$ such that for each $i \leq n$, we have $H_i^- \cap S \neq \emptyset \neq H_i^+ \cap S$.*

Proof. Let us define a coloring map $\phi\colon |\mathcal{K}^n| \to \{0, 1, \dots, n\}$ by

$$\phi(s) := \max\left\{ j : s \in \bigcap_{i=0}^{j} F_i \right\},$$

where $F_0 = |\mathcal{K}^n|$, and $F_i = H_i^+ \setminus |\mathcal{F}_i^-|$ for $0 < i \leq n$.

If $s \in |\mathcal{F}_i^-|$, then $\phi(s) < i$, and if $s \in |\mathcal{F}_i^+|$, then $\phi(s) \neq i-1$. It follows that for each $(n-1)$-face $T \in \mathcal{K}^n$ such that $\phi(T \cap |\mathcal{F}_i^\varepsilon|) = \{0, \dots, n-1\}$, we have $i = n$ and $\varepsilon = -$. Moreover, the fact $H_i^- \cup H_i^+ = |\mathcal{K}^n|$ yields that if $\phi(s) = i-1$, then $s \in H_i^-$. Obviously, if $\phi(s) = i$, then $s \in H_i^+$.

The lemma will be proved if the number of n-colored n-simplexes is odd. Our proof will be by induction on the dimension of $\mathcal{K}^n$. The number of n-colored n-simplexes is odd for $n = 1$ (we leave it as an exercise). Now, let us consider those $(n-1)$-faces $T \in \partial\mathcal{K}^n$ for which $\phi(T) = \{0, 1, \dots, n-1\}$. It is known that $T \in \mathcal{F}_n^-$. From condition (b_3), the set $\mathcal{F}_n^-$ is an $(n-1)$-cube-like complex, and $\mathcal{F}_n^- \cap \mathcal{F}_i^-$ and $\mathcal{F}_n^- \cap \mathcal{F}_i^+$ are its i-th opposite faces for $i < n$. By the inductive assumption, there is an odd number of $(n-1)$-colored $(n-1)$-faces in $\mathcal{F}_n^-$.

Now let us consider all maximal chains of n-simplexes determined by $(n-1)$-colored $(n-1)$-faces from $\mathcal{F}_n^-$. There are two possibilities: the first and the last n-simplex of the maximal chain has $(n-1)$-colored $(n-1)$-face in $\mathcal{F}_n^-$, or the last n-simplex is n-colored. Since the first type of the maximal chains occupy even number of $(n-1)$-colored $(n-1)$-faces in $\mathcal{F}_n^-$, we get odd number of n-colored n-simplexes determined by the second type of maximal chains. Moreover, each maximal chain that starts at n-colored n-simplex, that is not counted above, must have an n-colored n-simplex at the end. $\square$

The intuition of reasoning presented in the proof of Lemma 3.9 is illustrated under Figure 1.

Let us observe that Lemma 3.9 can be formulated in the following way.

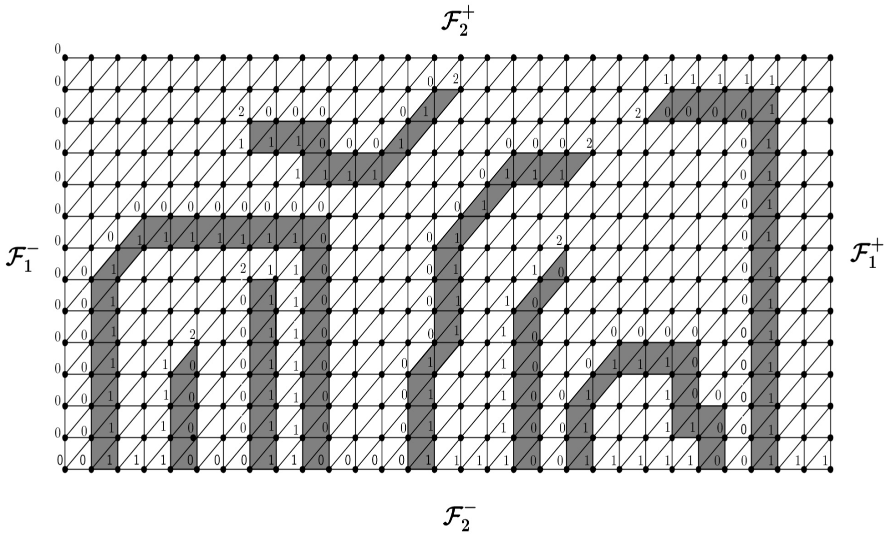

FIGURE 1. The illustration of the maximal chains.

