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SOME ELEMENTARY GENERAL PRINCIPLES OF CONVEX ANALYSIS

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Dedicated, with admiration, to Ky Fan

1. Introduction

In a recent paper [6], the authors presented a new geometric approach in the theory of minimax inequalities, which has numerous applications in different areas of mathematics. In this note, we complement and elucidate the above approach within the context of complete metric spaces.

More precisely, we concentrate on super-reflexive Banach spaces and show that a large part of the theory of these spaces (and, in particular, Hilbert spaces) can be obtained in a very elementary way, without using weak topology or compactness.

In Section 2, we give an elementary proof of the basic intersection property of closed convex bounded sets and give applications in Section 3. In Section 4, we describe the KKM property and in Section 5 we prove the fundamental intersection property of KKM-maps with closed convex bounded values. The remaining sections are devoted to applications to variational inequalities, theory of games, systems of inequalities and maximal monotone operator theory, respectively.

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2. Intersection Principle

We recall that a Banach space $(E, \|\cdot\|)$ is *uniformly convex* provided its norm $\|\cdot\|$ (also called uniformly convex) has the following property: If (x_n), (y_n) are sequences in E such that the three sequences $\|x_n\|$, $\|y_n\|$, and $\frac{1}{2}\|x_n + y_n\|$ converge to 1, then $\|x_n - y_n\| \to 0$. From the parallelogram identity in a Hilbert space it follows at once that any Hilbert space is uniformly convex. A Banach space $(E, |\cdot|)$ is called *super-reflexive* provided it admits an equivalent uniformly convex norm.

In what follows E stands for a super-reflexive Banach space and H denotes a Hilbert space.

Lemma 2.1. *Let $(E, \|\cdot\|)$ be super-reflexive and let (C_n) be a decreasing sequence of nonempty closed convex subsets of E. Suppose that $d = \sup_n d(0, C_n)$ is finite. Then there exists a unique point $\overline{x}$ in E such that $\overline{x} \in \bigcap_n C_n$ and $\|\overline{x}\| = d$.*

Proof. For the proof, we may suppose without loss of generality that the norm $\|\cdot\|$ in $(E, \|\cdot\|)$ is uniformly convex. For each n, let $P_n = C_n \cap B(0, d+1/n)$. Then (P_n) is a decreasing sequence of nonempty closed sets. We claim that $\lim_{n\to\infty} \delta(P_n) = 0$, where $\delta(P_n)$ denotes the diameter of the set P_n. Indeed, for any n, let x_n and y_n be arbitrary points in P_n. Since P_n is convex, the point $(x_n + y_n)/2$ lies in P_n, so the values $\|x_n\|$, $\|y_n\|$, and $\frac{1}{2}\|x_n + y_n\|$ are between $d + 1/n$ and $d(0, C_n)$, and therefore the three sequences converge to d. Because the norm $\|\cdot\|$ is uniformly convex, we infer that $\|x_n - y_n\| \to 0$, and consequently $\lim_{n\to\infty} \delta(P_n) = 0$. It follows from Cantor's theorem that the intersection of the sets P_n contains a unique point $\overline{x}$. This point satisfies $d(0, C_n) \le \|\overline{x}\| \le d$ for every n, so that $\|\overline{x}\| = d$. The proof is complete. $\square$

Theorem 2.2 (Intersection Principle). *Let $(E, \|\cdot\|)$ be super-reflexive and let $\{C_i \mid i \in I\}$ be a family of closed convex sets in E with the finite intersection property. If C_{i_0} is bounded for some $i_0 \in I$, then the intersection $\bigcap\{C_i \mid i \in I\}$ is not empty.*

Proof. Let $\langle I \rangle$ be the set of all finite subsets of I containing i_0. For $J \in \langle I \rangle$, let $C_J = \bigcap\{C_j \mid j \in J\}$. By hypothesis, the C_J's are nonempty closed convex subsets of H and $d = \sup_{J\in\langle I\rangle} d(0, C_J)$ is finite since the C_J's are contained in the bounded set C_{i_0}.

Let (J_n) be an increasing sequence in $\langle I \rangle$ such that $d(0, C_{J_n}) \ge d - 1/n$. Then (C_{J_n}) is a decreasing sequence of nonempty closed convex sets in E such that $d = \sup_n d(0, C_{J_n})$. By Lemma 2.1, there is a unique point $\overline{x}$ in $\bigcap_n C_{J_n}$ with $\|\overline{x}\| = d$.

Now let $J \in \langle I \rangle$ be arbitrary, and let $C_n = C_J \cap C_{J_n}$. Again, (C_n) is a decreasing sequence of nonempty closed convex sets in E such that $d = \sup_n d(0, C_n)$, so by Lemma 2.1, there is a (unique) point $\overline{x}'$ in $\bigcap_n C_n = C_J \cap \bigcap_n C_{J_n}$ with $\|\overline{x}'\| = d$. We derive that $\overline{x} = \overline{x}'$ belongs to C_J.

