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EXISTENCE OF PURE EQUILIBRIA IN GAMES WITH NONATOMIC SPACE OF PLAYERS

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Abstract. In this paper known results on the existence of pure Nash equilibria in games with nonatomic measure space of players are generalized and also a simple proof is offered. The relaxed assumptions include metrizability of the space of actions, measurability of payoff functions and available strategy correspondences.

1. Introduction

Schmeidler in [20] defined a notion of nonatomic game as a game with a set of players endowed with nonatomic measure, and defined what is understood as equilibrium. He showed (Theorem 1) that in a game with players constituting interval $[0,1]$ with Lebesgue measure and finite set of strategies there exists a mixed equilibrium. The proof of existence of a pure equilibrium when each player's payoff depends only on his own strategy and the mean of the profile was based on this result.

Rath in [17] proved Schmeidler's theorem without using the existence of mixed equilibrium. His proof was based on properties of the integral of a correspondence and Kakutani fixed point theorem. This approach turned out to be fruitful – the same routine could be used to prove a more general result: existence of a pure equilibrium in a game with compact set of strategies.

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Mas–Colell in [16] reformulated the model and the definition of the equilibrium: instead of measurable functions from the space of players into the space of strategies, strategy profiles were represented as distributions on the product of the space of characteristics (i.e. continuous utility functions) endowed with the supremum norm, and the space of actions. He gave a simple proof of the existence of an equilibrium (what he defined was a pure equilibrium).

There were many generalizations of the models of Schmeidler and Mas–Colell, e.g. Khan [9] and [10], Khan and Sun [11], Khan and Rustichini [12], Balder [5] and [6]. Extensions included weakening the assumptions on the continuity of payoff functions and the compactness of a strategy space, the completeness of a space of players and using preference relations instead of payoff functions.

From the point of view of the present paper, especially interesting is Balder [6], since the existence theorem of this paper is a straightforward generalization of one presented in [6]. Balder proved as a main result (Theorem 2.1) a general theorem on existence of a mixed equilibrium in games with a measure space of players. It is the most general result for the case with payoff functions. Existence of a pure equilibrium in games with a nonatomic space of players (Theorem 3.4.1) was shown as a consequence of Theorem 2.1 and a Lemma from his earlier paper ([4, Lemma III]) concerning optimal control theory. The proof of the latter one was long and complicated.

The author's work on the existence theorem was started by the thesis [21], which contains various results on games with infinitely many players and a certain economic application of them. The results contained in this paper are used in [22]–[26].

The theorem presented in this paper not only generalizes the result of Balder ([6, Theorem 3.4.1]), but also gives a completely different, more direct and simpler proof of it. For comparison, Balder's result will be presented in Section 3.3.

Although the main theorem of this paper may appear very abstract (the measurability seems to be a very natural assumption), replacing the measurability with analyticity is natural. If we consider sets being projections or measurable (or even continuous) images of measurable sets, we cannot expect more than analyticity. Moreover, for a function with a measurable graph, the only thing we can assume about the inverse images of measurable sets is that they are analytic.

Therefore in the models, which are in fact only projections of a measurable but very complex real world into a simpler reality of the model, or measurable images of the world, with functions represented by their graphs – measurable sets of pairs, a theorem assuming analyticity may often turn out to be useful.

2. General model

A *game with measure space of players* is a system

$$((\Omega, \Im, \mu), (\mathbb{S}, \mathcal{S}), S, \{\Pi_\omega\}_{\omega\in\Omega})$$

a space of players, a correspondence S of (pure) strategies available to the players (acting into the space $\mathbb{S}$) and players' payoff functions. All of them are defined below.

A measure space $(\Omega, \Im, \mu)$ with a finite measure μ will be our *space of players*.

A measurable space $(\mathbb{S}, \mathcal{S})$ is a *space of* (*pure*) *strategies*.

The set $\mathbb{S}$ is topologized with a Hausdorff topology. All the topological assumptions about objects defined on $\mathbb{S}$ refer to this topology. The σ-field $\mathcal{S}$ is not assumed to coincide with $\mathcal{B}(\mathbb{S})$ – the Borel σ-field of $\mathbb{S}$.