REMARK 3.10. Let $\mathcal{K}^n$ be an n-cube-like complex, and let $\phi\colon |\mathcal{K}| \to \{0,1\}^n$ be a map such that for each $i \le n$, we have $\phi_i(|\mathcal{F}_i^-|) = \{0\}$ and $\phi_i(|\mathcal{F}_i^+|) = \{1\}$. Then there exists an n-simplex $S \in \mathcal{K}^n$ such that for each $i \le n$, we have $\phi_i(S) = \{0,1\}$.

3.4. Algorithmic proof of Lemma 3.9. In this section we present a method of finding the simplex described in Lemma 3.9. Let A be a finite set.

DEFINITION 3.11. Let $S = \{v_0, \dots, v_n\} \subset A$ be an n-simplex. An abstract complex $\mathcal{K}(\mathcal{F}) \subset \mathcal{P}(\{v_0, \dots, v_n\} \times \{0,1\})$ generated by the family of $(n+1)$-simplexes

$$\mathcal{F} = \big\{\{(v_0,0), \dots, (v_i,0), (v_i,1), \dots, (v_n,1)\} : i = 0, \dots, n\big\}$$

is called an *S-doubled complex* and it is denoted by $\mathrm{dc}(S)$.

Let $\mathcal{K}^n$ be an n-cube-like complex. Let $|\mathcal{K}^n| = \{w_0, \dots, w_m\}$ be a fixed enumeration of its vertices. Each k-simplex $T \in \mathcal{K}^n$ has uniquely determined orientation $T = \{w_{i_0}, \dots, w_{i_k}\}$, where $0 \leqslant i_0 < \ldots < i_k \leqslant m$. From now on, we assume that the orientation of each k-simplex $T = \{v_0, \dots, v_k\} \in \mathcal{K}^n$ is consistent with the enumeration of vertices of $\mathcal{K}^n$, i.e. $v_0 = w_{i_0}, v_1 = w_{i_1}, \dots, v_k = w_{i_k}$.

DEFINITION 3.12. Let $\mathcal{K}^n$ be an n-cube-like complex with $|\mathcal{K}^n| = \{w_0, \dots, w_m\}$. The set

$$\mathcal{C}\mathcal{K}^n := (\mathcal{K}^n \times \{0\}) \cup \bigcup_{S \in \mathcal{F}_n^-} \mathrm{dc}(S)$$

is called an *extension* of $\mathcal{K}^n$.

EXAMPLE 3.13. The extension of an n-cube-like complex $\mathcal{K}^n$ $(n = 2)$.

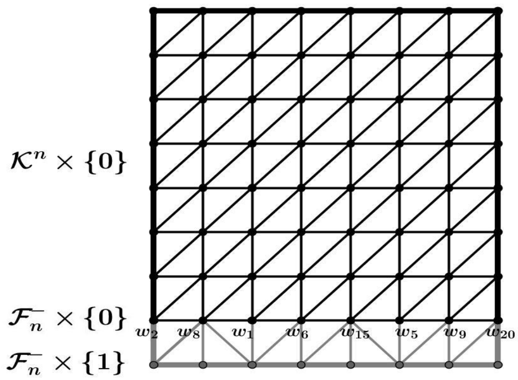

FIGURE 2. The illustration of a concept described in Definition 3.12.

OBSERVATION 3.14. If $\mathcal{K}^n$ is an n-cube-like complex, then $\mathcal{CK}^n$ is also an n-cube-like complex.

PROOF. The complex $\mathcal{CK}^n$ is generated by the family of n-simplexes. It suffices to show that conditions (a), (b) of Definition 3.4 are satisfied.

(a) The proof is similar to the one given in [8, Lemma 4.9].

(b) Let $\mathcal{F}_i^-, \mathcal{F}_i^+$ be i-th opposite faces of $\mathcal{K}^n$. The faces of the complex $\mathcal{CK}^n$ are defined as follows:

$$\begin{aligned}
\widetilde{\mathcal{F}}_i^- &:= \mathcal{C}\mathcal{F}_i^- && \text{for } i \in \{1, \ldots, n-1\},\\
\widetilde{\mathcal{F}}_i^+ &:= \mathcal{C}\mathcal{F}_i^+ && \text{for } i \in \{1, \ldots, n-1\},\\
\widetilde{\mathcal{F}}_n^- &:= \mathcal{F}_n^- \times \{1\},\\
\widetilde{\mathcal{F}}_n^+ &:= \mathcal{F}_n^+ \times \{0\}.
\end{aligned}$$

Let us check the conditions.