Finally, the point $\overline{x}$ belongs to C_J for any $J \in \langle I \rangle$, which proves that $\bigcap\{C_i \mid i \in I\} = \bigcap\{C_J \mid J \in \langle I \rangle\}$ is not empty. □

3. Minimization of quasiconvex functions

Let X be a nonempty subset of a super-reflexive Banach space E. We recall that a function $\varphi : X \to \mathbb{R}$ is said to be *quasiconvex* (*lower semicontinuous, coercive* respectively) if the sections $S(\varphi, \lambda) = \{x \in X \mid \varphi(x) \le \lambda\}$, $\lambda \in \mathbb{R}$, are convex (closed, bounded respectively) in X. A function $\varphi : X \to \mathbb{R}$ is *upper semicontinuous* (= u.s.c.) provided $-\varphi$ is lower semicontinuous (= l.s.c.).

THEOREM 3.1. *Let E be super-reflexive, X be a nonempty closed convex subset of E and $\varphi : X \to \mathbb{R}$ be a quasiconvex lower semicontinuous coercive function. Then φ attains its minimum on X.*

PROOF. For each $x \in X$, let $C(x) = \{y \in X \mid \varphi(y) \le \varphi(x)\}$. We have to show that the family $\{C(x) \mid x \in X\}$ has a nonempty intersection. This readily follows from Theorem 2.2, since the sets $C(x)$ are convex, closed and bounded in E, and for any $\{x_1, \dots, x_n\} \subset X$, the intersection

$$\bigcap_{i=1}^{n} C(x_i) = \{y \in X \mid \varphi(y) \le \min_{i=1,\dots,n} \varphi(x_i)\}$$

is not empty. □

We now give a few immediate consequences of Theorem 3.1 in the theory of Hilbert spaces. We recall that a bilinear form $a : H \times H \to \mathbb{R}$ is said to be *coercive* if the function $y \mapsto a(y, y)/\|y\|$ is coercive on $H \setminus \{0\}$. When specialized to quadratic forms, Theorem 3.1 yields

THEOREM 3.2. *Let X be a nonempty closed convex subset of a Hilbert space H, $a : H \times H \to \mathbb{R}$ a continuous coercive bilinear form, and $\ell : H \to \mathbb{R}$ a continuous linear form. Then there exists a unique point $y_0 \in X$ having the following equivalent properties:*

(A) $\frac{1}{2}a(y_0, y_0) - \ell(y_0) = \min\left\{\frac{1}{2}a(x, x) - \ell(x) \mid x \in X\right\}$,
(B) $\frac{1}{2}(a(y_0, y_0 - x) + a(y_0 - x, y_0)) \le \ell(y_0 - x)$ *for all* $x \in X$.

PROOF. The coercivity of a implies $a(x, x) > 0$ for all $x \ne 0$. A routine calculation then shows that the above properties are equivalent and that (B) has at most one solution. The existence of a solution for (A) follows from Theorem 3.1 applied to the convex continuous coercive function $x \mapsto \frac{1}{2}a(x, x) - \ell(x)$. □

Whenever the bilinear form a is symmetric, property (B) in Theorem 3.2 reduces to the *variational inequality*

$$a(y_0, y_0 - x) \le \ell(y_0 - x) \qquad \text{for all } x \in X,$$

whose study in its full generality is the object of Section 6.

Theorem 3.2 includes basic results from Hilbert space theory, let us recall but a few:

Corollary 3.3 (F. Riesz representation theorem). *For any continuous linear form $\ell : H \to \mathbb{R}$ there exists a unique $y_0 \in H$ such that $\langle y_0, x\rangle = \ell(x)$ for all $x \in X$.*

Proof. Apply Theorem 3.2 (B) to the set $X = H$, the bilinear form $a(\cdot,\cdot) = \langle\cdot,\cdot\rangle$ and the given linear form ℓ. $\square$

Corollary 3.4 (Projection on closed convex sets). *Let X be a nonempty closed convex subset of H and $x_0 \in H$. Then there is a unique point $P_X x_0 \in X$ satisfying $\|x_0 - P_X x_0\| = d(x_0, X)$. Moreover, this point is characterized by the variational inequality $\langle P_X x_0 - x_0, P_X x_0 - x\rangle \le 0$ for all $x \in X$.*

Proof. Apply Theorem 3.2 to the set $X - x_0$, the bilinear form $a(\cdot,\cdot) = \langle\cdot,\cdot\rangle$ and the linear form $\ell = 0$. $\square$

Corollary 3.5 (Separation of closed convex sets). *Let X be a nonempty closed convex subset of H and $x_0 \notin X$. Then there exist $y_0 \in H$ and $r \in \mathbb{R}$ such that $\langle y_0, x_0\rangle < r < \langle y_0, x\rangle$ for all $x \in X$.*

Proof. The projection of x_0 on X yields a point $P_X x_0 \in X$ satisfying $\langle P_X x_0 - x_0, P_X x_0 - x\rangle \le 0$ for all $x \in X$; put $y_0 = P_X x_0 - x_0 \in H$ and $r = \frac{1}{2}\|y_0\|^2 + \langle y_0, x_0\rangle$. $\square$

4. Definition and examples of KKM-maps

In the process of a proof based on the intersection property of convex sets, the crucial point in general is to verify that the family of sets has the finite intersection property. In this section, we describe a very simple condition that is frequently met in applications. And in the next section, we show that this condition is sufficient for a family of closed convex sets to have the finite intersection property, thus enlarging considerably the domain of applications of Theorem 2.2.

