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CORRESPONDENCE FOR $\mathcal{D}$-MODULES AND PENROSE TRANSFORM

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Dedicated to Jean Leray

1. Introduction

Sheaf theory was created by Jean Leray during the second world war. It is one of the most powerful tools of modern mathematics, and we hope to convince the reader of this fact by applying sheaf theory, together with $\mathcal{D}$-module theory, to the study of the Penrose transform.

The Penrose correspondence is an integral transformation which interchanges global sections of line bundles on open subsets U of the complex projective space $\mathbb{P} := \mathbb{P}^3(\mathbb{C})$ and holomorphic solutions of some partial differential equations (like the wave equation) on corresponding open subsets $\widehat{U}$ of the compactified complex Minkowski space $\mathbb{M}$ (see Section 2 below for a more precise statement). This correspondence has been studied by many authors and generalized in various directions (let us mention in particular the books [12], [1], [18] and the papers [5], [4], [19], [20], [7]).

In this paper (which is an extended and corrected version of [2], and which announces the main results of [3]), we will describe a new approach to the Penrose transform in the language of sheaves and $\mathcal{D}$-modules, in a general geometrical framework.

In our opinion, this approach has several advantages: first, it allows us to distinguish between two kinds of problems arising in this correspondence, one of a topological nature (sheaves), and the other one of an analytical sort ($\mathcal{D}$-modules). Moreover, working in derived categories clarifies many problems, and we will see for instance how some difficulties encountered by many authors are due to the fact that the Penrose transform of a $\mathcal{D}$-module is, in general, not concentrated in degree 0.

Let us describe the main features of this paper. Let

$$
\begin{array}{ccc} & Y & \\ {}^{g}\swarrow & & \searrow^{f} \\ Z & & X \end{array} \tag{1}
$$

be a double morphism of complex manifolds, satisfying suitable hypotheses. Our principal results are the following:

a) We give a general formula which identifies the (generalized) solutions of a $\mathcal{D}$-module $\mathcal{N}$ on Z and the corresponding solutions of the Penrose transform of $\mathcal{N}$. This formula is exemplified in Section 5 where we easily recover the results of [20] on hyperfunction solutions.

b) We give a criterion which ensures that the Penrose transform of a $\mathcal{D}$-module is concentrated in degree zero. In the particular case of the twistor correspondence, this shows that the Penrose transform of a line bundle $\mathcal{O}_{BbbP}(k)$ is concentrated in degree zero if and only if k is negative.

c) We give (under suitable hypotheses) an equivalence of categories between coherent $\mathcal{D}$-modules on Z (modulo flat connections) and coherent $\mathcal{D}$-modules on X with regular singularities along an involutive submanifold V of $\dot{T}^*Z$ (modulo flat connections).

2. The twistor correspondence

Let $\mathbb{T}$ be a complex four-dimensional vector space, $\mathbb{F} = F_{12}(\mathbb{T})$ its five-dimensional flag manifold of type $(1,2)$, $\mathbb{P} = F_1(\mathbb{T})$ the projective three-space, and $\mathbb{M} = F_2(\mathbb{T})$ the four-dimensional Grassmannian of two-dimensional vector subspaces. The manifold $\mathbb{M}$ is identified with a conformal compactification of the complexified Minkowski space M^4, and it is possible to consider the family of massless field equations on M^4 as a family of differential operators acting between holomorphic bundles on $\mathbb{M}$. This family, denoted here by $\square_h$, is parameterized by a half-integer h called helicity, and includes Maxwell's wave equation, Dirac-Weyl's neutrino equations and Einstein linearized zero rest mass equations.

The Penrose transform is an integral transform associated with the double fibration

$$\begin{array}{ccc} & \mathbb{F} & \\ {}^{g}\swarrow & & \searrow^{f} \\ \mathbb{P} & & \mathbb{M} \end{array} \tag{2}$$

(where $f(L_1, L_2) = L_2$ and $g(L_1, L_2) = L_1$ for $L_1 \subset L_2 \subset \mathbb{T}$ complex subspaces of dimension 1 and 2) allowing one to find the holomorphic solutions of the equation $\Box_h \phi = 0$ on some open subsets $U \subset \mathbb{M}$ in terms of cohomology classes of line bundles on $\widehat{U} = gf^{-1}(U) \subset \mathbb{P}$.