(b_1) It follows easily from Definition 3.12.

(b_2) Since $\mathcal{F}_i^- \cap \mathcal{F}_i^+ = \emptyset$, we receive $\widetilde{\mathcal{F}}_i^- \cap \widetilde{\mathcal{F}}_i^+ = \emptyset$ for $i \in \{1, \ldots, n-1\}$. Moreover, we have $\widetilde{\mathcal{F}}_n^- \cap \widetilde{\mathcal{F}}_n^+ = (\mathcal{F}_n^- \times \{1\}) \cap (\mathcal{F}_n^+ \times \{0\}) = \emptyset$.

(b_3) We proceed by induction on n. For $n = 0$, we have $\mathcal{K}^0 = \{a\}$ for some $a \in A$. Then $\mathcal{CK}^0 = \{(a, 0)\}$ is obviously a 0-cube-like complex. Assume that $\mathcal{CK}^k$ is a k-cube-like complex for $k < n$. Let us consider the complex $\mathcal{CK}^n$. For each $i_0 \in \{1, \ldots, n-1\}$, the sets $\mathcal{F}_{i_0}^-, \mathcal{F}_{i_0}^+$ are $(n-1)$-cube-like complexes. Then by the inductive assumption, the sets $\widetilde{\mathcal{F}}_{i_0}^-, \widetilde{\mathcal{F}}_{i_0}^+$ are $(n-1)$-cube-like complexes. Moreover, the sets $\widetilde{\mathcal{F}}_n^-$ and $\widetilde{\mathcal{F}}_n^+$ are copies of $\mathcal{F}_n^-$ and $\mathcal{F}_n^+$, respectively, and thus they are $(n-1)$-cube-like complexes. Let us note that for each $i_0 \in \{1, \ldots, n-1\}$ and each $\varepsilon \in \{-, +\}$, we obtain $\widetilde{\mathcal{F}}_{i_0}^{\varepsilon} \cap \widetilde{\mathcal{F}}_i^{\delta} = \mathcal{C}(\mathcal{F}_{i_0}^{\varepsilon} \cap \mathcal{F}_i^{\delta})$ for $i < n$, $i \neq i_0$,

$\delta \in \{-,+\}$. Since $\mathcal{F}_{i_0}^{\varepsilon} \cap \mathcal{F}_i^{\delta}$ is a face of the $(n-1)$-cube-like complex $\mathcal{F}_{i_0}^{\varepsilon}$, the set $\widetilde{\mathcal{F}}_{i_0}^{\varepsilon} \cap \widetilde{\mathcal{F}}_i^{\delta}$ is a face of the $(n-1)$-cube-like complex $\widetilde{\mathcal{F}}_{i_0}^{\varepsilon}$. Moreover, we have $\widetilde{\mathcal{F}}_{i_0}^{\varepsilon} \cap \widetilde{\mathcal{F}}_n^{-} = (\mathcal{F}_{i_0}^{\varepsilon} \cap \mathcal{F}_n^{-}) \times \{1\}$, and $\widetilde{\mathcal{F}}_{i_0}^{\varepsilon} \cap \widetilde{\mathcal{F}}_n^{+} = (\mathcal{F}_{i_0}^{\varepsilon} \cap \mathcal{F}_n^{+}) \times \{0\}$. It suffices to prove that the second part of condition (b_3) is true for the complexes $\widetilde{\mathcal{F}}_n^{-}, \widetilde{\mathcal{F}}_n^{+}$. For each $i < n$, $\varepsilon \in \{-,+\}$, we have:

$$\widetilde{\mathcal{F}}_n^{-} \cap \widetilde{\mathcal{F}}_i^{\varepsilon} = (\mathcal{F}_n^{-} \cap \mathcal{F}_i^{\varepsilon}) \times \{1\}, \qquad \widetilde{\mathcal{F}}_n^{+} \cap \widetilde{\mathcal{F}}_i^{\varepsilon} = (\mathcal{F}_n^{-} \cap \mathcal{F}_i^{\varepsilon}) \times \{0\}. \qquad \square$$