We begin with some notations and terminology. By a *space* we shall understand a metric space and by a *map* a set-valued transformation. Given a map $T : X \to Y$ between two sets X and Y, its *inverse* $T^{-1} : Y \to X$ is given by $T^{-1}y = \{x \in X \mid y \in Tx\}$ and its *dual* $T^* : Y \to X$ is given by $T^*y = X \setminus T^{-1}y$. The sets Tx are the *values* of T, the sets $T^{-1}y$ are the *fibers* of T and the sets

T^*y are the *cofibers* of T. The set $\Gamma_T = \{(x, y) \in X \times Y \mid y \in Tx\}$ is the *graph* of T.

In what follows, given a vector space E and $A \subset E$, we use the abbreviation $[A] = \operatorname{conv} A$ for the convex hull of A. For each positive integer n, we set $[n] = \{i \in \mathbb{N} \mid 1 \le i \le n\}$.

Definition 4.1. Let E be a vector space and $X \subset E$ an arbitrary subset. A map $G : X \to E$ is called a *Knaster–Kuratowski–Mazurkiewicz map* or simply a *KKM-map* provided for each finite subset $A = \{x_1, \dots, x_n\}$ of X we have

$$[A] = \operatorname{conv}\{x_1, \dots, x_n\} \subset G(A) = \bigcup_{i=1}^{n} Gx_i.$$

We say that G is *strongly KKM* provided (i) $x \in Gx$ for each $x \in X$, and (ii) the cofibers G^*y of G are convex.

Proposition 4.2. *If $X \subset E$ is convex and $G : X \to E$ is strongly KKM, then G is a KKM-map.*

Proof. Let $A = \{x_1, \dots, x_n\} \subset X$ and let $y_0 \in [A]$. We have to show that $y_0 \in \bigcup_{i=1}^n Gx_i$. Since $y_0 \in Gy_0$, we see that $y_0 \notin G^*y_0$ and therefore $[A]$ is not contained in G^*y_0. Since the set G^*y_0 is convex, at least one point x_i of A does not belong to G^*y_0, which means that $y_0 \in Gx_i$. $\square$

We now give a few examples of KKM-maps. Examples 1–2 are special cases of Example 3. In these examples, the domain of the map is convex, so it is enough to show that the map is strongly KKM. This will not be the case in Example 4.

Example 1. Let E be a vector space and $C \subset E$ a convex subset of E. Assume that we are given a bilinear form $a : E \times E \to \mathbb{R}$ and a linear form $\ell \in E'$. Then the map $G : C \to E$ defined by

$$Gx = \{y \in C \mid a(y, y - x) \le \ell(y - x)\}$$

is strongly KKM: indeed, $x \in Gx$ for each x and the cofibers $G^*y = \{x \in C \mid a(y, y - x) > \ell(y - x)\}$ of G are convex.

Example 2. Let $E = (H, \langle\ \rangle)$ be a pre-Hilbert space, C a convex subset of H and $\varphi : C \to H$ any function. Then the map $G : C \to E$ defined by

$$Gx = \{y \in E \mid \langle \varphi(y), y - x\rangle \le 0\}$$

is strongly KKM: indeed, $x \in Gx$ for each x and the cofibers $G^*y = \{x \in C \mid \langle \varphi(y), y - x\rangle > 0\}$ of G are convex.

Example 3. Let C be a convex subset of a vector space E and $g : C \times C \to \mathbb{R}$ be a function such that

(a) $g(x,x) \le 0$ for each $x \in C$,
(b) $x \mapsto g(x,y)$ is quasiconcave on C for each $y \in C$.

Then the map $G : C \to C$ given by $Gx = \{y \in C \mid g(x,y) \le 0\}$ is strongly KKM. Indeed, it follows from (a) that $x \in Gx$ for each $x \in C$ and it follows from (b) that the cofibers $G^*y = \{x \in C \mid g(x,y) > 0\}$ of G are convex.

Example 4. Let C be a convex subset of a vector space E, $D \subset C$ be an arbitrary set, and $g : D \times C \to \mathbb{R}$ be a function such that

(a) $g(x,y) + g(y,x) \le 0$ for each $(x,y) \in D \times D$,
(b) $y \mapsto g(x,y)$ is convex on C for each $x \in D$.

