Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 3, 1994, 179–187

REMARKS ON MULTIPLE SOLUTIONS FOR ASYMPTOTICALLY LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS[1]

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Dedicated to Jean Leray

1. Introduction

It is known that the critical groups are useful in distinguishing critical points (cf. [5]). We shall present here a few examples from semilinear elliptic boundary value problems showing how they work in the study of multiple solutions. Let us consider the problem

$$(1.1)\qquad \begin{cases} -\Delta u = g(x,u), \\ u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. Let λ_j be the j-th eigenvalue of $-\Delta$ with zero Dirichlet boundary data. We assume:

(g_1) $g \in C^1(\overline{\Omega} \times \mathbb{R}^1, \mathbb{R}^1)$, $\quad g(x,0) = 0$,

(g_2) $g'(x,0) < \lambda_1 \;\; \forall x \in \overline{\Omega}$

(g_3) $\lim_{|t|\to\infty} g(x,t)/t \triangleq g_\infty > \lambda_2$.

g_∞ satisfies one of the following three conditions:

(i) $g_\infty \notin \sigma(-\Delta)$, the spectrum of $-\Delta$.

[1]Partially supported by the Chinese National Science Foundation.
The authors would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

(ii) $g_\infty \in \sigma(-\Delta)$ and $\phi(x,u) \triangleq g(x,u) - g_\infty u$ is bounded and satisfies the Landesman-Lazer condition

$$\int_\Omega \Phi\bigg(x, \sum_{j=1}^m t_j \varphi_j(x)\bigg)\,dx \to \infty \qquad \text{as } \sum_{j=1}^m t_j^2 \to \infty$$

where $\Phi(x,t) = \int_0^t \phi(x,s)\,ds$, $\operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\} = \operatorname{Ker}(-\Delta - g_\infty I)$.

(iii) $g_\infty \in \sigma(-\Delta)$ and ϕ satisfies the strong resonance condition: for all $\xi_j \in \mathbb{R}^m$ with $|\xi_j| \to \infty$ for all $u_j \to u$ in $H_0^1(\Omega)$ and for all $v \in H_0^1(\Omega)$, we have

$$\lim_{j\to\infty} \int_\Omega \phi\bigg(x, u_j(x) + \sum_{i=1}^m \xi_j^i e_i(x)\bigg) v(x)\,dx = 0$$

and

$$\lim_{j\to\infty} \int_\Omega \Phi\bigg(x, u_j(x) + \sum_{i=1}^m \xi_j^i e_i(x)\bigg) v(x)\,dx = 0$$

where $\{e_i(x)\}_{i=1}^m$ is an orthonormal basis of the eigenspace $\operatorname{Ker}(-\Delta - g_\infty I)$, and $\xi_j = (\xi_j^1, \xi_j^2, \dots, \xi_j^m)$.

Our first result is

THEOREM A. *Assume g satisfies* (g_1)—(g_3). *Then* (1.1) *has at least three nontrivial solutions.*

For the second result, we assume that g satisfies

(g_4) $\lambda_1 < g'(x,0) < \lambda_k < g_\infty$ for all $x \in \overline{\Omega}$, where g_∞ satisfies one of the conditions (i), (ii), (iii) given in (g_3).

Then we have

THEOREM B. *Assume g satisfies* (g_1), (g_4). *Assume also that there exists $t_0 \neq 0$ such that $g(x,t_0) = 0$ for all $x \in \overline{\Omega}$. Then* (1.1) *has at least three nontrivial solutions. Moreover, if we replace* (g_4) *by*

(g_4)$'$ $\lambda_2 < g'(x,0) < \lambda_k < g_\infty$ *for all* $x \in \overline{\Omega}$,

then (1.1) *has at least four nontrivial solutions.*

COROLLARY. *Assume g satisfies* (g_1), (g_4). *Moreover, assume that there exists $t_1 < 0$, $t_2 > 0$ such that $g(x,t_i) = 0$ for all $x \in \overline{\Omega}$, $i = 1, 2$. Then* (1.1) *has at least five nontrivial solutions.*

REMARK. Many authors have made contributions to this problem. The results for at least one nontrivial solution were obtained in [2], [11], [9], for at least two nontrivial solutions in [3], [1] and [10] under the assumption $g_\infty < \lambda_1$. Further results have been given by many authors (see e.g. [5] and references therein).