Observation 3.15 ([5, Observation 2]). Let $\mathcal{K}^n$ be an n-cube-like complex, and let $\mathcal{F}_i^{-}, \mathcal{F}_i^{+}$, $i \in \{1, \ldots, n\}$, be its i-th opposite faces. Let $\psi \colon |\mathcal{K}^n| \to \{0, \ldots, n\}$ be a coloring function defined by

$$\psi(v) := \begin{cases} n & \text{for } v \in |\mathcal{K}^n \setminus \mathcal{F}_n^{-}|, \\ i & \text{for } v \in |(\mathcal{F}_n^{-} \cap \ldots \cap \mathcal{F}_{i+1}^{-}) \setminus \mathcal{F}_i^{-}|, \\ 0 & \text{for } v \in |\mathcal{F}_n^{-} \cap \ldots \cap \mathcal{F}_1^{-}|. \end{cases}$$

Then there exists exactly one n-colored n-simplex in $\mathcal{K}^n$.

The algorithm. Let $\mathcal{K}^n$ be an n-cube-like complex, and let $\mathcal{F}_i^{-}, \mathcal{F}_i^{+}$, $i \le n$, be its i-th opposite faces. Let us define a map $\phi \colon |\mathcal{K}^n| \to \{0, \ldots, n\}$ by

$$\phi(s) = \max \Big\{ j \colon s \in \bigcap_{i=0}^{j} F_i \Big\},$$

where $F_0 = |\mathcal{K}^n|$ and $F_i = H_i^{+} \setminus |\mathcal{F}_i^{-}|$ for $i \le n$.

Consider the extension $\mathcal{CK}^n$ of the n-cube-like complex $\mathcal{K}^n$. By Observation 3.14, it is also an n-cube-like complex and its faces are defined as follows:

$$\begin{aligned} \widetilde{\mathcal{F}}_i^{-} &:= \mathcal{C}\mathcal{F}_i^{-} && \text{for } i < n, \\ \widetilde{\mathcal{F}}_i^{+} &:= \mathcal{C}\mathcal{F}_i^{+} && \text{for } i < n, \\ \widetilde{\mathcal{F}}_n^{-} &:= \mathcal{F}_n^{-} \times \{1\}, \\ \widetilde{\mathcal{F}}_n^{+} &:= \mathcal{F}_n^{+} \times \{0\}. \end{aligned}$$

Since $\widetilde{\mathcal{F}}_n^{-}$ is $(n-1)$-cube-like we can define a map $\psi \colon |\widetilde{\mathcal{F}}_n^{-}| \to \{0, \ldots, n-1\}$, similarly as in Observation 3.15. Let us define a coloring function $\Phi \colon |\mathcal{CK}^n| \to \{0, \ldots, n\}$ by the formula:

$$\Phi((v,t)) := \begin{cases} \phi(v) & \text{for } (v,t) \in |\mathcal{K}^n \times \{0\}|, \\ \psi(v) & \text{for } (v,t) \in |\mathcal{F}_n^{-} \times \{1\}|. \end{cases}$$

Observation 3.15 implies that there exists exactly one $(n-1)$-colored $(n-1)$-simplex $T_0 \in \widetilde{\mathcal{F}}_n^{-}$. Since $T_0 \in \partial \mathcal{CK}^n$, there exists exactly one n-simplex $S_0 \in \mathcal{CK}^n$ such that T_0 is its $(n-1)$-face. Since $\phi(\mathcal{F}_n^{-} \times \{0\}) = \{0, \ldots, n-1\}$, the set S_0 is $(n-1)$-colored n-simplex. It implies that S_0 contains exactly two $(n-1)$-colored $(n-1)$-faces. Let us denote the second one by T_1. Since exactly one

$(n-1)$-colored $(n-1)$-face (it is T_0) lies on the boundary of the $\mathcal{CK}^n$, we have $T_1 \in \mathcal{CK}^n \setminus \partial\mathcal{CK}^n$, and there exists exactly one n-simplex S_1 such that $S_0 \cap S_1 = T_1$. If S_1 is n-colored, then we finish the procedure. Otherwise, the n-simplex S_1 has two $(n-1)$-colored $(n-1)$-faces. Let us denote the second one by T_2. Now we continue this procedure for the $(n-1)$-face T_2.