Then the map $\Gamma : D \to C$ given by $\Gamma x = \{y \in C \mid g(x,y) \le 0\}$ is KKM. For, let $\{x_1, \dots, x_n\} \subset D$ and let $y_0 = \sum_{i=1}^n \lambda_i\, x_i$ be a convex combination of the x_i's. It follows from (a) that

$$g(x_i, x_j) + g(x_j, x_i) \le 0 \qquad \text{for every } i, j \in [n],$$

so, multiplying by λ_i and summing over i, we find

$$\sum_{i=1}^n \lambda_i\, g(x_i, x_j) + \sum_{i=1}^n \lambda_i\, g(x_j, x_i) \le 0 \qquad \text{for every } j \in [n],$$

and since by (b) the function $y \mapsto g(x,y)$ is convex, we obtain

$$\sum_{i=1}^n \lambda_i\, g(x_i, x_j) + g(x_j, y_0) \le 0 \qquad \text{for every } j \in [n].$$

By applying the same operations on these inequalities (multiplication by λ_j, addition over j, use of the convexity of $y \mapsto g(x,y)$), we finally get

$$\sum_{i=1}^n \lambda_i\, g(x_i, y_0) + \sum_{j=1}^n \lambda_j\, g(x_j, y_0) \le 0.$$

It follows that $g(x_i, y_0) \le 0$ for at least one point x_i, which means that y_0 lies in $\bigcup\{\Gamma x_i \mid i \in [n]\}$ and proves that Γ is KKM.

5. Elementary KKM Principle

The next theorem gives a sufficient condition for a family of closed convex sets to have the finite intersection property.

Theorem 5.1. *Let E be super-reflexive, X be a nonempty subset of E and $G : X \to E$ be a KKM-map with convex closed values. Then the family $\{Gx\}_{x \in X}$ has the finite intersection property.*

Proof. In fact, given a finite subset $A = \{x_1, \dots, x_n\}$ of X, we are going to show that

$$\bigcap\{Gx_i \mid i \in [n]\} \cap [A] \neq \emptyset. \tag{1}$$

The proof is by induction. For any set consisting of a single element our statement holds, because $x \in Gx$ for any $x \in X$. Assuming that the statement is true for any set containing $n-1$ elements, we are going to show that (1) holds. To this end, we define $y_1, \dots, y_n$ by picking up, for each $j \in [n]$, an element

$$y_j \in \bigcap\{Gx_i \mid i \in [n],\ i \neq j\} \cap [A \setminus \{x_j\}], \tag{2}$$

where the set in (2) is nonempty by inductive hypothesis.

Let $Y = [y_1, \dots, y_n]$. We now define real-valued functions $\varphi_i : Y \to \mathbb{R}$ by $\varphi_i(y) = d(y, Gx_i)$ for $i = 1, \dots, n$, and the function $\varphi : Y \to \mathbb{R}$ by $\varphi(y) = \max\{\varphi_1(y), \dots, \varphi_n(y)\}$. Because all the functions φ_i are continuous and convex, so is φ. By Theorem 3.1, φ admits a minimum at a point $\widehat{y} \in Y$, i.e.

$$\varphi(\widehat{y}) \leq \varphi(y) \qquad \text{for all } y \in Y. \tag{3}$$

Clearly, because the sets $Gx_1, \dots, Gx_n$ are closed, it is enough to show that $\varphi(\widehat{y}) = 0$. Suppose to the contrary that

$$0 < \varphi(\widehat{y}) = \varepsilon. \tag{4}$$

Since $Y = [y_1, \dots, y_n] \subset [x_1, \dots, x_n]$ and G is KKM, the family $\{Gx_i\}_{i=1}^n$ covers Y; thus $\widehat{y}$ belongs to one of the sets Gx_i, say Gx_n.

We now evaluate each of the functions φ_i at points of the segment $[\widehat{y}, y_n] = \{z_t \in E \mid z_t = t\widehat{y} + (1-t)y_n\}$. For $i = n$, because $\varphi_n(\widehat{y}) = 0$, we have

$$\varphi_n(z_t) \leq t\varphi_n(\widehat{y}) + (1-t)\varphi_n(y_n) = (1-t)\varphi_n(y_n).$$

Since $(1-t)\varphi_n(y_n) \to 0$ as $t \to 1$, it follows that for some $t_0 \in (0,1)$ sufficiently close to 1 we have

$$\varphi_n(z_{t_0}) \leq (1-t_0)\varphi_n(y_n) < \varepsilon = \varphi(\widehat{y}). \tag{5}$$

On the other hand, for any $i \in [n-1]$, because $\varphi_i(y_n) = 0$, we have

$$\varphi_i(z_{t_0}) \leq t_0\varphi_i(\widehat{y}) + (1-t_0)\varphi_i(y_n) < \varphi_i(\widehat{y}) \leq \varphi(\widehat{y}). \tag{6}$$

Thus $\varphi(z_{t_0}) = \max\{\varphi_1(z_{t_0}), \dots, \varphi_n(z_{t_0})\} < \varphi(\widehat{y})$, which, in view of (3), gives us a contradiction. The proof is complete. □

By combining Theorem 2.2 with Theorem 5.1, we immediately get the main result of this note:

Theorem 5.2 (Elementary Principle of KKM-maps). *Let E be super-reflexive, X be a nonempty subset of E and $G : X \to E$ be a KKM-map with convex closed values. Assume, furthermore, that one of the following conditions is satisfied:*

(i) *X is bounded,*
(ii) *all Gx are bounded,*
(iii) *Gx_0 is bounded for some $x_0 \in X$.*

Then the intersection $\bigcap\{Gx \mid x \in X\}$ is not empty.