A correspondence $S : \Omega \multimap \mathbb{S}$ is called *correspondence of players' available strategies*; S_ω denotes the set of strategies available to player ω; any strategy $d \in S_\omega$ is called *player ω's individual strategy.*

Pure profiles are measurable functions $\delta : \Omega \to \mathbb{S}$ such that for almost every ω we have $\delta(\omega) \in S_\omega$. The set of all pure profiles is denoted by $\mathcal{R}$.

Generally, the *payoff functions* Π_ω are assumed to act from $\mathcal{R}$ into $[-\infty, \infty)$. However, we assume that there is a specific *internal-external representation* of the payoffs consisting of a topological space $\mathbb{Y}$ (called *space of profile statistics*), functions $P_\omega : S_\omega \times \mathbb{Y} \to [-\infty, \infty)$ (*reduced payoff functions*) and a mapping $e : \mathcal{R} \to \mathbb{Y}$ (*externality mapping*) such that the payoff functions Π_ω have the form

$$\Pi_\omega(\delta) = P_\omega(\delta(\omega), e(\delta)).$$

In this paper the externality mapping is assumed to have the form

$$e(\delta) = \left[\int_\Omega g_i(\omega, \delta(\omega))\, d\mu(\omega)\right]_{i=1}^r$$

where the function $g : \mathrm{Gr}(S) \to \mathbb{R}^r$ is measurable and the family of functions $\{g(\,\cdot\,, d)\}_{d\in\mathbb{S}}$ is integrably bounded.

A *pure Nash equilibrium* is a pure profile δ such that

$$\delta(\omega) \in \mathrm{Argmax}_{d\in S_\omega} P_\omega(d, e(\delta)) \quad \text{for a.e. } \omega.$$

3. Existence of pure equilibria in games with nonatomic space of players

We will use the symbol $\mathbb{U}$ for the image $e(\mathcal{R})$ and $(\Omega, \overline{\Im}, \overline{\mu})$ for the completion of $(\Omega, \Im, \mu)$.

As usual we understand $\Im$-analytic sets as obtained by the Souslin $\mathcal{A}$-operation (e.g. [13] or [19]) performed on the sets belonging to $\Im$. Besides, as we noted in Section 1, $\Im$-analytic sets appear as projections or measurable images of measurable sets, or inverse images of measurable sets by a function with a measurable graph.

We denote the diagonal $\{(s,s) : s \in \mathbb{S}\}$ by $\mathrm{diag}(\mathbb{S})$.

The following assumptions will be used in the existence Theorem:

(A1) The space $\mathbb{S}$ is such that $\mathrm{diag}(\mathbb{S})$ is $\mathcal{S} \otimes \mathcal{S}$-measurable and there exists a measurable space $(\mathbb{Z}, \mathcal{Z})$ and a measurable function $F : \mathbb{Z} \xrightarrow{\text{onto}} \mathbb{S}$ such that $\mathbb{Z}$ is an analytic subset of a measurable a space $(\mathbb{W}, \mathcal{W})$ (with $\mathcal{Z} = \{W \cap \mathbb{Z} : W \in \mathcal{W}\}$) such that $\mathcal{W}$ is generated (in the sense of taking countable unions and intersections) by a compact countable family of sets (compact means that for every sequence of sets $\{F_n\}_{n\in\mathbb{N}}$ with finite intersection property, $\bigcap_{n\in\mathbb{N}} F_n$ is nonempty).

(A1') The space $\mathbb{S}$ is such that $\mathrm{diag}(\mathbb{S})$ is $\mathcal{S} \otimes \mathcal{S}$-measurable and there exists a measurable space $(\mathbb{Z}, \mathcal{Z})$ and a measurable function $F : \mathbb{Z} \xrightarrow{\text{onto}} \mathbb{S}$ such that $\mathbb{Z}$ is an analytic subset of a separable compact metrizable topological space $\mathbb{W}$ (with $\mathcal{Z} = \{W \cap \mathbb{Z} : W \in \mathcal{B}(\mathbb{W})\}$).

