Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 15, 2000, 235–250

POSITIVE SOLUTIONS OF A HAMMERSTEIN INTEGRAL EQUATION WITH A SINGULAR NONLINEAR TERM

Mario Michele Coclite

Abstract. In this paper the existence of a positive measurable solution of the Hammerstein equation of the first kind with a singular nonlinear term at the origin is presented.

0. Introduction

The literature on the Hammerstein equations with the reciprocal of the solution in the integrand is rather limited, although there are many applications. For example, the equation

$$u(x) = \int_0^1 K(x,y)\frac{1}{u(y)}\,dy, \quad 0 \le x \le 1, \tag{1}$$

arises in some mathematical models of the Signal Theory (see [12], [13]). The more general Hammerstein equation on $\Omega \subset \mathbb{R}^N$, $1 \le N$,

$$u(x) = \int_\Omega K(x,y)g(y,u(y))\,dy, \quad x \in \Omega, \tag{2}$$

2000 *Mathematics Subject Classification.* 45E99, 45G10, 45L99.

Key words and phrases. Hammerstein integral equations, singular nonlinearity, existence of positive solutions.

Work supported by M.U.R.S.T. Italy (fondi 40%, 60%) and by the G.N.A.F.A. of C.N.R.

where $K(x,y)$ is of potential type and $g(y,s)$ does not converge as $s \to 0^+$ is usefull to establish the existence of the solutions of semilinear homogeneous boundary problems (see [11]) with a nonlinear term depending on the reciprocal of the solution (see [1]–[3], [5], [6]).

Nowosad in [13] proved the existence of one continuous, positive solution of (1) assuming $K(x,y)$ continuous, non-negative and symmetric. Later Karlin and Nirenberg, in [10], improved this result considering $K(x,y)$ continuous, non-negative and $K(x,x) > 0$, $0 \le x \le 1$. They proved the existence of one continuous, positive eigenfunction for the eigenvalue problem related to (2), where $\Omega = [0,1]$ and $g(y,s)$ is continuous and positive in $[0,1] \times]0,\infty[$, bounded as $s \to \infty$ and $0 < c_0 \le g(y,s) \le c_1 s^{-\alpha}$, $0 \le y \le 1$, $0 < s < 2$, $\alpha > 0$. In particular, if $g(y,s) = 1/s^{\alpha}$, they proved that 1 will be an eigenvalue and, consequently, (2) will have a positive solution (unique if $0 < \alpha \le 1$). More recently Karapetyants and Yakubov, in [9], have weakened the previous assumptions about the kernel. They consider $K(x,y) = k(x-y)$, with $k \in L^1_{\mathrm{loc}}([0,\infty[)$ being non-negative and non-increasing, and have found that the convolution equation $u^{-\alpha} = k * u$, $\alpha > 1$ has a unique solution $u \in C(]0,\infty[)$ almost increasing and positive. In [4] we have proved that (2) has a non-negative summable solution assuming that $K(x,y)$ is measurable, non-negative and that there exists a finite covering $(E_i)_{1 \le i \le n}$ of Ω, with E_i being a measurable set, and $R > 0$, such that for every measurable set $F \subset \Omega$, whose measure is finite, it results that

$$R \cdot \operatorname{meas}(E_i \cap F) \le \int\limits_{E_i \cap F} K(x,y)\,dx, \quad y \in E_i \text{ a.e.}, \ 1 \le i \le n. \tag{3}$$

Moreover, $g(y,s)$ is a non-negative Carathéodory function in $\Omega \times]0,\infty[$ (i.e. $g(\,\cdot\,,s)$ is measurable in Ω, for all fixed $s > 0$ and $g(y,\,\cdot\,)$ is continuous in $]0,\infty[$, for almost fixed $y \in \Omega$) bounded with respect to s as $s \to \infty$. There is no hypothesis about the behaviour of $g(y,s)$ when $s \to 0^+$, and the following possibility is not excluded:

$$\varliminf_{s \to 0^+} g(y,s) = 0, \qquad \varlimsup_{s \to 0^+} g(y,s) = \infty.$$

In this work we advance some steps forward. The (3) is still satisfied but is considered a countable covering of Ω and $R_i > 0$ instead of R. We hypothesize nothing regarding $\inf_i R_i$, it can be equal to 0. This assumption also permits us to consider kernels which are not strictly positive over the diagonal set of $\overline{\Omega} \times \overline{\Omega}$. We consider only kernels whose support is a subset of a neighbourhood of the diagonal of $\Omega \times \Omega$. We prove that there exists a measurable non-negative function u_0 that *ignors the behaviour of* $g(y,s)$ *when* $s \to 0^+$, in the sense that it satisfies the following equality

(4)
$$u_0(x) = \int_{0<u_0} K(x,y)g(y,u_0(y))\,dy,$$

where the symbol $0 < u_0$ denotes the set $\{x \in \Omega \mid 0 < u_0(x)\}$, and then it verifies the following alternative

$$u_0 = 0 \text{ a.e. in } \Omega \text{ or } u_0 > 0 \text{ a.e. in } \Omega,$$

(see Theorem 1).

Equation (4) always has the trivial solution, however, if there exists $\varphi \in L^1_{\mathrm{loc}}(\Omega)$, $\varphi > 0$ a.e. in Ω, such that

$$\lim_{s\to 0} \frac{g(y,s)}{\varphi(y)s} = \infty,$$

uniformly with respect to y in every E_i, there exists $u_0 > 0$, a.e. in Ω, solution of (4) and then of (2) (see Theorem 2). The result of Karlin and Nirenbereg is a particular case of this result (see [10]).