6. Variational inequalities

In this section, as a first application of Theorem 5.2, we establish some basic results in the theory of variational inequalities. The case of a symmetric bilinear form was already treated in Section 3. The direct use of Theorem 5.2 enables us to eliminate this assumption:

Theorem 6.1 (Stampacchia). *Let X be a nonempty closed convex subset of a Hilbert space H, $a : H \times H \to \mathbb{R}$ a continuous coercive bilinear form, and $\ell : H \to \mathbb{R}$ a continuous linear form. Then there exists a unique point $y_0 \in X$ such that $a(y_0, y_0 - x) \le \ell(y_0 - x)$ for all $x \in X$.*

Proof. It follows from the coercivity of a that $a(x, x) > 0$ for all $x \neq 0$, so there can be at most one solution. Consider the map $G : X \to X$ given by

$$Gx = \{y \in X \mid a(y, y - x) \le \ell(y - x)\}.$$

Its values are closed, convex (because $x \mapsto a(x, x)$ is convex) and bounded (because a is coercive), and we know from Example 1 that it is a KKM-map. Therefore, by Theorem 5.2, there is a $y_0 \in \bigcap_{x\in X} Gx$, which was to be proved. □

We recall that the special case $X = H$ in Theorem 6.1 leads to a generalization of the Riesz representation theorem:

Corollary 6.2 (Lax–Milgram). *Let $a : H \times H \to \mathbb{R}$ be a continuous coercive bilinear form. Then for any continuous linear form $\ell : H \to \mathbb{R}$ there exists a unique point $y_0 \in H$ such that $a(y_0, x) = \ell(x)$ for all $x \in H$.*

The theorems of Stampacchia and Lax–Milgram extend to a certain class of nonlinear operators which we describe now. Let X be a nonempty subset of H. We recall that an operator $\varphi : X \to H$ is said to be *monotone* if for all $x, y \in X$ one has $\langle\varphi(y) - \varphi(x), y - x\rangle \ge 0$, *hemicontinuous* if for all $x, y \in X$ the function $t \in [0, 1] \mapsto \langle\varphi(y + t(x - y)), x - y\rangle$ is continuous at 0, and *coercive* if for some $x_0 \in X$ the function $x \mapsto \langle\varphi(x), x - x_0\rangle/\|x - x_0\|$ is coercive on $X \setminus \{x_0\}$.

Theorem 6.3 (Hartman–Stampacchia). *Let X be a nonempty closed convex subset of H, $\varphi : X \to H$ a monotone hemicontinuous coercive operator, and $\ell : H \to \mathbb{R}$ a continuous linear form. Then there exists a point $y_0 \in X$ such that $\langle \varphi(y_0), y_0 - x\rangle \le \ell(y_0 - x)$ for all $x \in X$.*

Proof. We consider only the case of a bounded X; an easy proof of the general case is left to the reader. Define two set-valued maps $G, F : X \to X$ by

$$\begin{aligned} Gx &= \{y \in X \mid \langle \varphi(y), y - x\rangle \le \ell(y - x)\},\\ Fx &= \{y \in X \mid \langle \varphi(x), y - x\rangle \le \ell(y - x)\}. \end{aligned}$$

Because φ is monotone we have

$$\langle \varphi(y), y - x\rangle \ge \langle \varphi(x), y - x\rangle \qquad \text{for all } x, y \in X$$

and therefore $Gx \subset Fx$ for each $x \in X$. In Example 2, we observed that G is a KKM-map, consequently so is the map F. Since by definition the values of F are convex and closed, we infer by Theorem 5.2 that for some $y_0 \in X$ we have $y_0 \in \bigcap_{x\in X} Fx$ and thus

$$\langle \varphi(z), y_0 - z\rangle \le \ell(y_0 - z) \qquad \text{for all } z \in X.$$

Fix $x \in X$ and consider points $z_t = y_0 + t(x - y_0)$, $t \in [0, 1]$. We have

$$\langle \varphi(y_0 + t(x - y_0)), y_0 - x\rangle \le \ell(y_0 - x) \qquad \text{for } t > 0,$$

and thus, letting $t \to 0$, by hemicontinuity of φ we obtain $\langle \varphi(y_0), y_0 - x\rangle \le \ell(y_0 - x)$. Since x was arbitrary, the conclusion follows. □

The special case $X = H$ in Theorem 6.3 reads as follows:

Corollary 6.4 (Minty–Browder). *Let $\varphi : H \to H$ be a monotone hemicontinuous coercive operator. Then for any continuous linear form $\ell : H \to \mathbb{R}$ there exists a point $y_0 \in H$ such that $\langle \varphi(y_0), x\rangle = \ell(x)$ for all $x \in H$.*

As another consequence of Theorem 6.3, we get the following fixed point theorem for nonexpansive operators:

Corollary 6.5 (Browder–Goehde–Kirk). *Let X be a nonempty closed convex bounded subset of H and let $f : X \to H$ be a nonexpansive operator (i.e., $\|f(x) - f(y)\| \le \|x - y\|$ for all $x, y \in X$). Suppose that for each $x \in X$ there exists $t > 0$ such that $x + t(f(x) - x) \in X$. Then f has a fixed point.*

Proof. Since f is nonexpansive, the operator $\varphi(x) = x - f(x)$ from X to H is monotone continuous. Applying Theorem 6.3 we get a point $y_0 \in X$ such that $\langle y_0 - f(y_0), y_0 - x\rangle \le 0$ for all $x \in X$. Since for some $t > 0$ the point $y_0 + t(f(y_0) - y_0)$ lies in X, we can insert that value into the above inequality to get $\langle y_0 - f(y_0), f(y_0) - y_0\rangle \ge 0$, showing that y_0 is a fixed point for f. □

7. Theorem of von Neumann

We now establish a classical result in the theory of games.