Our theorems deal with the case $g_\infty > \lambda_2$. Theorem A is quite similar to the superlinear case (see [13]). But Theorem B and its Corollary are more delicate.

2. Proof of Theorem A

Set

$$f(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - G(x,u)\right] dx$$

for $u \in X \triangleq H_0^1(\Omega)$, where $G(x,t) = \int_0^t g(x,\tau)\,d\tau$. It is well known that $f \in C^2(X,\mathbb{R})$ satisfies the Palais-Smale condition. Any critical point of f corresponds to a (weak) solution of (1.1). Without loss of generality, we assume that f has only a finite number of critical points.

Theorem A is proved in the following two steps:

STEP 1. (1.1) has two nontrivial solutions, one is positive, another is negative.

Set

$$g_1(x,t) = \begin{cases} g(x,t), & t \geq 0, \\ 0, & t \leq 0, \end{cases} \tag{2.1}$$

and consider the modified problem

$$\begin{cases} -\Delta u = g_1(x,u), \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

We define

$$f_1(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - G_1(x,u)\right] dx$$

where $G_1(x,t) = \int_0^t g_1(x,\tau)\,d\tau$. Then $f_1 \in C^{2-0}(X,\mathbb{R})$. We claim that f_1 satisfies (PS). Let (u_n) be a sequence such that

$$|f_1(u_n)| < c$$

and

$$\nabla f_1(u_n) \to \theta \qquad \text{as } n \to \infty. \tag{2.3}$$

From (g_3) and (2.1) we get

$$g_1(x,t) = g_\infty t + O(t) \qquad \text{for } t > 0 \text{ large.} \tag{2.4}$$

(2.3) implies that for all $\varphi \in X$

$$\int_\Omega [\nabla u_n \nabla \varphi - g_1(x, u_n)\varphi]\, dx \to 0 \qquad \text{as } n \to \infty. \tag{2.5}$$

Set $\varphi = u_n$; we have

$$\|u_n\|^2 \leq \int_\Omega g_1(x, u_n)u_n\, dx + O(\|u_n\|) \leq C + C\|u_n\|^2_{L^2} + O(\|u_n\|).$$

If $\|u_n\|_{L_2}$ is bounded, then so is $\|u_n\|$. Otherwise, $\|u_n\|_{L^2} \to +\infty$. Let $v_n = u_n/\|u_n\|_{L^2}$. Then $\|v_n\|_{L^2} = 1$ and $\|v_n\|$ is bounded. A subsequence of v_n converges to v with $\|v\|_{L^2} = 1$, strongly in L^2 and weakly in H^1_0. From (2.5) it follows that

$$\int_\Omega [\nabla v \nabla \varphi - g_\infty v^+ \varphi]\, dx = 0, \qquad \forall \varphi \in H^1_0, \tag{2.6}$$

where

$$v^+ = \max\{0, v\}.$$

The regularity theory implies

$$\begin{cases} \Delta v + g_\infty\, v^+ = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

By the maximum principle $v = v^+ \geq 0$. But $g_\infty \neq \lambda_1$, and hence $v \equiv 0$ which contradicts $\|v\|_{L^2} = 1$. A standard argument shows that (u_n) has a convergent subsequence. (PS) is true for f_1.

From (g_2) there exist $\rho > 0, \delta > 0$ such that

$$f_1(u) \geq \delta \qquad \forall u \in S_\rho = \{u \in X \mid \|u\| = \rho\}$$

and from $g_\infty > \lambda_2$ we can take t large enough so that

$$f_1(t\varphi_1) < 0,$$

where φ_1 is the first eigenfunction of $-\Delta$ with zero Dirichlet boundary data. Consequently, by the mountain pass lemma, (2.2) has a weak solution u_1. By the maximum principle and regularity of solution of elliptic BVP$_S$, we know that the solution u_1 of (2.2) is classical and $u_1 > 0$ for $x \in \Omega$ and the outward directional derivative $\partial u(x)/\partial n < 0$ for $x \in \partial\Omega$. Therefore u_1 is a solution of (1.1).

Similarly, we get a negative solution u_2 of (1.1).