Since the number of n-simplexes in $\mathcal{CK}^n$ is finite, the procedure will end up. We obtain the sequence $S_0, \dots, S_l$. It is easy to observe that $S_l = \{v_0 \times \{0\}, \dots, v_n \times \{0\}\}$ for some $v_0, \dots, v_n \in |\mathcal{K}^n|$ and $\Phi(S_l) = \{0, \dots, n\}$. Let $S = \{v_0, \dots, v_n\}$. We get $S \in \mathcal{K}^n$ and $\phi(S) = \{0, \dots, n\}$. $\square$

EXAMPLE 3.16. The illustration of an algorithm finding an n-colored n-simplex.

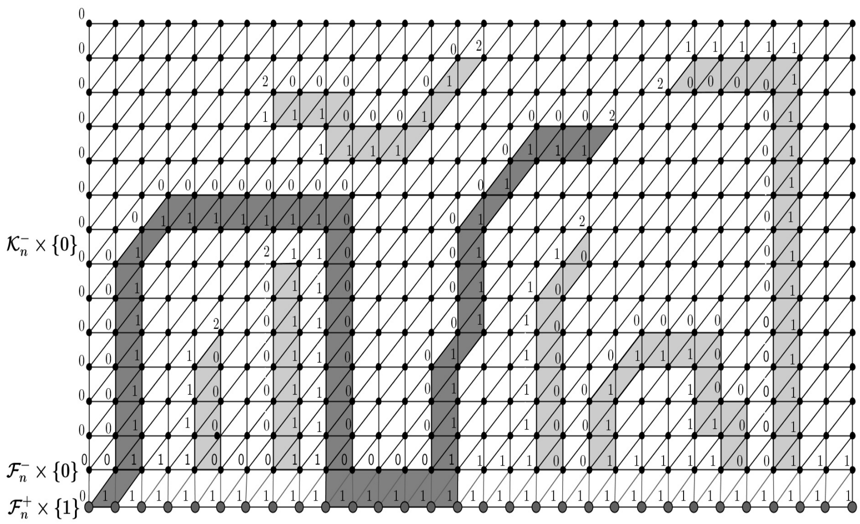

FIGURE 3. The sequence of n-simplexes $S_0, \dots, S_l$.

4. The Bolzano property for n-cube-like polyhedrons

THEOREM 4.1. *Let $(\overline{K}, \overline{\mathcal{K}})$ be an n-cube-like polyhedron in R^m. Then $\overline{K}$ has the n-dimensional Bolzano property.*

PROOF. Let n-cube-like complex $\mathcal{K}^n$ be the vertex-scheme of $(\overline{K}, \overline{\mathcal{K}})$. For $i \le n$, put $A_i = \overline{F_i^-}$ and $B_i = \overline{F_i^+}$. Let the family $\{H_i^-, H_i^+ : i = 1, \dots, n\}$ of pairs of subsets of $\overline{K}$ be as required. For each $k \in N$, let us consider subdivision of $\mathcal{K}^n$ such that $\mathrm{mesh}(\overline{\mathcal{K}_k}) := \max\{\mathrm{diam}(S) \colon S \in \overline{\mathcal{K}_k}\} < 1/k$. By Lemma 3.9, we get an n-simplex S_k such that for each $i \le n$, we have $H_i^- \cap S_k \ne \emptyset \ne H_i^+ \cap S_k$. Since $\overline{K}$ is a compact space and $\lim_{k\to\infty} \mathrm{diam}(S_k) = 0$, we may assume that for each sequence $\{x_k \in S_k, k \in N\}$ we have $\lim_{k\to\infty} x_k = x \in \overline{K}$. Moreover, the sets $\{H_i^-, H_i^+ : i = 1, \ldots, n\}$ are closed. Thus, $x \in \bigcap\{H_i^- \cap H_i^+ : i = 1, \ldots, n\} \neq \emptyset$.$\square$

Theorems 2.2 and 4.1 imply the following result.