More precisely, recall that for $k \in \mathbb{Z}$, the line bundles on $\mathbb{P}$ are given by the $-k$-th tensor powers $\mathcal{O}_{\mathbb{P}}(k)$ of the tautological bundle. Set $h(k) = -(1 + k/2)$, and for $z \in Z$, set $\widehat{z} = fg^{-1}(z)$. We then have the following result of Eastwood, Penrose and Wells [5]:

THEOREM 2.1. *Let $U \subset \mathbb{M}$ be an open subset such that*

$$U \cap \widehat{z} \text{ is connected and simply connected for every } z \in \widehat{U}. \tag{3}$$

Then, for $k < 0$, the natural morphism associated with (2) which maps a 1-form on $\widehat{U}$ to the integral along the fibers of f of its inverse image by g induces an isomorphism

$$\mathcal{P} : H^1(\widehat{U}; \mathcal{O}_{\mathbb{P}}(k)) \simeq \ker(U; \Box_{h(k)}).$$

3. Correspondence for sheaves and $\mathcal{D}$-modules

Let X be a complex manifold, and denote by d_X its complex dimension. We denote by $\mathbf{D}^{\mathrm{b}}(X)$ the derived category of the category of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on X, and refer to [11] for a detailed exposition of sheaf theory in the framework of derived categories. We make use of the six classical operations on sheaves, Rf_*, f^{-1}, $Rf_!$, $f^!$, $\otimes$, $R\mathcal{H}om$, and we use the notation $D'_X(\cdot) = R\mathcal{H}om(\cdot, \mathbb{C}_X)$.

From now on, we shall consider a correspondence of complex analytic manifolds

$$\begin{array}{ccc} & Y & \\ {}^{g}\swarrow & & \searrow^{f} \\ Z & & X \end{array} \tag{4}$$

for which we will consider the following hypotheses:

(H.1) f is proper and g is smooth,

(H.2) $(g, f) : Y \hookrightarrow Z \times X$ is a closed embedding.

Definition 3.1. Let $H \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(Z))$. We set

$$\mathcal{P}H = Rf_! g^{-1}(H), \tag{5}$$

$$\widetilde{\mathcal{P}}H = Rf_* g^!(H), \tag{6}$$

and we similarly define $\mathcal{P}F$ and $\widetilde{\mathcal{P}}F$ for $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(X))$.

(This definition is not the same as that of [2].)

Note that assuming (H.1), we get a natural isomorphism for $H \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(Z))$:

$$\widetilde{\mathcal{P}}H \simeq \mathcal{P}H[2d_Y - 2d_Z].$$

By classical adjunction formulas, such as the Poincaré-Verdier duality formula, one gets:

Proposition 3.2.

(i) *The functor* $\widetilde{\mathcal{P}}$ *is a right adjoint to* $\mathcal{P}$, *i.e. for* $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(X))$ *and* $H \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(Z))$ *we have an isomorphism:*

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(Z)}(\mathcal{P}F, H) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(X)}(F, \widetilde{\mathcal{P}}H).$$

(ii) *Assume* (H.1). *Let* $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(X))$, $H \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(Z))$. *Then we have the following commutative diagram whose lines are isomorphisms:*

$$\begin{array}{ccc} \mathrm{R}\Gamma_c(Z; \mathcal{P}D'_Z F \otimes H) & \xrightarrow{\sim} & \mathrm{R}\Gamma_c(X; D'_Z F \otimes \mathcal{P}H) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma(Z; R\mathcal{H}om(\mathcal{P}F[2d_Y - 2d_Z], H)) & \xrightarrow{\sim} & \mathrm{R}\Gamma(X; R\mathcal{H}om(F, \mathcal{P}H)). \end{array}$$

Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X, $\mathcal{D}_X$ the sheaf of rings of differential operators. Denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ the derived category of the category of bounded complexes of left $\mathcal{D}_X$-modules. Following [16] (see also [13]), we say that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is *good* if, in a neighborhood of any compact subset of X, $\mathcal{M}$ admits a finite filtration by coherent $\mathcal{D}_X$-submodules $\mathcal{M}_k$ ($k = 1, \dots, l$) such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ can be endowed with a good filtration. We denote by $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ consisting of good $\mathcal{D}_X$-modules. This definition ensures that $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ is the smallest thick subcategory of $\mathrm{Mod}(\mathcal{D}_X)$ containing the modules which can be endowed with good filtrations on a neighborhood of any compact subset of X. Note that in the algebraic case, coherent $\mathcal{D}$-modules are good.

Denote by $\underline{f}_!$, $\underline{f}_*$ and $\underline{f}^{-1}$ the direct and inverse image functors for $\mathcal{D}$-modules. (Refer to [9], [14] or see [15] for a detailed exposition on $\mathcal{D}$-modules.)

DEFINITION 3.3. Let $\mathcal{N} \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(\mathcal{D}_Z))$, $\mathcal{M} \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X))$. We define the functors $\underline{\mathcal{P}}$ and $\underline{\widetilde{\mathcal{P}}}$ by

$$\underline{\mathcal{P}}\mathcal{N} = \underline{f}_! \underline{g}^{-1}\mathcal{N},$$
$$\underline{\widetilde{\mathcal{P}}}\mathcal{M} = \underline{g}_* \underline{f}^{-1}\mathcal{M}[d_Z - d_X].$$

We define similarly $\underline{\mathcal{P}}\mathcal{M}$ and $\underline{\widetilde{\mathcal{P}}}\mathcal{N}$. We call $\underline{\mathcal{P}}\mathcal{N}$ the *Penrose transform* of $\mathcal{N}$.