Let us note that (A1) implies (A1').

(A2) For a.e. ω, the set S_ω is nonempty and compact.

(A3) The function P_ω is upper semicontinuous on $S_\omega \times \mathbb{U}$ for a.e. ω.

(A4) The graph of S is $\overline{\Im} \otimes \mathcal{S}$-analytic.

(A5) The function $P_\omega(d, \cdot\,)$ is continuous on $\mathbb{U}$ for a.e. ω and every $d \in S_\omega$.

(A6) For every $u \in \mathbb{U}$, the function $P_\cdot(\,\cdot\,, u) : \mathrm{Gr}(S) \to [-\infty, \infty)$ is such that inverse images of Borel sets are $\overline{\Im} \otimes \mathcal{S}$-analytic.

(A7) The functions g_i are measurable, integrably bounded and such that $g_i(\omega, \cdot\,)$ is continuous on S_ω for every i and a.e. ω.

Now it is time to formulate the main result:

Theorem 3.1.

(a) *If* $(\Omega, \Im, \mu)$ *is nonatomic complete and assumptions* (A1), (A2)–(A7) *are fulfilled then there exists a pure strategy Nash equilibrium profile.*

(b) *If* $(\Omega, \Im, \mu)$ *is nonatomic, assumptions* (A1'), (A2)–(A7) *are fulfilled then there exists a pure strategy Nash equilibrium profile.*

3.1. Useful facts concerning measurability. Let $(\mathbb{S}, \mathcal{S})$ be any measurable space. If $\mathbb{S}$ is any measurable space such that the σ-field $\mathcal{S}$ contains a countable, separating points family of sets, then $\mathrm{diag}(\mathbb{S})$ is $\mathcal{S} \otimes \mathcal{S}$-measurable: let $\{A_i\}_{i\in\mathbb{N}}$ be a family separating points, then

$$\mathrm{diag}(\mathbb{S}) = (\mathbb{S} \times \mathbb{S}) \setminus \bigcup_{i,j\in\mathbb{N}} ((A_i \times A_j) \setminus ((A_i \times A_i) \cup (A_j \times A_j))).$$

To prove Theorem 3.1 we will work with $(\overline{\Im}, \overline{\mu})$ – the completion of $(\Im, \mu)$. The following facts explain why such a procedure will lead to satisfactory results.

It is obvious, that for an arbitrary σ-finite measure space $(\Omega, \Im, \mu)$ and for every $\Im$-measurable function $f : \Omega \to R$, integrably bounded from above or below, we have $\int_\Omega f(\omega)\, d\mu(\omega) = \int_\Omega f(\omega)\, d\overline{\mu}(\omega)$.

The remaining facts are not so immediate:

Proposition 3.2. *Let $(\Omega, \Im, \mu)$ be any σ-finite measure space and let* (A1') *be fulfilled. If $\overline{f} : \Omega \to S$ is an $\overline{\Im}$-measurable function, then there exists an $\Im$-measurable function $f : \Omega \to S$ almost everywhere equal to $\overline{f}$.*

Proof. We shall start from checking measurability of the graph of the correspondence $F^{-1} \circ \overline{f}$ (where F is the function appearing in assumption A1'). We have

$$\begin{aligned}\mathrm{Gr}(F^{-1} \circ \overline{f}) &= \{(\omega, x) : x \in \overline{F}^{-1}(\overline{f}(\omega))\} = (\mathrm{Id}_\Omega, F)^{-1}(\mathrm{Gr}(\overline{f})) \\ &= ((\mathrm{Id}_\Omega, F) \circ (\overline{f}, \mathrm{Id}_{\mathbb{S}}))^{-1}(\mathrm{diag}(\mathbb{S})).\end{aligned}$$

The diagonal is measurable, both functions (Id_Ω, F) and $(\overline{f}, \mathrm{Id}_{\mathbb{S}})$ are measurable (with respect to the corresponding σ-fields), so their composition is measurable, therefore the graph of $(F^{-1} \circ \overline{f})$, as the inverse image of a measurable set is $\overline{\Im} \otimes \mathcal{Z}$-measurable, therefore it is $\overline{\Im} \otimes \mathcal{B}(\mathbb{W})$-analytic.