Theorem 4.2 (generalization of the Poincaré–Miranda theorem). *Let $(\overline{K}, \overline{\mathcal{K}})$ be an n-cube-like polyhedron in R^m, $f\colon \overline{K} \to R^n$, $f = (f_1, \ldots, f_n)$, be a continuous map such that $f_i(\overline{F_i^-}) \subset (-\infty, 0]$ and $f_i(\overline{F_i^+}) \subset [0, \infty)$ for $i \le n$. Then there exists $c \in \overline{K}$ such that $f(c) = (0, \ldots, 0)$.*

5. Inverse system

Let us consider the inverse system $\{X_\sigma, \pi_\rho^\sigma, \Sigma\}$, where

(i) for all $\sigma \in \Sigma$, X_σ is a compact Hausdorff space with an n-dimensional boundary system $\{(A_i^\sigma, B_i^\sigma) : i = 1, \ldots, n\}$;
(ii) for all $\sigma, \rho \in \Sigma$, $\rho \le \sigma$, the map $\pi_\rho^\sigma\colon X_\sigma \to X_\rho$ is a surjection such that $\pi_\rho^\sigma(A_i^\sigma) = A_i^\rho$, $\pi_\rho^\sigma(B_i^\sigma) = B_i^\rho$.

Theorem 5.1. *The space $X = \varprojlim\{X_\sigma, \pi_\rho^\sigma, \Sigma\}$ has the n-dimensional Bolzano property.*

Proof. For each $i = 1, \ldots, n$ put

$$A_i = \varprojlim\{A_i^\sigma, \pi_\rho^\sigma : A_i^\sigma, \Sigma\}, \qquad B_i = \varprojlim\{B_i^\sigma, \pi_\rho^\sigma : B_i^\sigma, \Sigma\}.$$

Let $\{(H_i^-, H_i^+) : i = 1, \ldots, n\}$ be as required. For each $i \le n$ and each $\sigma \in \Sigma$ let us define closed sets

$$H_{i,\sigma}^- := p_\sigma(H_i^-), \qquad H_{i,\sigma}^+ := p_\sigma(H_i^+),$$

where $p_\sigma\colon X \to X_\sigma$ is a projection map. Since $A_i \subset H_i^-$, $B_i \subset H_i^+$, $X = H_i^- \cup H_i^+$ and maps p_σ are onto, we have

$$A_i^\sigma \subset H_{i,\sigma}^-, \qquad B_i^\sigma \subset H_{i,\sigma}^+, \qquad X_\sigma = H_{i,\sigma}^- \cup H_{i,\sigma}^+.$$

For each $\sigma \in \Sigma$, since the space X_σ has the n-dimensional Bolzano property, the set

$$C_\sigma := \bigcap\{H_{i,\sigma}^- \cap H_{i,\sigma}^+ : i = 1, \ldots, n\}$$

is nonempty. Let us observe that for each $\sigma, \rho \in \Sigma$, $\rho \le \sigma$, we have $\pi_\rho^\sigma(C_\sigma) \subset C_\rho$ and the set

$$C_\rho^\sigma := \Big\{x \in \prod_{\tau\in\Sigma} X_\tau : p_\sigma(x) \in C_\sigma\Big\} \cap \Big\{x \in \prod_{\tau\in\Sigma} X_\tau : \pi_\rho^\sigma(p_\sigma(x)) = p_\rho(x)\Big\}$$

is closed. The family $\{C_\rho^\sigma : \rho, \sigma \in \Sigma, \rho \le \sigma\}$ is centered: Let us consider its finite subfamily $\{C_{\rho_1}^{\sigma_1}, \ldots, C_{\rho_k}^{\sigma_k}\}$. There exists $\tau \in \Sigma$ such that for each $i \le k$, we have $\sigma_i \le \tau$. Choose $x \in \prod_{\sigma\in\Sigma} X_\sigma$ such that $p_\tau(x) \in C_\tau$, and $\pi_{\sigma_i}^\tau(p_\tau(x)) = p_{\sigma_i}(x)$, $\pi_{\rho_i}^{\sigma_i}(p_{\sigma_i}(x)) = p_{\rho_i}(x)$ for $i \le k$. It is obvious that $x \in \bigcap\{C_{\rho_i}^{\sigma_i} : i = 1, \ldots, k\}$.

Since the space $\prod_{\sigma\in\Sigma} X_\sigma$ is compact, the set $C := \bigcap\{C^\sigma_\rho : \rho,\sigma\in\Sigma,\rho\le\sigma\}$ is nonempty. It is clear that $C\subset X$, and $C\subset\bigcap\{H^-_i\cap H^+_i : i=1,\dots,n\}$. □

Remark 5.2. Observe that if for each $\sigma\in\Sigma$, we have $X_\sigma = I^n$ and the maps $\pi^\sigma_\rho\colon X_\sigma\to X_\rho$ are such that $\pi^\sigma_\rho(I^-_i)\subset I^-_i$, $\pi^\sigma_\rho(I^+_i)\subset I^+_i$, then the maps π^σ_ρ are onto. Therefore, we easily see that Theorem 5.1 is a generalization of the Bolzano theorem [6, p. 90].