PROPOSITION 3.4. *Assume* (H.1). *Then, for* $\mathcal{N} \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Z))$, $\mathcal{M} \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X))$, *the following adjunction formula holds:*

$$\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\underline{\mathcal{P}}\mathcal{N}, \mathcal{M})) \simeq \mathrm{R}\Gamma(Z; R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N}, \underline{\widetilde{\mathcal{P}}}\mathcal{M})).$$

By the Cauchy-Kowalevski-Kashiwara theorem applied to g, and the direct image theorem applied to f (cf. [9], [17]), we get:

PROPOSITION 3.5. *Assume* (H.1). *Let* $\mathcal{N} \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Z))$. *Then there is a canonical isomorphism*

$$\mathcal{P}\, R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N}, \mathcal{O}_Z) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\underline{\mathcal{P}}\mathcal{N}, \mathcal{O}_X)[d_X - d_Y].$$

THEOREM 3.6. *Assume hypothesis* (H.1). *Let* $\mathcal{N} \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Z))$, *and let* $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}(X))$. *Then, setting*

$$\begin{aligned}
A &= \mathrm{R}\Gamma_c(Z; R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N}, \mathcal{P}D'_X F \otimes \mathcal{O}_Z)),\\
B &= \mathrm{R}\Gamma_c(X; R\mathcal{H}om_{\mathcal{D}_X}(\underline{\mathcal{P}}\mathcal{N}, D'_X F \otimes \mathcal{O}_X))[d_X - d_Y],\\
C &= \mathrm{R}\Gamma(Z; R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N} \otimes \mathcal{P}F[2d_Y - 2d_Z], \mathcal{O}_Z)),\\
D &= \mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{D}_X}(\underline{\mathcal{P}}\mathcal{N} \otimes F, \mathcal{O}_X))[d_X - d_Y],
\end{aligned}$$

we have the following commutative diagram whose lines are isomorphisms:

$$\begin{array}{ccc}
A & \xrightarrow{\sim} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{\sim} & D
\end{array}$$

PROOF. Apply Proposition 3.2 with $H = R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N}, \mathcal{O}_Z)$, then apply Proposition 3.5. □

REMARK 3.7. The formulation of Proposition 2.2 and Theorem 2.5(i) of [2] was slightly different from that of Proposition 3.2 and Theorem 3.6, and a hypothesis was missing: in [2] we should have assumed that f is smooth.

This result allows us to distinguish between two kind of problems arising in the Penrose transform:

- to compute the sheaf theoretical Penrose transform of F,
- to compute the $\mathcal{D}$-module Penrose transform of $\mathcal{N}$.

The first problem is of a topological nature, and under reasonable hypotheses is not very difficult to solve (cf. Section 5 below for an example). The transform $\underline{\mathcal{P}}\mathcal{N}$ is much more difficult to compute. For example, $\underline{\mathcal{P}}\mathcal{N}$ is a complex of $\mathcal{D}$-modules, and it is not necessarily concentrated in degree zero. This does not affect the formulas as far as we use derived categories, but things may become rather complicated when computing explicitly cohomology groups.

In the next section we will study the transform $\underline{\mathcal{P}}\mathcal{N}$. We begin here with an easy corollary of Theorem 3.6.

For $x \in X$, $z \in Z$ and $U \subset X$, set $\widehat{x} = gf^{-1}(x)$, $\widehat{z} = fg^{-1}(z)$, $\widehat{U} = gf^{-1}(U)$.

COROLLARY 3.8. *Assume* (H.1) *and* (H.2). *Let* $\mathcal{N} \in \mathrm{Ob}(\mathrm{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Z))$. *Then*

(i) (Germ formula) *For $x \in X$ one has*

$$\mathrm{R}\Gamma(\widehat{x}; R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N}, \mathcal{O}_Z)) \simeq \mathrm{RHom}_{\mathcal{D}_X}(\underline{\mathcal{P}}\mathcal{N}, \mathcal{O}_X)_x[d_X - d_Y].$$

(ii) (Holomorphic solutions) *Let $U \subset X$ be an open set such that*

$$U \cap \widehat{z} \text{ is contractible for every } z \in \widehat{U}. \tag{7}$$

Then

$$\mathrm{R}\Gamma(\widehat{U}; R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{N}, \mathcal{O}_Z)) \simeq \mathrm{R}\Gamma(U; R\mathcal{H}om_{\mathcal{D}_X}(\underline{\mathcal{P}}\mathcal{N}, \mathcal{O}_X))[d_X - d_Y].$$

PROOF. (i) By Theorem 3.6, it is enough to check that $\mathcal{P}\mathbb{C}_x \simeq \mathbb{C}_{\widehat{x}}$, which follows from hypothesis (H.2) since g is an isomorphism from $f^{-1}(x)$ to $\widehat{x}$.