By a generalization of Aumann's measurable selection Theorem ([14, Theorem 5.5], see also [2]) there exists $\overline{h}$ – an $\overline{\Im}$-measurable a.e. selection from the correspondence $(F^{-1} \circ \overline{f})$. Let $\{A_i\}_{i \in \mathbb{N}}$ be a countable family of generators of $\mathcal{B}(\mathbb{W})$, x any element of $\mathbb{W}$ and $C_i = \overline{h}^{-1}(A_i)$. Since $C_i \in \overline{\Im}$, there exist sets $C_{i-} \subset C_i \subset C_{i+}$ such that $C_{i+}, C_{i-} \in \Im$ and $\mu(C_{i+} \setminus C_{i-}) = 0$. Let C denote $\bigcup_{i \in \mathbb{N}} (C_{i+} \setminus C_{i-})$. Note that $\mu(C) = 0$. We define a function h as follows:

$$h(\omega) = \begin{cases} x & \text{if } \omega \in C, \\ \overline{h}(\omega) & \text{if } \omega \notin C. \end{cases}$$

The function h is measurable, since

$$h^{-1}(A_i) = \begin{cases} C_{i-} & \text{if } x \notin A_i, \\ C_{i-} \cup C & \text{if } x \in A_i. \end{cases}$$

Therefore $f = F \circ h$ is $\Im$-measurable and $f|_{\Omega \setminus C} = (F \circ F^{-1})\overline{f}|_{\Omega \setminus C} = \overline{f}|_{\Omega \setminus C}$, which completes the proof. $\square$

Lemma 3.3. *If $(\Omega, \Im, \mu)$ is any σ-finite measure space, $(\mathbb{S}, \mathcal{S})$ – any measurable space fulfilling assumption* (A1)*, a correspondence $S : \Omega \multimap \mathbb{S}$ has an $\overline{\Im} \otimes \mathcal{S}$-analytic graph and a function $f : \Omega \times \mathbb{S} \to \overline{\mathbb{R}}$ is such that the inverse* *images of Borel subsets of* $\overline{\mathbb{R}}$ *are* $\overline{\Im} \otimes \mathcal{S}$*-analytic, then the function* $H : \Omega \to \overline{\mathbb{R}}$ *defined by* $H(\omega) = \max_{d \in S_\omega} f(\omega, d)$ *is* $\overline{\Im}$*-measurable.*

Proof. Let us note that $H(\omega) = \max_{x \in F^{-1}(S_\omega)} f(\omega, F(x))$. It is enough to show that sets A_a being inverse images of intervals $[a, \infty)$ are $\Im$-measurable for every $a \in \overline{\mathbb{R}}$. We have

$$\begin{aligned} A_a &= \mathrm{Proj}_\Omega\{(\omega, x, t) : t = f(\omega, F(x)),\ t \geq a,\ x \in F^{-1}(S_\omega)\} \\ &= \mathrm{Proj}_\Omega(\{(\omega, x, t) : t = f(\omega, F(x))\} \\ &\quad \cap \{(\omega, x, t) : t \geq a\} \cap \{(\omega, x, t) : x \in F^{-1}(S_\omega)\}). \end{aligned}$$

We have $\mathrm{Gr}(F^{-1} \circ S) = (\mathrm{Id}_\Omega, F)^{-1}(\mathrm{Gr}(S))$, therefore $\mathrm{Gr}(F^{-1} \circ S)$ is $\overline{\Im} \otimes \mathcal{Z}$-analytic.