Theorem 7.1 (von Neumann). *Let X and Y be two nonempty closed bounded convex subsets of two super-reflexive spaces E_X and E_Y. Let $f : X \times Y \to \mathbb{R}$ be a real-valued function satisfying*

(i) *$x \mapsto f(x, y)$ is convex and l.s.c. for each $y \in Y$,*
(ii) *$y \mapsto f(x, y)$ is concave and u.s.c. for each $x \in X$.*

Then:

(A) *There exists a saddle point for f, i.e. a point $(x_0, y_0) \in X \times Y$ such that*

$$f(x_0, y) \leq f(x, y_0) \qquad \textit{for all } (x, y) \in X \times Y.$$

(B) $\min_{y \in Y} \max_{x \in X} f(x, y) = \max_{x \in X} \min_{y \in Y} f(x, y)$.

Proof. Since (B) follows at once from (A), we only need to establish (A). To this end, define a set-valued map G of $X \times Y \subset E_X \times E_Y$ into itself by putting

$$G(x, y) = \{(x', y') \in X \times Y \mid f(x', y) - f(x, y') \leq 0\}.$$

We claim that G is strongly KKM. Indeed, we have $(x, y) \in G(x, y)$ for each $(x, y) \in X \times Y$ and, since the function $(x, y) \mapsto f(x', y) - f(x, y')$ is concave, the cofibers

$$G^*(x', y') = \{(x, y) \in X \times Y \mid f(x', y) - f(x, y') > 0\}$$

of G are convex. From this, because $X \times Y$ is convex, we conclude that G is a KKM-map. On the other hand, because for each $(x, y) \in X \times Y$ the function $(x', y') \mapsto f(x', y) - f(x, y')$ is convex and l.s.c., we conclude that all the sets $G(x, y)$ are convex and closed. Consequently, by Theorem 5.2, there exists (x_0, y_0) such that $(x_0, y_0) \in G(x, y)$ for all $(x, y) \in X \times Y$; this means exactly that (x_0, y_0) is a saddle point for f. The proof is complete. □

8. Systems of inequalities

Let X be a convex bounded closed subset of a super-reflexive space E, and let $\Phi = \{\varphi\}$ be a nonempty family of real-valued functions $\varphi : X \to \mathbb{R}$ which are convex and lower semicontinuous. To formulate a general result we let $[\Phi]$ be the convex hull of Φ in the vector space $\mathbb{R}^X$; we are concerned with the following two problems:

($\mathcal{P}_1$) There exists $x_0 \in X$ such that $\varphi(x_0) \leq 0$ for all $\varphi \in \Phi$.
($\mathcal{P}_2$) For each $\psi \in [\Phi]$ there exists $\widehat{x} \in X$ such that $\psi(\widehat{x}) \leq 0$.

Theorem 8.1. *Under the above assumptions, the problems* $(\mathcal{P}_1)$ *and* $(\mathcal{P}_2)$ *are equivalent. In other words, we have the following alternative: Either* (a) *there is* $x_0 \in X$ *satisfying* $\varphi(x_0) \leq 0$ *for all* $\varphi \in \Phi$*, or* (b) *there is* $\psi \in [\Phi]$ *such that* $\psi(x) > 0$ *for all* $x \in X$*.*

Proof. Clearly, it is enough to show that $(\mathcal{P}_2) \Rightarrow (\mathcal{P}_1)$. Assume that $(\mathcal{P}_2)$ holds and let $S(\varphi) = \{x \in X \mid \varphi(x) \leq 0\}$. To establish our claim, we have to show that $\bigcap_{\varphi \in \Phi} S(\varphi)$ is not empty. Since the sets $S(\varphi)$ are convex and closed, in view of Theorem 2.2 this reduces to showing that the family $\{S(\varphi) \mid \varphi \in \Phi\}$ has the finite intersection property. To this end, let $\varphi_1, \dots, \varphi_n \in \Phi$. Set

$$\Lambda = \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \;\middle|\; \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^n \lambda_i = 1 \right\}$$

and consider the real-valued function $f : X \times \Lambda \to \mathbb{R}$ given by

$$f(x, \lambda) = \sum_{i=1}^n \lambda_i \varphi_i(x).$$

Observe that all the conditions of Theorem 7.1 are satisfied. Consequently, there exists $(x_0, \mu) \in X \times \Lambda$ such that $f(x_0, \lambda) \leq f(x, \mu)$ for all $(x, \lambda) \in X \times \Lambda$. Said differently, there exist $x_0 \in X$ and $\psi = \sum_{i=1}^n \mu_i \varphi_i \in [\Phi]$ such that $\varphi_i(x_0) \leq \psi(x)$ for all $i = 1, \dots, n$ and $x \in X$. Now, by $(\mathcal{P}_2)$, there exists $\widehat{x} \in X$ such that $\psi(\widehat{x}) \leq 0$, so $\varphi_i(x_0) \leq 0$ for all $i = 1, \dots, n$; that gives $x_0 \in \bigcap_{i=1}^n S(\varphi_i)$ and the proof is complete. $\square$