(ii) By Theorem 3.6, it is enough to check that $\mathcal{P}\mathbb{C}_U[2d_Y - 2d_Z] \simeq \mathbb{C}_{\widehat{U}}$. By hypothesis (H.2) there is an isomorphism $U \cap \widehat{z} \simeq f^{-1}(U) \cap g^{-1}(z)$, and hence (7) implies that the fibers of g are topologically trivial on $f^{-1}(U)$. It remains to use [11, Remark 3.3.10]. □

Let $\mathcal{G}$ be a holomorphic bundle on Z, $\mathcal{G}^* = \mathrm{hom}_{\mathcal{O}_Z}(\mathcal{G}, \mathcal{O}_Z)$ its dual, and consider the locally free $\mathcal{D}_Z$-module

$$\mathcal{D}\mathcal{G}^* = \mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{G}^*.$$

It is possible to recover $\mathcal{G}$ as the sheaf of holomorphic solutions to $\mathcal{D}\mathcal{G}^*$. In fact,

$$\mathcal{G} \simeq R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{D}\mathcal{G}^*, \mathcal{O}_Z).$$

Let us apply the above results to the particular case of the twistor correspondence described in Section 2. Applying Corollary 3.8 to $\mathcal{N} = \mathcal{D}\mathcal{G}^*$, $\mathcal{G} = \mathcal{O}_{\mathbb{P}}(k)$, we get the isomorphisms

$$\begin{aligned} \mathrm{R}\Gamma(\widehat{x}; \mathcal{O}_{\mathbb{P}}(k)) &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\underline{\mathcal{P}}\mathcal{D}_{\mathbb{P}}(-k), \mathcal{O}_X)_x[1], \\ \mathrm{R}\Gamma(\widehat{U}; \mathcal{O}_{\mathbb{P}}(k)) &\simeq \mathrm{R}\Gamma(U; R\mathcal{H}om_{\mathcal{D}_X}(\underline{\mathcal{P}}\mathcal{D}_{\mathbb{P}}(-k), \mathcal{O}_X))[1], \end{aligned}$$

where we set $\mathcal{D}_{\mathbb{P}}(-k) = \mathcal{D}\mathcal{O}_{\mathbb{P}}(-k)$. Theorem 2.1 appears now as a corollary of the following result:

THEOREM 3.9. *For $k < 0$, the Penrose transform of the $\mathcal{D}$-module $\mathcal{D}_{\mathbb{P}}(-k)$ is the $\mathcal{D}$-module associated with the operator $\square_h$.*

For instance, if $k = -2$ then $\underline{\mathcal{P}}\mathcal{D}_Z(2)$ is the $\mathcal{D}_X$-module associated with the Maxwell wave equation. The proof of Theorem 3.9 is implicit in [5]. In fact, Eastwood et al. work with complexes of locally free $\mathcal{O}$-modules and $\mathcal{D}$-linear morphisms. This category is equivalent to that of filtered $\mathcal{D}$-modules as proved by Saito [13]. This shows that these authors indeed use $\mathcal{D}$-module theory.

Note that the authors of [5] not being interested in computing all cohomology groups, could weaken the topological hypothesis on U and simply assume that $U \cap \widehat{z}$ is connected and simply connected for every $z \in \widehat{U}$.

4. Equivalence of $\mathcal{D}$-modules

Let $\dot{T}^*X$ be the cotangent bundle to X with the zero-section removed. Assuming (H.2), let $\Lambda = T^*_Y(Z \times X) \cap (\dot{T}^*Z \times \dot{T}^*X)$ be the conormal bundle to $Y \hookrightarrow Z \times X$, and consider the diagram:

$$\begin{array}{ccccccc} & \Lambda & & & & \dot{T}^*Z \times \dot{T}^*X & \\ {}^{p_1^a}\swarrow & & \searrow^{p_2} & \hookrightarrow & {}^{p_1^a}\swarrow & & \searrow^{p_2} \\ \dot{T}^*Z & & \dot{T}^*X & & \dot{T}^*Z & & \dot{T}^*X \end{array}$$

where p_1, p_2 are the projections, and p_1^a is the composite of p_1 and the antipodal map. The manifold Λ being Lagrangian, p_1 is smooth on Λ if and only if p_2^a is immersive. We will assume:

(H.3) $p_2|_\Lambda$ is a closed embedding and identifies Λ with a closed regular involutive submanifold V of $\dot{T}^*X$ and $p_1^a|_\Lambda$ is smooth and surjective on $\dot{T}^*Z$.