By assumptions about $(\mathbb{S}, \mathcal{S})$ and f, the inverse images of measurable sets by the function $((f \circ (\mathrm{Id}_\Omega, F)), \mathrm{Id}_{\overline{\mathbb{R}}})$ are $\overline{\Im} \otimes \mathcal{Z} \otimes \mathcal{B}(\overline{\mathbb{R}})$-analytic. Moreover, the diagonal $\mathrm{diag}(\overline{\mathbb{R}})$ is measurable, therefore

$$\mathrm{Gr}(f \circ (\mathrm{Id}_\Omega, F)) = ((f \circ (\mathrm{Id}_\Omega, F)), \mathrm{Id}_{\overline{\mathbb{R}}})^{-1}(\mathrm{diag}(\overline{\mathbb{R}}))$$

is $\overline{\Im} \otimes \mathcal{Z} \otimes \mathcal{B}(\overline{\mathbb{R}})$-analytic. The sets $\{(\omega, x, t) : t = f(\omega, F(x))\}$, $\{(\omega, x, t) : t \geq a\}$ and $\{(\omega, x, t) : x \in F^{-1}(S_\omega)\}$ are $\overline{\Im} \otimes \mathcal{Z} \otimes \mathcal{B}(\overline{\mathbb{R}})$-analytic, therefore their intersection is $\overline{\Im} \otimes \mathcal{Z} \otimes \mathcal{B}(\overline{\mathbb{R}})$-analytic, too. This implies that it is $\overline{\Im} \otimes \mathcal{W} \otimes \mathcal{B}(\overline{\mathbb{R}})$-analytic.

Since the σ-field $\mathcal{W}$ is generated by a compact family of sets, which implies the same for $\mathcal{B}(\overline{\mathbb{R}}) \otimes \mathcal{W}$, by the projection theorem of Marczewski and Ryll–Nardzewski ([15]), the sets A_a are $\overline{\Im}$-analytic. By a theorem of Saks (Theorem 5.5, p. 50 in [19]), analytic sets are universally measurable (i.e. measurable with respect to the completion of every measure on $\Im$), therefore they belong to $\overline{\Im}$, which completes the proof of $\Im$-measurability of H. □

3.2. Proof of the main result. We introduce some notation and definitions:

$$B_\omega(u) = \mathrm{Argmax}_{d \in S_\omega} P_\omega(d, u)$$

(B_ω is called the *best response correspondence of player* ω) and

$$\overline{B}(u) = \left[\int_\Omega g_i(\omega, B_\omega(u)) d\overline{\mu}(\omega)\right]_{i=1}^r$$

(*statistic of the best response*).

Proposition 3.4. *If* $(\Omega, \Im, \mu)$ *is any measure space and* (A1') *is fulfilled, then the existence of a pure equilibrium is equivalent to the existence of a fixed point of the statistic of the best response correspondence* $\overline{B}$.

PROOF. Let $u \in \overline{B}$. By the definition of Aumann's integral, there exists an $\overline{\Im}$-measurable function $\overline{f} : \Omega \to \mathbb{S}$ such that $\left[\int_\Omega g_i(\omega, \overline{f}(\omega))\, d\overline{\mu}(\omega)\right]_{i=1}^r = u$ and for almost every ω, $\overline{f}(\omega) \in B_\omega(u)$.

By Proposition 3.2, there exists an $\Im$-measurable function f such that

$$\left[\int\limits_\Omega g_i(\omega, f(\omega)) d\mu(\omega)\right]_{i=1}^r = u$$

and for a.e. ω $f(\omega) \in B_\omega(u)$. By definition of B, the profile f is a Nash equilibrium. □

Before the proof of Theorem 3.1, we shall formulate a sequence of necessary lemmata.

LEMMA 3.5. *If* $(\Omega, \Im, \mu)$ *is a measure space with nonatomic, finite measure and assumptions* (A2) *and* (A7) *are fulfilled, then* $\mathbb{U}$ *is convex and compact.*

PROOF. Since μ is nonatomic, the integral of every correspondence, in particular of S, is convex (see e.g. Richter [18]).

The values of S are closed and g_i are integrably bounded, so $\mathbb{U}$ is compact (by a known theorem, see Aumann [1], Theorem 4 or Hildenbrand [8, Proposition 7, p. 73]). □

LEMMA 3.6. *If* $\mathbb{S}$ *is a Hausdorff topological space, assumption* (A2) *is fulfilled, for a.e.* ω, $g(\omega, \cdot)$ *is continuous on* S_ω *and for a.e.* ω *and every* u, $P_\omega(\cdot, u)$ *is upper semicontinuous on* S_ω, *then for a.e.* ω *the values of* $g(\omega, B_\omega(\cdot))$ *are nonempty and compact.*

PROOF. Let us take any $u \in \mathbb{U}$ and any ω for which the required properties of P_ω and S_ω hold.