Thus we may restrict ourselves to the space C_0^1. Since $f_1|_{C_0^1}$ is C^2 in a C_0^1-neighborhood of u_1, the Splitting Lemma, the Shifting Theorem and the characterization of a mountain pass point carry over to $f_1|_{C_0^1}$ (cf. [5]). Again, according to [6], the deformation flow remains in C_0^1. Consequently,

$$\operatorname{rank} C_q(f_1, u_1) = \delta_{q1} \tag{2.8}$$

where $C_q(f, u)$ denotes the q-th critical group of f at u. Thus,

$$\operatorname{rank} C_q(f|_{C_0^1(\overline{\Omega})}, u_1) = \operatorname{rank} C_q(f_1|_{C_0^1(\overline{\Omega})}, u_1) = \delta_{q1} \qquad \forall q = 0, 1, 2, \dots \tag{2.9}$$

By the same method, we have

$$\operatorname{rank} C_q(f|_{C_0^1}, u_2) = \delta_{q1} \qquad \forall q = 0, 1, 2, \dots \tag{2.10}$$

STEP 2. The existence of the third solution.

Let X^- (X^+) be the negative (resp. positive) subspace of $-\Delta - g_\infty I$. Then, there exists $R > 0$ such that

$$\sup_{u \in X^-, \|u\| \geq R} f(u) < \inf_{v \in X^+} f(v).$$

According to [12], we know that f possesses a critical point u satisfying

$$\operatorname{rank} C_m(f, u) \neq 0, \tag{2.11}$$

where $2 \leq m = \dim X^-$. By Chapter III, Theorem 1.1 and Corollary 1.2 of [5], we have

$$C_m(f|_{C_0^1}, u) = C_m(f, u).$$

Since θ is a local mimimizer, we have

$$\operatorname{rank} C_q(f|_{C_0^1}, \theta) = \operatorname{rank} C_q(f, \theta) = \delta_{q0}. \tag{2.12}$$

Combining (2.9)–(2.12) shows that u is a third nontrivial solution of (1.1). Theorem A is proved.

REMARK 2.1. When g_∞ satisfies condition (iii), f satisfies the $(\mathrm{PS})_c$ condition for all $c \neq 0$. The first deformation theorem can be extended to study the fake critical set (see [7]).

3. Proof of Theorem B

The proof is divided into three steps. We suppose $t_0 > 0$; the proof for $t_0 < 0$ is similar.

STEP 1. Let us define

$$\widetilde{g}(x,t) = \begin{cases} 0, & t < 0, \\ g(x,t), & t \in [0, t_0], \\ 0, & t > t_0, \end{cases}$$

and

$$\widetilde{f}(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - \widetilde{G}(x,u)\right] dx, \tag{3.1}$$

where $\widetilde{G}(x,t) = \int_0^t \widetilde{g}(x,\tau)\,d\tau$. Since $\widetilde{f}$ is bounded below and satisfies (PS), there is a minimizer u^1 of $\widetilde{f}$. According to the maximum principle, we obtain: either $u^1 \equiv 0$ or $0 < u^1(x) < t_0$ for all $x \in \Omega$, and $\partial u^1/\partial n\big|_{\partial\Omega} < 0$. But by assumption $g'(x,0) > \lambda_1$, θ is not a minimizer, i.e., $u^1 \not\equiv \theta$. Thus u^1 must be a local minimizer of the functionals f and f_1 (the latter was defined previously in Theorem A), in the $C_0^1(\overline{\Omega})$ topology. However, according to Chapter III, Theorem 1.1 and Corollary 1.2 of [5] (as well as [4]), one concludes that u^1 is also a local minimizer of f in $H_0^1(\Omega)$ topology. Thus

$$\operatorname{rank} C_q(f, u^1) = \delta_{q0}. \tag{3.2}$$

STEP 2. As in the proof of Theorem A, we obtain a critical point u^3 satisfying

$$\operatorname{rank} C_m(f, u^3) \neq 0 \tag{3.3}$$

with $m = \dim\ X^- \geq 2$.

We only want to show $u^3 \neq \theta$. Indeed,

$$\operatorname{rank} C_q(f, \theta) = 0 \tag{3.4}$$

for all $q > \dim \bigoplus_{1\leq i\leq k-1} \ker(-\Delta - \lambda_i I)$, because $g'(x,0) < \lambda_k$.