Let $\mathcal{E}_X$ be the sheaf of microdifferential operators of finite order on T^*X (cf. [14], [8], or [15] for a detailed exposition). If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, we set $\mathcal{E}\mathcal{M} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$, where π denotes the projection from T^*X to X. We say that $\mathcal{M}$ has *regular singularities on* V, if so has $\mathcal{E}\mathcal{M}$. Let us recall that the notion of regular singularities was introduced in [10]. Denote by $\mathrm{Mod}_{RS(V)}(\mathcal{D}_X)$ the thick subcategory of $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_X)$ whose objects have regular singularities on V, and by $\mathbf{D}^{\mathrm{b}}_{RS(V)}(\mathcal{D}_X)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ whose objects have cohomology groups belonging to $\mathrm{Mod}_{RS(V)}(\mathcal{D}_X)$.

THEOREM 4.1. *Assume* (H.1)–(H.3).

(i) *Let* $\mathcal{N} \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_Z))$. *Then* $\underline{\mathcal{P}}\mathcal{N}$ *belongs to* $\mathbf{D}^{\mathrm{b}}_{RS(V)}(\mathcal{D}_X)$.

(ii) *If* $\mathcal{N}$ *is concentrated in degree zero (i.e. is a* $\mathcal{D}_Z$*-module), then for* $j \neq 0$, $H^j(\underline{\mathcal{P}}\mathcal{N})$ *has its characteristic variety contained in the zero-section (in other words, it is a flat holomorphic connection).*

(iii) *Let* $\mathcal{G}$ *be a holomorphic vector bundle, and recall that* $\mathcal{D}\mathcal{G} = \mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{G}$. *Then,* X *being connected,* $\underline{\mathcal{P}}\mathcal{D}\mathcal{G}$ *is concentrated in degree zero if and only if there exists* $x \in X$ *such that* $H^j(\widehat{x}; \mathcal{G}) = 0$ *for every* $j \neq d_Y - d_X$.

PROOF. One first shows that $\underline{\mathcal{P}}$ "commutes with microlocalization". More precisely, denoting by $\underline{f}_!$, $\underline{f}_*$ and $\underline{f}^{-1}$ the direct and inverse image functors on $\mathcal{E}$-modules, one extends $\underline{\mathcal{P}}$ to the category $\mathbf{D}^{\mathrm{b}}(\mathcal{E}_Z)$ using the same formula as in Definition 3.3, and using (H.3) one proves that

$$\underline{\mathcal{P}}(\mathcal{E}\mathcal{M}) = \mathcal{E}\underline{\mathcal{P}}(\mathcal{M}).$$

To prove (ii), one notices that in view of hypothesis (H.3), the Penrose transform acting on $\mathcal{E}_Z$-modules is an exact functor outside the zero-section. Assertion (iii) is proved using (ii) and the germ formula of Corollary 3.8(i).

NOTATION 4.2. Denote by $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_Z; \mathcal{O}_Z)$ the localization of the abelian category $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_Z)$ by the subcategory of modules whose characteristic variety is contained in the zero-section. Note that the objects of $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_Z; \mathcal{O}_Z)$ are the good $\mathcal{D}_Z$-modules but a morphism u of $\mathcal{D}_Z$-modules becomes an isomorphism in this new category if $\ker(u)$ and $\mathrm{coker}(u)$ have their characteristic varieties contained in the zero-section. We similarly define $\mathrm{Mod}_{RS(V)}(\mathcal{D}_X; \mathcal{O}_X)$ by localizing $\mathrm{Mod}_{RS(V)}(\mathcal{D}_X)$.

Let $\underline{\mathcal{P}}^0$ be the functor $H^0(*) \circ \underline{\mathcal{P}}$, "zero-th cohomology of the Penrose transform". This functor is well defined from $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_Z)$ to $\mathrm{Mod}_{RS(V)}(\mathcal{D}_X)$, and sends modules whose characteristic variety is contained in the zero-section to modules on X with the same property.

THEOREM 4.3. *Assume* (H.1)–(H.3) *and also:*

(H.4) f *is smooth,* g *is proper,*

(H.5) g *has connected and simply connected fibers.*

Then the functor

$$\underline{\mathcal{P}}^0 : \mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_Z; \mathcal{O}_Z) \to \mathrm{Mod}_{RS(V)}(\mathcal{D}_X; \mathcal{O}_X)$$

is an equivalence of categories.