Since S_ω is compact and $P_\omega(\cdot, u)$ is upper semicontinuous on S_ω, the supremum $M := \sup_{d \in S_\omega} P_\omega(d, u)$ is attained, therefore $g(\omega, B_\omega(\cdot))$ is nonempty.

The function $P_\omega(\cdot, u)$ is upper semicontinuous on S_ω, which is compact, so for every $r \in \mathbb{R} \cup \{-\infty\}$, the set $\{d \in S_\omega : P_\omega(\cdot, u) \geq r\}$ is compact; so is the set

$$B_\omega(u) = \{d \in S_\omega : P_\omega(\cdot, u) \geq M\}.$$

The function g is continuous in the latter variable, therefore the set $g(\omega, B_\omega(u))$ is compact. □

LEMMA 3.7. *If* $(\Omega, \Im, \mu)$ *is any* σ*-finite measure space, assumptions* (A1)), (A4), (A6) and the functions g_i are $\overline{\Im} \otimes \mathcal{S}$-measurable are fulfilled, then for every u, the correspondence $g \circ (\mathrm{Id}_\Omega \times B.(u))$ has an $\overline{\Im} \otimes \mathcal{B}(\mathbb{R}^r)$-analytic graph.

Proof. Let us take an arbitrary u. The graph of $g\circ(\mathrm{Id}_\Omega\times B.(u))$ is described by the equation

$$\mathrm{Gr}(g\circ(\mathrm{Id}_\Omega\times B.(u)))=\{(\omega,y)\in\Omega\times\mathbb{R}^r:\ \exists\overline{d}\in S_\omega \text{ such that } P_\omega(\overline{d},u)=\max_{d\in S_\omega}P_\omega(d,u)\text{ and } g(\omega,\overline{d})=y\}.$$

We can put it another way:

$$\mathrm{Gr}(g\circ(\mathrm{Id}_\Omega\times B.(u)))=\{(\omega,y)\in\Omega\times\mathbb{R}^r:\ \exists\overline{x}\in F^{-1}(S_\omega)$$

such that $P_\omega(F(\overline{x}),u)=\max_{x\in F^{-1}(S_\omega)}P_\omega(F(x),u)$ and $g(\omega,F(\overline{x}))=y\}$.

By Lemma 3.3, the function H defined by $H(\omega)=\max_{d\in S_\omega}P_\omega(d,u)$, is $\overline{\Im}$-measurable.

We have

$$\begin{aligned}
&\mathrm{Gr}(g\circ(\mathrm{Id}_\Omega\times B.(u)))\\
&\quad=\mathrm{Proj}_{\Omega\times\mathbb{R}^r}\{(\omega,x,y)\in\Omega\times\mathbb{Z}\times\mathbb{R}^r: P_\omega(F(x),u)-H(\omega)=0,\\
&\qquad x\in F^{-1}(S_\omega),\ y=g(\omega,F(x))\}\\
&\quad=\mathrm{Proj}_{\Omega\times\mathbb{R}^r}(\{(\omega,x,y)\in\Omega\times\mathbb{Z}\times\mathbb{R}^r: P_\omega(F(x),u)-H(\omega)=0\}\\
&\qquad\cap\{(\omega,x,y)\in\Omega\times\mathbb{Z}\times\mathbb{R}^r:\ x\in F^{-1}(S_\omega)\}\\
&\qquad\cap\{(\omega,x,y)\in\Omega\times\mathbb{Z}\times\mathbb{R}^r:\ y=g(\omega,F(x))\})\\
&\quad=\mathrm{Proj}_{\Omega\times\mathbb{R}^r}(G^{-1}(0)\times\mathbb{R}^r\cap\mathrm{Gr}((\mathrm{Id}_\Omega,F)^{-1}(\mathrm{Gr}(S)))\times\mathbb{R}^r\\
&\qquad\cap\mathrm{Gr}(g\circ(\mathrm{Id}_\Omega\times F))),
\end{aligned}$$

where the function G is defined by $G(\omega,x)=P_\omega(F(x),u)-H(\omega)$.