Let X be a set and $\Phi = \{\varphi\}$ be a nonempty family of real-valued functions $\varphi : X \to \mathbb{R}$. We say that Φ is *concave in the sense of Ky Fan* (or simply *F-concave*) provided for any convex combination $\sum_{i=1}^n \lambda_i \varphi_i$ of $\varphi_1, \dots, \varphi_n \in \Phi$ there is a $\varphi \in \Phi$ such that $\varphi(x) \geq \sum_{i=1}^n \lambda_i \varphi_i(x)$ for each $x \in X$.

Theorem 8.2. *Let* X *be a nonempty convex bounded closed subset of a super-reflexive Banach space* E *and* $\Phi = \{\varphi\}$ *an F-concave family of convex l.s.c. real-valued functions* $\varphi : X \to \mathbb{R}$*. Then the following two conditions are equivalent:*

(A) *There exists* $x_0 \in X$ *such that* $\varphi(x_0) \leq 0$ *for all* $\varphi \in \Phi$*.*
(B) *For each* $\varphi \in \Phi$ *there exists* $\widehat{x} \in X$ *such that* $\varphi(\widehat{x}) \leq 0$*.*

Proof. Clearly it is enough to show that (B)$\Rightarrow$(A). To the contrary, suppose that (A) does not hold. Then by Theorem 8.1 there is a convex combination $\sum_{i=1}^n \lambda_i \varphi_i \in [\Phi]$ such that

$$\sum_{i=1}^n \lambda_i \varphi_i(x) > 0 \qquad \text{for all } x \in X$$

and hence, by definition of the F-concave family, we have, for some $\varphi \in \Phi$,

$$\varphi(x) \geq \sum_{i=1}^{n} \lambda_i \varphi_i(x) > 0 \qquad \text{for all } x \in X.$$

This contradicts (B) and the proof is complete. □

As an immediate consequence we obtain

Theorem 8.3. *Under the assumptions of Theorem* 8.2 *we have*

$$\alpha = \min_{x \in X} \sup_{\varphi \in \Phi} \varphi(x) = \sup_{\varphi \in \Phi} \min_{x \in X} \varphi(x) = \beta.$$

Proof. Since always $\beta \leq \alpha$ we show $\alpha \leq \beta$. Assume $\alpha > \beta$ and consider the family $\Psi = \{\psi\}$ of functions $\psi : X \to \mathbb{R}$ given by

$$\psi(x) = \phi(x) - \lambda, \qquad x \in X,$$

where $\beta < \lambda < \alpha$. Clearly, Ψ satisfies the conditions of Theorem 8.2, so that either

(i) $\psi(x_0) = \phi(x_0) - \lambda \leq 0$ for all $\phi \in \Phi$, or
(ii) $\psi_0(x) = \phi_0(x) - \lambda > 0$ for all $x \in X$.

In case (i) we have $\sup_{\varphi \in \Phi} \varphi(x_0) \leq \lambda$, and therefore

$$\alpha = \min_{x \in X} \sup_{\varphi \in \Phi} \varphi(x) \leq \lambda,$$

a contradiction. Similarly, in case (ii) we get

$$\beta = \sup_{\varphi \in \Phi} \min_{x \in X} \varphi(x) \geq \min_{x \in X} \varphi_0(x) \geq \lambda,$$

a contradiction. The proof is complete. □

Corollary 8.4 (Kneser–Fan). *Let X and Y be two convex subsets of super-reflexive spaces E_X and E_Y and assume that Y is closed and bounded. Let $f : X \times Y \to \mathbb{R}$ be a real-valued function such that*

(i) *$x \mapsto f(x,y)$ is concave for each $y \in Y$,*
(ii) *$y \mapsto f(x,y)$ is l.s.c. and convex for each $x \in X$.*

Then

$$\min_{y \in Y} \sup_{x \in X} f(x,y) = \sup_{x \in X} \min_{y \in Y} f(x,y).$$

9. Maximal monotone operators

We now establish some basic results in the theory of maximal monotone operators in a Hilbert space H. Recall that a set-valued operator $T : H \to H$ is said to be *monotone* if $\langle y^* - x^*, y - x\rangle \geq 0$ whenever $x^* \in Tx$ and $y^* \in Ty$, and *maximal monotone* if it is monotone and maximal in the set of all monotone operators from H into H. In what follows we denote by D the *domain* of T, i.e. $D = \{y \in H \mid Ty \neq \emptyset\}$, and by C the closed convex hull of D. It is clear from the definitions that

1) if T is monotone, then for all $x \in D$ and $y \in C$, $\sup_{x^* \in Tx}\langle x^*, y - x\rangle$ is finite,
2) if T is maximal monotone, then $y^* \in Ty$ whenever $\langle x^* - y^*, x - y\rangle \geq 0$ for all $x \in D$ and $x^* \in Tx$.