In other words, modulo flat connections, every good $\mathcal{D}_X$-module with regular singularities along V is the Penrose transform of a unique good $\mathcal{D}_Z$-module. Notice that on a simply connected space, the sheaf of holomorphic solutions of a flat connection is a constant sheaf of finite rank. In this sense one can say that $\underline{\mathcal{P}}^0$ is almost an equivalence of categories between $\mathrm{Mod}_{\mathrm{good}}(\mathcal{D}_Z)$ and $\mathrm{Mod}_{RS(V)}(\mathcal{D}_X)$.

5. Back to the twistor case

Let us now return to the twistor correspondence (2)

$$\begin{array}{ccc} & \mathbb{F} & \\ {}^{g}\swarrow & & \searrow^{f} \\ \mathbb{P} & & \mathbb{M} \end{array} \tag{9}$$

where $\mathbb{F} = F_{12}(\mathbb{T})$, $\mathbb{P} = F_1(\mathbb{T})$, and $\mathbb{M} = F_2(\mathbb{T})$. Hypotheses (H.1), (H.2), (H.4), and (H.5) are clearly satisfied.

Let us show that hypothesis (H.3) is also satisfied. Choose local coordinates (x_1, x_2, x_3, x_4), (z_1, z_2, z_3) on affine charts of $\mathbb{M}$ and $\mathbb{P}$ respectively and denote by $(x; \xi)$, $(z; \zeta)$ the associated coordinates on $T^*\mathbb{M}$ and $T^*\mathbb{P}$ respectively. Here (x_1, x_2, x_3, x_4) corresponds to the two-plane of $\mathbb{T}$ generated by the vectors $(x_1, x_3, 1, 0)$ and $(x_2, x_4, 0, 1)$ and (z_1, z_2, z_3) corresponds to the line generated by $(1, z_1, z_2, z_3) \in \mathbb{T}$. The submanifold $\mathbb{F}$ of $\mathbb{P} \times \mathbb{M}$ is given by the system of equations

$$\begin{pmatrix} x_1 \\ x_3 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} x_2 \\ x_4 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = 0.$$

On the open set $x_1 \neq 0$ we find the independent equations

$$\begin{cases} z_1 - z_2 x_3 - z_3 x_4 = 0, \\ 1 - z_2 x_1 - z_3 x_2 = 0. \end{cases}$$

The fiber of $\Lambda = T^*_{\mathbb{F}}(\mathbb{P} \times \mathbb{M})$ at (z, x) is given by

$$\lambda dz_1 + (-\lambda x_3 - \mu x_1)dz_2 + (-\lambda x_4 - \mu x_2)dz_3 - \mu z_2 dx_1 - \mu z_3 dx_2 - \lambda z_2 dx_3 - \lambda z_3 dx_4$$

for $\lambda, \mu \in \mathbb{C}$. Then one checks that $p_2|_\Lambda$ is an embedding and $V = p_2(\Lambda)$ is given by the equation

$$V \cap \pi^{-1}(\{x;\ x_1 \neq 0\}) = \{(x;\xi);\ \xi_1\xi_4 = \xi_2\xi_3\}.$$

In particular, one notices that V is indeed the characteristic variety of the wave equation. Applying Theorem 4.3, we find that any coherent $\mathcal{D}_{\mathbb{M}}$-module $\mathcal{M}$ with regular singularities on $\dot{V}$ is (up to a flat connection) the Penrose transform of a unique coherent $\mathcal{D}_{\mathbb{P}}$-module.

In order to apply Theorem 4.1, note that for $x \in \mathbb{M}$, the set $\widehat{x}$ is identified with a projective space $\mathbb{P}^1$ linearly embedded in $\mathbb{P}$.

LEMMA 5.1. *For $x \in \mathbb{M}$ one has:*

$$H^0(\widehat{x}; \mathcal{O}_{\mathbb{P}}(k)) = \begin{cases} 0 & \textit{for } k < 0, \\ \neq 0 & \textit{and finite dimensional for } k \geq 0, \end{cases}$$

$H^1(\widehat{x}; \mathcal{O}_{\mathbb{P}}(k))$ *is infinite dimensional for every* k,

$H^j(\widehat{x}; \mathcal{O}_{\mathbb{P}}(k)) = 0$ *for* $j \neq 0,\ 1$ *and for every* k.

Although this result should be well-known to specialists, we give here a proof.

PROOF. Choose homogeneous coordinates $[t_0, t_1, t_2, t_3] \in \mathbb{P}$ so that $\widehat{x} \subset \mathbb{P}$ is given by the equations $t_2 = t_3 = 0$. For $0 < a < b$ let $U_0 = \{[t];\ |t_1/t_0| \geq b\}$ and $U_1 = \{[t];\ |t_0/t_1| \geq a\}$ with coordinates $(x_1, x_2, x_3) = (t_1/t_0, t_2/t_0, t_3/t_0)$ and $(y_1, y_2, y_3) = (t_0/t_1, t_2/t_1, t_3/t_1)$ respectively. This is a Leray covering for $\mathcal{O}_{\mathbb{P}}(k)|_{\widehat{x}}$ formed by two sets, and hence the third assertion follows.