Inverse images of intervals by G are $\overline{\Im}\otimes\mathcal{Z}$-analytic, so the set $G^{-1}(\{0\})$ is $\overline{\Im}\otimes\mathcal{Z}$-analytic. The sets $\mathrm{Gr}((\mathrm{Id}_\Omega,F)^{-1}(\mathrm{Gr}(S)))\times\mathbb{R}^r$ and $\mathrm{Gr}(g\circ(\mathrm{Id}_\Omega\times F))$ are $\overline{\Im}\otimes\mathcal{Z}\otimes\mathcal{B}(\mathbb{R}^r)$-analytic, therefore they are $\overline{\Im}\otimes\mathcal{W}\otimes\mathcal{B}(\mathbb{R}^r)$-analytic. Hence the set $G^{-1}(0)\times\mathbb{R}^r\cap\mathrm{Gr}((\mathrm{Id}_\Omega,F)^{-1}(\mathrm{Gr}(S)))\times\mathbb{R}^r\cap\mathrm{Gr}(g\circ(\mathrm{Id}_\Omega\times F)$ is $\overline{\Im}\otimes\mathcal{W}\otimes\mathcal{B}(\mathbb{R}^r)$-analytic. Since the σ-field $\mathcal{W}$ is generated by a compact family of sets, by the projection Theorem of Marczewski and Ryll–Nardzewski, the projection is $\overline{\Im}\otimes\mathcal{B}(\mathbb{R}^r)$-analytic.

So we have proved that the graph of $g\circ(\mathrm{Id}_\Omega\times B.(u))$ is $\overline{\Im}\otimes\mathcal{B}(\mathbb{R}^r)$-analytic.□

Lemma 3.8. *If $\mathbb{S}$ is a topological Hausdorff space, assumptions* (A2), (A3) *and* (A5) *are fulfilled, and for a.e. ω, $g(\omega,\cdot)$ is continuous on S_ω then for a.e. ω, the graph of the correspondence $g(\omega,B_\omega(\cdot))$ is compact.*

Proof. We fix some ω for which the assumed properties hold.

At first let us prove that the graph of $B_\omega(\cdot)$ is compact. Since it is a subset of a compact set, it is enough to show closedness.

Let us suppose, contrary to our claim, that for this ω the correspondence B_ω is not closed. Let $(u_n,d_n)\in\mathrm{Gr}(B_\omega)$ and $(u_n,d_n)\to(\overline{u},\overline{d})$.

Let $m = P_\omega(\overline{d}, \overline{u})$, $M = \max_{d \in S_\omega} P_\omega(d, \overline{u}) = P_\omega(\widetilde{d}, \overline{u})$ (such $\widetilde{d}$ exists because $B_\omega(u)$ is nonempty) and $\varepsilon = M - m > 0$.

We first assume that $m > -\infty$. Since P_ω is upper semicontinuous and for d in S_ω, the function $P_\omega(d, \cdot)$ is continuous, for n large enough the following inequalities are simultaneously fulfilled:

$$P_\omega(d_n, u_n) - m < \frac{\varepsilon}{2} \quad \text{and} \quad |M - P_\omega(\widetilde{d}, u_n)| < \frac{\varepsilon}{2}.$$

So $M - \varepsilon/2 < P_\omega(\widetilde{d}, u_n) \leq P_\omega(d_n, u_n) < m + \varepsilon/2$, which is a contradiction.

Now let $m = -\infty$ and $M \in \mathbb{R}$. There exists $\eta \in \mathbb{R}$ such that for n large enough, $P_\omega(\widetilde{d}, u_n) > M - \eta$. Therefore we have $M - \eta < P_\omega(\widetilde{d}, u_n) \leq P_\omega(d_n, u_n) \to -\infty$, which is a contradiction in this case.