Theorem 9.1. *Let $T : H \to H$ be a monotone operator and $u : H \to H$ be a single-valued, linear, monotone, bounded operator. Assume that for some point $x_0 \in D$ the set $\{y \in C \mid \sup_{x_0^* \in Tx_0}\langle u(y) + x_0^*, y - x_0\rangle \leq 0\}$ is bounded. Then there is a point $y_0 \in C$ such that*

$$\sup_{x^* \in Tx}\langle u(y_0) + x^*, y_0 - x\rangle \leq 0 \qquad \text{for all } x \in D.$$

Proof. We show that the map $G : D \to C$ defined by

$$Gx = \{y \in C \mid \sup_{x^* \in Tx}\langle u(y) + x^*, y - x\rangle \leq 0\} \qquad \text{for } x \in D$$

satisfies all the conditions of Theorem 5.2.

First, we are going to show that G is a KKM-map. To this end consider the function $f : C \times D \times C \to \mathbb{R}$ given by

$$f(\zeta, x, y) = \sup_{x^* \in Tx}\langle u(\zeta) + x^*, y - x\rangle.$$

Since T is monotone, f is well defined and satisfies the following conditions:

(a) $f(\zeta, x, y) + f(\zeta, y, x) \leq 0$ for each $(x, y) \in D \times D$ and each $\zeta \in C$,
(b) $y \mapsto f(\zeta, x, y)$ is convex on C for each $x \in D$ and each $\zeta \in C$.

Now we observe that, using the function f, the map $G : D \to C$ can be equivalently described as

$$Gx = \{y \in C \mid f(y, x, y) \leq 0\}.$$

To show that G is KKM, let $A = \{x_1, \dots, x_n\} \subset D$ and let $y_0 \in [A]$. Define $g : A \times [A] \to \mathbb{R}$ by letting $g(x, y) = f(y_0, x, y)$. It follows from (a) and (b) that g satisfies the conditions of Example 4, so the map $\Gamma : A \to [A]$ given by $\Gamma x = \{y \in [A] \mid g(x, y) \leq 0\}$ is KKM. This implies in particular that $y_0 \in \Gamma x_i$ for some $x_i \in A$, which means that $f(y_0, x_i, y_0) = g(x_i, y_0) \leq 0$, that is, $y_0 \in Gx_i$. The proof that G is KKM is complete.

On the other hand, the values of G are closed and convex since the function $y \mapsto \langle u(y), y\rangle$ is continuous and convex on C, and the value Gx_0 is bounded by assumption. We may therefore invoke Theorem 5.2 to deduce that $\bigcap\{Gx \mid x \in D\}$ is not empty, which was to be proved. □

Corollary 9.2 (Minty). *If $T : H \to H$ is maximal monotone with bounded domain, then T is onto.*

Proof. It is clearly enough to show that $0 \in T(H)$. Since T is maximal, D is not empty, and therefore C is closed, convex, bounded and nonempty. Applying Theorem 9.1 with $u \equiv 0$ we find $y_0 \in C$ such that

$$\langle x^*, x - y_0\rangle \geq 0 \qquad \text{for all } x \in D \text{ and } x^* \in Tx.$$

Since T is maximal monotone, this implies that $0 \in Ty_0$. □

Corollary 9.3 (Minty). *If $T : H \to H$ is maximal monotone, then $I + T$ is onto* (I *denotes the identity on* H).

Proof. As above, it is enough to show that $0 \in (I + T)(H)$. Invoke Theorem 9.1 with $u \equiv I$. We find $y_0 \in C$ such that

$$\langle x^* - (-y_0), x - y_0\rangle \geq 0 \qquad \text{for all } x \in D \text{ and } x^* \in Tx.$$

Since T is maximal monotone, we derive that $-y_0 \in Ty_0$, or equivalently that $0 \in y_0 + Ty_0$. □

10. Remarks

1. The authors thank Marcin Bownik for useful comments related to the proof of Lemma 2.1.

2. The reader may observe that Theorem 3.1 can be deduced directly from Lemma 2.1. Theorem 5.1 was formulated in other terms in [1]; it is equivalent to the result of V. Klee [7] (see also [2]).

3. The KKM property appears for the first time in the paper of Knaster–Kuratowski–Mazurkiewicz [8], devoted to the combinatorial proof of the Brouwer Fixed Point Theorem.

4. Since, in weak topology, closed convex bounded subsets of a super-reflexive Banach space are compact, the Elementary KKM Principle is contained in the Geometric KKM Principle, established by the authors in [6]. It follows from the Geometric KKM Principle that the main results of the present note are valid for arbitrary reflexive spaces, but their proofs require the theory of weak topology and compactness. On the other hand, the reader may observe that, using the elementary tools of this note, the results 6.3, 6.4 and 9.1, 9.2 (formulated for simplicity of exposition in the context of a Hilbert space) can be established in arbitrary super-reflexive spaces.

5. The Elementary and Geometric KKM Principles are special cases of the Topological Principle of KKM-maps established by Ky Fan [4], [5] (for its numerous applications see also [3] and [9]). The topological principle is, in fact, equivalent to the Brouwer Fixed Point Theorem.

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