Taking (x_1, x_2, x_3) as coordinates on $U_0 \cap U_1$, the restriction maps

$$\Gamma(U_0 \cap \widehat{x}; \mathcal{O}_{\mathbb{P}}(k)) \xrightarrow[r_0]{} \Gamma(U_0 \cap U_1 \cap \widehat{x}; \mathcal{O}_{\mathbb{P}}(k)) \xleftarrow[r_1]{} \Gamma(U_1 \cap \widehat{x}; \mathcal{O}_{\mathbb{P}}(k))$$

are given by

$$r_0(f(x_1, x_2, x_3)) = f(x_1, x_2, x_3),$$
$$r_1(g(y_1, y_2, y_3)) = x_1^k g(x_1^{-1}, x_2 x_1^{-1}, x_3 x_1^{-1}).$$

Take sections

$$\text{(10)} \quad \begin{aligned} \Gamma(U_0 \cap \widehat{x}; \mathcal{O}_{\mathbb{P}}(k)) \ni f(x_1, x_2, x_3) &= \sum_{\alpha \in \mathbb{N}^3} a_\alpha\, x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \\ \Gamma(U_1 \cap \widehat{x}; \mathcal{O}_{\mathbb{P}}(k)) \ni g(y_1, y_2, y_3) &= \sum_{\beta \in \mathbb{N}^3} b_\beta\, y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3}. \end{aligned}$$

The pair (f, g) represents an element of $\Gamma(U_0 \cap U_1 \cap \widehat{x}; \mathcal{O}_{\mathbb{P}}(k))$ if and only if $r_0(f) = r_1(g)$, i.e. if and only if

$$\sum_{\alpha \in \mathbb{N}^3} a_\alpha \, x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} = \sum_{\beta \in \mathbb{N}^3} b_\beta \, x_1^{k-\beta_1-\beta_2-\beta_3} x_2^{\beta_2} x_3^{\beta_3}.$$

In particular, we get $k - \beta_1 - \beta_2 - \beta_3 \geq 0$ and hence the first assertion follows.

One has

$$H^1(\widehat{x}; \mathcal{O}_{\mathbb{P}}(k)) \simeq \frac{\Gamma(U_0 \cap U_1 \cap \widehat{x}; \mathcal{O}_{\mathbb{P}}(k))}{r_0 \Gamma(U_0 \cap \widehat{x}; \mathcal{O}_{\mathbb{P}}(k)) + r_1 \Gamma(U_1 \cap \widehat{x}; \mathcal{O}_{\mathbb{P}}(k))}.$$

Take

$$\Gamma(U_0 \cap U_1 \cap \widehat{x}; \mathcal{O}_{\mathbb{P}}(k)) \ni h(x_1, x_2, x_3) = \sum_{\gamma \in \mathbb{Z} \times \mathbb{N}^2} c_\gamma \, x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3};$$

then $[h] = 0$ in $H^1(\widehat{x}; \mathcal{O}_{\mathbb{P}}(k))$ if and only if there exist f and g as in (10) such that $h = f + g$, i.e. if and only if

$$\sum_{\gamma \in \mathbb{Z} \times \mathbb{N}^2} c_\gamma \, x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} = \sum_{\alpha \in \mathbb{N}^3} a_\alpha \, x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} + \sum_{\beta \in \mathbb{N}^3} b_\beta \, x_1^{k-\beta_1-\beta_2-\beta_3} x_2^{\beta_2} x_3^{\beta_3}.$$

In particular, we notice that the elements of $H^1(\widehat{x}; \mathcal{O}_{\mathbb{P}}(k))$ whose representatives are given by x^γ are different from zero (and different from each other) for $k - \gamma_2 - \gamma_3 < \gamma_1 < 0$. The second assertion follows. □

It is then possible to characterize those line bundles of $\mathbb{P}$ whose Penrose transform is concentrated in degree zero.

PROPOSITION 5.2. *The complex $\underline{P}\mathcal{D}_{\mathbb{P}}(-k)$ is concentrated in degree zero if and only if $k < 0$.*

PROOF. In view of the previous lemma, this is a consequence of Theorem 4.1(iii). □

This last result explains why many authors restrict their study to the case of positive helicity (i.e. $k < 0$).

It would be interesting to prove that if $\mathcal{M}$ has simple characteristics on $\dot{V}$ (in the sense of [15, Ch. 1, Def. 6.2.2]) then it is the image of a locally free $\mathcal{D}_{\mathbb{P}}$-module of rank one, i.e. of $\mathcal{D}_{\mathbb{P}}\mathcal{G}$, for $\mathcal{G}$ a line bundle. This would better explain Penrose's result.