The graph of $g(\omega, B_\omega(\cdot))$ is compact, since it is equal to $(\mathrm{Id}_{\mathbb{U}} \times g)(\mathrm{Gr}(B_\omega))$ (an image of a compact set by a continuous map). □

Proof of the Theorem. The set $\mathbb{U}$ is compact and convex by Lemma 3.5. The family of correspondences $\{g \circ (\mathrm{Id}_\Omega \times B.(u))\}_{u \in \mathbb{U}}$ is integrably bounded. The values of $\overline{B}$ are convex (see Richter [18]).

By Lemmata 3.6 and 3.7, for every u and i, the correspondence

$$g_i \circ (\mathrm{Id}_\Omega \times B.(u))$$

is integrably bounded, it has nonempty values a.e. and an $\overline{\Im} \otimes \mathcal{B}(\mathbb{R}^r)$-analytic graph. Therefore by a measurable selection Theorem (Leese [14, Theorem 5.5]) for a correspondence with an analytic graph, there exists an $\overline{\Im}$-measurable a.e. selection from $g_i \circ (\mathrm{Id}_\Omega \times B.(u))$. Therefore the values of $\overline{B}$ are nonempty.

By Lemma 3.6, for every u and i, the correspondence $g \circ (\mathrm{Id}_\Omega \times B.(u))$ is integrably bounded and it has closed values a.e., so the values of $\overline{B}$ are compact (the same reasoning as in the proof of Lemma 3.5).

By Lemma 3.8, the correspondences $g(\omega, B_\omega(\cdot))$ are closed and bounded by the same integrable function, so $\overline{B}$ is a closed correspondence (Aumann [3]).

Therefore $\overline{B}$ is a closed correspondence of a compact, convex set into itself with nonempty, compact, convex values. By the Kakutani theorem, there exists a fixed point $\overline{u} \in \overline{B}(\overline{u})$.

This completes the proof of (a). If case (b) the existence of an equilibrium follows from (a) and Proposition 3.4. □

3.3. Balder's results. As announced, we recall, for comparison, the mentioned result of Balder [6] concerning the existence of pure equilibria in large games rephrased to fit our framework.

Assumptions:

(B1) The strategy space $\mathbb{S}$ is a Souslin metric space.

(B2) The set S_ω is nonempty and compact for every ω.

(B3) The function P_ω is upper semicontinuous on $S_\omega \times \mathbb{Y}$ for every ω.

(B4) The correspondence S has an $\Im \otimes \mathcal{B}(\mathbb{S})$-measurable graph.

(B5) The function $P_\omega(d, \cdot)$ is continuous on $\mathbb{Y}$ for every ω and every $d \in S_\omega$.

(B6) The function $P_\cdot(\cdot, u) : \mathrm{Gr}(S) \to [-\infty, \infty)$ is measurable for every $u \in \mathbb{Y}$.

(B7) The functions g_i are measurable, integrably bounded and such that $g_i(\omega, \cdot)$ is continuous on S_ω for every ω, i.

Theorem 3.9 (Balder [6]). *If* $(\Omega, \Im, \mu)$ *is nonatomic and assumptions* (B1)–(B7) *are fulfilled, then there exists a pure strategy Nash equilibrium profile.*

Let us note that Theorem 3.1 is a generalization of Theorem 3.9, since Polish spaces from topological point of view coincide with G_δ subsets of the Hilbert cube $[0,1]^{\mathbb{N}}$ (see e.g. [13, p. 430]), which is compact, and in the Borel σ-field of a Souslin metric space there exists a countable family of sets separating points (see e.g. [7, p. 81]), therefore the diagonal is in the product σ-field.

Certainly, the assumptions (A1) and (A1') are essentially weaker than (B1) while (A4) and (A6) than, respectively, (B4) and (B6).

The original proof of Balder was based on a Theorem on existence of mixed equilibria (the main result of [6], using properties of Young measures), a simple rule for purification and a Lemma (an adjusted version of a very complicated Lemma III from a paper on optimal control [4]).

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