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EIGENVALUES OF THE LAPLACIAN FOR SPHERE BUNDLES

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Dedicated to Ky Fan on the occasion of his 80th birthday

Let $\pi : Z \to Y$ be a fiber bundle, where Z and Y are compact Riemannian manifolds without boundary. Let $N(\lambda, p, Z)$ and $N(\lambda, p, Y)$ be the eigenspaces of the p-form valued Laplacians Δ_p^Z and Δ_p^Y on Z and Y. We say ψ is *harmonic* if $\psi \in N(0, p, \cdot)$. We wish to know circumstances when there exists a non-zero harmonic p-form Φ on Y so that $\pi^*\Phi \in N(\mu, p, Z)$ for $\mu > 0$. In [1] we studied Riemannian submersions and answered this question for $p \neq 1$ by showing

Theorem 1.

(a) *Let $\pi : Z \to Y$ be a Riemannian submersion. If $0 \neq \Phi \in N(\lambda, 0, Y)$ and if $\pi^*\Phi \in N(\mu, 0, Z)$, then $\mu = \lambda$.*
(b) *For any $p \geq 2$, there exists a Riemannian submersion $\pi : Z \to Y$ and $0 \neq \Phi$ harmonic on Y so that $\pi^*\Phi$ belongs to $N(\mu, p, Z)$ for $\mu > 0$.*

Remark. We say π^* *preserves the eigenforms of the Laplacian* if $\pi^* N(\lambda, p, Y)$ is contained in $N(\mu(\lambda), p, Z)$ for all λ. Let π be a Riemannian submersion. If $p = 0$, then π^* preserves the eigenforms of the Laplacian if and only if the fibers of π are minimal; if $p > 0$, then π^* preserves the eigenforms of the Laplacian if and only if the fibers of π are minimal and the horizontal distribution of π is

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integrable. In this setting $\mu(\lambda) = \lambda$ for all λ so the eigenvalues do not change. We refer to [1, 3, 5] for details.

We proved Theorem 1(b) by considering the Hopf fibration $S^1 \to S^3 \to S^2$ and taking Φ to be the volume element on S^2. Then

$$\Phi \in N(0, 2, S^2) \quad \text{and} \quad \pi^*\Phi \in N(\mu, 2, S^3) \quad \text{for } \mu > 0;$$

we used Riemannian products to handle the case $p > 2$. There are other examples where the fiber is a circle due to Muto [4], but Riemannian products and covering projections involving these examples are essentially the only known examples where this phenomenon occurs.

The situation is quite different for sphere bundles of higher rank. Let E be a real vector bundle of fiber dimension ν over M which is equipped with a Riemannian fiber metric. Let $S(E)$ be the unit sphere bundle of E. If ∇ is a Riemannian connection on E, then $S(E)$ is naturally equipped with a Riemannian metric and the natural projection π from $S(E)$ to Y is a Riemannian submersion. In [2] we showed:

Theorem 2. *Let E be a real vector bundle over Y of fiber dimension $\nu \geq 3$. Give the unit sphere bundle $S(E)$ of E a Riemannian metric induced by a Riemannian connection ∇ on E and let $\pi : S(E) \to Y$ be the natural projection. Let $0 \neq \Phi$ be a harmonic p-form on Y. If $\pi^*\Phi \in N(\mu, p, S(E))$, then $\mu = 0$.*

In this brief note, we will use topological methods to prove a variant of Theorem 2. We shall impose the restriction that ν is odd but drop the restriction that $\pi : S(E) \to Y$ is a Riemannian submersion.

Theorem 3. *Let E be a real vector bundle over Y of odd fiber dimension $\nu \geq 3$. Give the unit sphere bundle $S(E)$ of E an arbitrary Riemannian metric. Let $0 \neq \Phi$ be a harmonic p-form on Y. If $\pi^*\Phi \in N(\mu, p, S)$, then $\mu = 0$.*

Remark. This shows once again how special the case of circle bundles is in this theory. In contrast to the proof of Theorem 2 which was entirely local and differential geometric in nature, our proof here is global and rests on the machinery of algebraic topology. It is also extremely short and conceptual as contrasted to the proof given for Theorem 2 which was very computational in nature.

Proof of Theorem 3. Let $0 \neq \Phi$ be a smooth p-form on Y with $\Delta_p^Y \Phi = 0$. Let $\phi = \pi^*\Phi$ be the pull back of Φ to $S(E)$ and suppose $\Delta_p^{S(E)}\phi = \mu\phi$ for $\mu \neq 0$. We will prove the theorem by arguing for a contradiction.

Since Φ is harmonic, $d\Phi = 0$. Since $\pi^* d = d\pi^*$, $d\phi = 0$. We compute:

$$\phi = \lambda^{-1}\Delta_p^{S(E)}\phi = \lambda^{-1}(d\delta + \delta d)\phi = \lambda^{-1} d\delta\phi.$$

Thus in particular, $\phi \in \text{image}(d)$. This means the associated cohomology class $[\phi]$ in $H^p(S(E);\mathbb{C})$ vanishes; we have used the de Rham theorem to identify simplicial and de Rham cohomology. We may use the Hodge decomposition theorem to see $[\Phi]$ is a non-zero cohomology class in $H^p(Y;\mathbb{C})$. This shows that

$$0 \neq [\Phi] \in \ker\{\pi^* : H^p(Y;\mathbb{C}) \to H^p(S(E);\mathbb{C})\}. \tag{1}$$

Suppose that E is orientable. The Gysin sequence is a long exact sequence

$$\dots \to H^{p-\nu}(Y;\mathbb{C}) \xrightarrow{\cup e} H^p(Y;\mathbb{C}) \xrightarrow{\pi^*} H^p(S(E);\mathbb{C}) \xrightarrow{\varepsilon} H^{p-\nu+1}(Y;\mathbb{C}) \to \dots$$

where $\cup e$ denotes cup product with the Euler form, π^* is the pull back, and ε is the connecting homomorphism. The crucial point here is that the Euler form e can be computed in terms of the curvature of the bundle E using Chern–Weil theory and vanishes since the fiber dimension of E is odd. Thus the Gysin sequence yields

$$0 \to H^p(Y;\mathbb{C}) \xrightarrow{\pi^*} H^p(S(E);\mathbb{C})$$

so $\ker(\pi^*) = \{0\}$. This contradicts (1) above and completes the proof if E is orientable.

If E is not orientable, let $\sigma : Y_1 \to Y$ be the double cover defined by the orientation class of E and let $E_1 = \sigma^*(E)$ be the induced bundle over Y_1. Let $\pi_1 : S(E_1) \to Y_1$ be the natural projection and let $S(\sigma) : S(E_1) \to S(E)$ be the double cover. We have a commutative diagram

$$\begin{array}{ccc} S(E_1) & \xrightarrow{\pi_1} & Y_1 \\ {\scriptstyle S(\sigma)}\downarrow & & \uparrow{\scriptstyle \sigma} \\ S(E) & \xrightarrow{\pi} & Y \end{array}$$

We note that σ^* intertwines Δ_p^Y and $\Delta_p^{Y_1}$, that $S(\sigma)^*$ intertwines $\Delta_p^{S(E)}$ and $\Delta_p^{S(E_1)}$, that $\Phi_1 = \sigma^*\Phi$ is a harmonic p-form on Y_1, and that $\phi_1 = \pi_1^*\Phi$ is an eigenform of the Laplacian with eigenvalue $\mu \neq 0$ on $S(E_1)$. Since E_1 is orientable over Y_1, we apply the argument given above to complete the proof. □

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