To conclude, we will show how Theorem 3.6 allows us to easily recover the results of [20] on hyperfunction solutions.

Let ϕ be a Hermitian form on $\mathbb{T}$ of signature $(+,+,-,-)$. Let us choose a basis for $\mathbb{T}$ such that

$$\phi = \begin{pmatrix} 0 & iI_2 \\ -iI_2 & 0 \end{pmatrix}$$

where $I_2 \in M_2(\mathbb{C})$ denotes the identity matrix. For $A \in M_2(\mathbb{C})$, we have

$$(A^*, I_2)\phi \begin{pmatrix} A \\ I_2 \end{pmatrix} = 0 \qquad \text{iff } A \text{ is Hermitian.}$$

In other words, the local chart

$$\begin{array}{ccc} \mathbb{C}^4 & \to & \mathbb{M}, \\ (x_1, x_2, x_3, x_4) & \mapsto & \begin{pmatrix} x_3 - x_4 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 + x_4 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{array}$$

identifies the Minkowski space $M^4 = (\mathbb{R}^4, \phi)$ with an open subset of the completely real compact submanifold M of $\mathbb{M}$ defined by

$$M = \{L_2 \in \mathbb{M};\ \phi(v) = 0\ \forall v \in L_2\}.$$

Note that M is a conformal compactification of the Minkowski space M^4. Let us consider

$$F = \{(L_1, L_2) \in \mathbb{F};\ \phi(v) = 0\ \forall v \in L_2\},$$
$$P = \{L_1 \in \mathbb{P};\ \phi(v) = 0\ \forall v \in L_1\},$$

and the induced double fibration

$$\begin{array}{ccccccc} & F & & & & \mathbb{F} & \\ {\scriptstyle \widetilde{g}}\swarrow & & \searrow{\scriptstyle \widetilde{f}} & \hookrightarrow & {\scriptstyle g}\swarrow & & \searrow{\scriptstyle f} \\ P & & M & & \mathbb{P} & & \mathbb{M} \end{array}$$

Recall that $\mathbb{M}$ is a complexification of M, that P is a real hypersurface in $\mathbb{P}$ topologically isomorphic to $S^2 \times S^3$, and that $\widetilde{g}$ is locally isomorphic to a projection $P \times S^1 \to P$ (cf. [19]).

The sheaves $\mathcal{A}_M$ and $\mathcal{B}_M$ of analytic functions and Sato hyperfunctions respectively are given by

$$\mathcal{A}_M = \mathbb{C}_M \otimes \mathcal{O}_{\mathbb{M}},$$
$$\mathcal{B}_M = R\mathcal{H}om(D'_{\mathbb{M}}\mathbb{C}_M, \mathcal{O}_{\mathbb{M}}).$$

In order to apply Theorem 3.6 to $F = D'_{\mathbb{M}}\mathbb{C}_M$, let us calculate $\mathcal{P}\mathbb{C}_M$.

Since $f^{-1}(M) = N$, we have $f^{-1}\mathbb{C}_M = \mathbb{C}_N$. Moreover, since $\widetilde{g}$ is locally isomorphic to $P \times S^1 \to P$, we find that $\mathcal{P}\mathbb{C}_M = R\widetilde{g}_*\mathbb{C}_N$ is concentrated in degree $0, 1$ and $H^j(\mathcal{P}\mathbb{C}_M)$ is locally free of rank one for $j = 0$ or 1. Finally, since $P \simeq S^2 \times S^3$ is connected and simply connected, we have $H^0(\mathcal{P}\mathbb{C}_M) \simeq \mathbb{C}_P$, $H^1(\mathcal{P}\mathbb{C}_M) \simeq \mathbb{C}_P$.

Now we assume

$$k < 0 \tag{11}$$

so that $\underline{P}\mathcal{D}_{\mathbb{P}}(-k)$ is a coherent $\mathcal{D}_{\mathbb{M}}$-module (concentrated in degree zero) by Proposition 5.2.

Then, as an easy application of Theorem 3.6, we find the following commutative diagram whose lines are isomorphisms:

$$\begin{array}{ccc} H^1(P; \mathcal{O}_{\mathbb{P}}(k)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}_{\mathbb{M}}}(\underline{P}\mathcal{D}_{\mathbb{P}}(-k), \mathcal{A}_M) \\ \downarrow & & \downarrow \\ H^2_P(\mathbb{P}; \mathcal{O}_{\mathbb{P}}(k)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}_{\mathbb{M}}}(\underline{P}\mathcal{D}_{\mathbb{P}}(-k), \mathcal{B}_M) \end{array}$$

This is Theorem 6.1 of [20].

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