6. Characterization of the Bolzano property

Theorem 6.1. *Let X be a locally connected space. A family $\{(A_i,B_i) : i = 1,\dots,n\}$ of pairs of disjoint closed subsets is an n-dimensional boundary system if and only if for each open cover $\{U_i : i=1,\dots,n\}$ of X for some $i_0\le n$, there exists a connected set $W\subset U_{i_0}$ such that $W\cap A_{i_0}\ne\emptyset\ne W\cap B_{i_0}$.*

Proof. $(\Rightarrow)$ Assume that there exists an open cover $\{U_i : i=1,\dots,n\}$ such that for each $i_0\le n$, there is no connected set $W\subset U_{i_0}$ which links A_{i_0} and B_{i_0}. For each $i\le n$, consider the components of U_i. Let L^-_i be the union of all components which intersect the set A_i, and L^+_i be the union of all components which do not intersect the set A_i. Since the space X is locally connected, we see that disjoint sets L^-_i, L^+_i are open. Moreover, $L^-_i\cap B_i=\emptyset=L^+_i\cap A_i$. Now let us define a family $\{(H^-_i,H^+_i) : i=1,\dots,n\}$, where $H^-_i = X\setminus L^+_i$ and $H^+_i = X\setminus L^-_i$. Observe that for each $i\le n$, we have $A_i\subset H^-_i$, $B_i\subset H^+_i$ and $H^-_i\cup H^+_i = X$. However, $\{U_i : i=1,\dots,n\}$ covers the space X, so $\bigcap\{H^-_i\cap H^+_i : i=1,\dots,n\} = X\setminus\bigcup_{i=1}^n U_i=\emptyset$, which contradicts that the family $\{(A_i,B_i) : i=1,\dots,n\}$ is an n-dimensional boundary system.

$(\Leftarrow)$ Let $\{(H^-_i,H^+_i) : i=1,\dots,n\}$ be as required. Suppose, on the contrary, that $\bigcap\{H^-_i\cap H^+_i : i=1,\dots,n\}=\emptyset$. Define an open cover $\{U_i := X\setminus(H^-_i\cap H^+_i) : i=1,\dots,n\}$. By assumption, there exists $i_0\le n$ and the required set $W\subset U_{i_0} = (X\setminus H^-_{i_0})\cup(X\setminus H^+_{i_0})$. Since $W\cap A_{i_0}\ne\emptyset\ne W\cap B_{i_0}$ and $A_{i_0}\subset H^-_{i_0}$, $B_{i_0}\subset H^+_{i_0}$, the open sets $X\setminus H^-_{i_0}, X\setminus H^+_{i_0}$ are nonempty in W. Using the fact that W is connected, we deduce that $(X\setminus H^-_{i_0})\cap(X\setminus H^+_{i_0})\ne\emptyset$. On the other hand, $X = H^-_{i_0}\cup H^+_{i_0}$ so $(X\setminus H^-_{i_0})\cap(X\setminus H^+_{i_0}) = X\setminus(H^-_{i_0}\cup H^+_{i_0})=\emptyset$, a contradiction. □

Remark 6.2. Observe that the latter implication holds for an arbitrary topological space. Note that the assumption of the local connectivity of the space X is crucial to prove the equivalence in Theorem 6.1. This is illustrated in the example below.

Example 6.3. Let $X = \{(0,0),(0,1)\}\cup\bigcup_{n\in\mathbb{N}}(\{1/n\}\times[0,1])$ be a subspace of the plane. The reader can easily verify that the sets $A_1=\{(0,0)\}$, $B_1=\{(0,1)\}$ create a 1-dimensional boundary system. Each set W containing A_1 and B_1 is not connected.

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PRZEMYSŁAW TKACZ
Faculty of Mathematics and Natural Sciences
College of Science
Cardinal Stefan Wyszyński University
Wóycickiego 1/3
01-938 Warszawa, POLAND
E-mail address: p.tkacz@uksw.edu.pl