From $\lambda_k < g_\infty$, we obtain

$$C_m(f, \theta) = 0. \tag{3.5}$$

Therefore $u^3 \neq \theta$.

STEP 3. Under $(g_4)'$ we can get one more solution by the mountain pass lemma.

In Step 1 we have obtained a $u^1 \in C_X$ which is a local minimizer of $\widetilde{f}$, where $C_X = C \cap C_0^1(\overline{\Omega})$ and $C = \{u \in H_0^1(\Omega) \mid t\varphi_1 \le u \le t_0$, a.e. for t small$\}$. Therefore, $d^2\widetilde{f}(u^1)$ is nonnegative and we have

$$\mathrm{Id} - (-\Delta)^{-1} g'(x, u^1) = d^2\widetilde{f}(u^1) \ge 0. \tag{3.6}$$

Let

$$v = u - u^1, \qquad g_1(x, v) = g(x, v + u^1) - g(x, u^1).$$

We have

$$g_1'(x, \theta) = g'(x, u^1) \tag{3.7}$$

and (1.1) is equivalent to

$$\begin{cases} -\Delta v = g_1(x, v), \\ v|_{\partial\Omega} = 0. \end{cases} \tag{3.8}$$

Let

$$\widetilde{g}_1(x, t) = \begin{cases} g_1(x, t), & t \ge 0, \\ 0, & t \le 0, \end{cases} \tag{3.9}$$

and define

$$f_1(v) = \int_\Omega \left[\frac{1}{2}|\nabla v|^2 - G_1(x, v)\right] dx,$$

$$\widetilde{f}_1(v) = \int_\Omega \left[\frac{1}{2}|\nabla v|^2 - \widetilde{G}_1(x, v)\right] dx,$$

where $G_1(x, t) = \int_0^t g_1(x, \tau)\, d\tau$, $\widetilde{G}_1(x, t) = \int_0^t \widetilde{g}_1(x, \tau)\, d\tau$.

From (3.6) and (3.7) we get

$$-\Delta(-g_1'(x, \theta)) \ge 0. \tag{3.10}$$

From (3.10) we know that θ is a local minimizer of $\widetilde{f}_1$. By Step 1 in the proof of Theorem A, $\widetilde{f}_1$ satisfies (PS). Using the mountain pass lemma, we immediately get a critical point v^+ of $\widetilde{f}_1$. From (3.6), one shows $v^+ \ne \theta$. Consequently, by the Strong Maximum Principle, $v^+ > \theta$ and then, by the same argument used in (2.8), we get

$$\operatorname{rank} C_p(\widetilde{f}_1|_{C_0^1}, v^+) = \delta_{p1}. \tag{3.11}$$

Let $u^+ = v^+ + u^1$. Then $u^+ > u^1$. Note that

$$f_1(v) = f(u) + \text{const} \tag{3.12}$$

and thus

$$\operatorname{rank} C_p(f|_{C_0^1}, u^+) = \operatorname{rank} C_p(f_1|_{C_0^1}, v^+) = \operatorname{rank} C_p(\widetilde{f}_1|_{C_0^1}, v^+) = \delta_{p1}. \tag{3.13}$$

In a similar way, we get a negative solution v^- of (3.8). Let $u^- = v^- + u^1$. Then $u^- < u^1$ and u^- is a solution of (1.1) which satisfies

$$\operatorname{rank} C_p(f|_{C_0^1}, u^-) = \delta_{p1}. \tag{3.14}$$

By $\lambda_2 < g'(x, 0)$ we have

$$\operatorname{rank} C_p(f|_{C_0^1}, \theta) = \operatorname{rank} C_p(f, \theta) = \delta_{pk}, \qquad k \geq 2. \tag{3.15}$$

Now (3.14) and (3.15) imply $u^- \neq \theta$. Combining this with Step 2, we get four nontrivial solutions. Theorem B is proved.

PROOF OF COROLLARY. By the assumption that there exists $t_1 < 0$ such that $g(x, t_1) = 0$, we can proceed in the same way by using a cut-off function and obtain two more other negative solutions: a local minimizer, and a mountain pass point.

REMARK. After finishing this work, we found a recent paper by E. N. Dancer and Y. H. Du [8], in which some results are very similar to ours.

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Manuscript received January 5, 1994

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