To conclude, we observe that, since $g(y,s)$ may not be regular when $s \to 0^+$, as in [1], the proof of Theorem 1 should begin with the solutions of

$(2)_\varepsilon$
$$u(x) = \int_\Omega K(x,y)g(y,\varepsilon + u(y))\,dy, \quad x \in \Omega,\ \varepsilon > 0,$$

(see Appendix 2). Any family of these approximate solutions will have a subsequence converging to one solution of (4) in all spaces $L^1(\bigcup_{i=0}^n E_i)$, $n \in \mathbb{N}$.

In this paper, as in [4], [5], [9], [10], [12]–[15], the positivity of the solutions of (2) depends on that of $K(x,y)$ and $g(y,s)$. For other information on the positive solutions, the reader is refered to the monographs [7], [11].

The paper is organized as follows. In Section 1 we present the assumptions and the results obtained. In Sections 2, 3 and 4 we demonstrate the Theorems. Sections 5 and 6 are dedicated to Appendices 1 and 2.

1. Assumptions and results

For abbreviation, we write $\mathbb{R}_+$ and $\mathbb{R}_+^*$ instead of $[0,\infty[$ and $]0,\infty[$, respectively.

$(\mathcal{G})$: Let us suppose that $g : \Omega \times \mathbb{R}_+^* \to \mathbb{R}$ satisfies the following hypotheses:

(a) g is a Carathéodory function and $g \geq 0$ a.e. in $\Omega \times \mathbb{R}_+^*$.

(b) $y \mapsto \sup_{s\leq t} g(y,t)$ is summable over Ω, for all $s > 0$.

Consequently the function

$$g^*(y,s) := \sup_{s\leq t} g(y,t), \quad (y,s) \in \Omega \times \mathbb{R}_+^*,$$

verifies the Carathéodory condition, is decreasing with respect to s, is summable on Ω, for all s, and $g \le g^*$.

($\mathcal{K}$) Let $K : \Omega \times \Omega \to \mathbb{R}$ satisfy the following hypotheses:

(a) $K(x,y)$ is measurable and $K(x,y) \ge 0$, $(x,y) \in \Omega \times \Omega$ a.e.

(b) $\int_\Omega K(\,\cdot\,,y)g^*(y,s)\,dy$ is summable over Ω, for all $s > 0$.

(c) There exists $(R_i)_{i\in\mathbb{N}}$, with $R_i > 0$, and a covering $(E_i)_{i\in\mathbb{N}}$ of Ω, with E_i a measurable set, such that for all $F \subset \Omega$ measurable with $\operatorname{meas}(F) < \infty$, it results that

$$R_i \operatorname{meas}(E_i \cap F) \le \int\limits_{E_i\cap F} K(x,y)\,dx, \quad y \in E_i \text{ a.e., } i \in \mathbb{N}.$$

(d) Let $\Omega_n := \bigcup_{i=0}^n E_i$, $n \in \mathbb{N}$, then the map

$$u \mapsto K_n(u) = \int\limits_{\Omega_n} K(\,\cdot\,,y)u(y)\,dy$$

is compact from $L^1(\Omega_n)$ into itself.

(e) There exists $(\Omega'_n)_{n\in\mathbb{N}}$, which is an increasing covering of Ω, such that $\Omega'_n \subset \Omega_n$ and $K(x,y) = 0$, $\Omega'_n \times (\Omega \setminus \Omega_n)$, $n \in \mathbb{N}$.

These hypotheses are satisfied, for example, by the following kernel:

$$K(x,y) = \varphi(x)\psi(y)H(x,y)\chi_E(x,y), \quad x,y \in \mathbb{R}^+,$$

where $\varphi,\psi \in C(\mathbb{R}^+)$, are strictly positive in $\mathbb{R}^*_+$ and $\varphi(0) = \psi(0) = \lim_{x\to\infty}\varphi(x) = \lim_{x\to\infty}\psi(x) = 0$,

$$H(x,y) = \begin{cases} 2 & \text{for } x \le y, \\ 1 & \text{for } y < x, \end{cases}$$

and χ_E is the characteristic function of $E = \bigcup_{i=0}^\infty (E_i \times E_i)$ where $E_{2k} = [k+1, k+3]$, $E_{2k+1} = [1/(2k+1), 4/(2k+1)]$ (see Appendix 1).

Since K_n maps $L^1(\Omega_n)$ into itself, then it is continuous (see [17, Chapter V, Theorem 1.5]). Let $|K_n|_n$ be its norm. General conditions on $K(x,y)$, such that K_n maps $L^1(\Omega_n)$ into itself, are unknown except for some special cases (see for example [8], [16], [17] and the next Lemma 3).

REMARK. For ($\mathcal{K}_\mathrm{b}$), by the Tonelli Theorem, $K(x,y)g^*(y,1)$ is summable in $\Omega\times\Omega$. Then, by Fubini's Theorem, $K(x,y)g^*(y,1)$ is summable with respect to x. Consequently we conclude that $K(\,\cdot\,,y) \in L^1(\Omega)$, $y \in \Omega$ a.e.

We can now formulate our main results:

THEOREM 1. *If* $(\mathcal{K})$ *and* $(\mathcal{G})$ *hold, then:*

(i) *There exists a measurable and non-negative, a.e. in* Ω, *function* u_0 *such that*

$$u_0(x) = \int\limits_{0<u_0} K(x,y)g(y,u_0(y))\,dy. \tag{5}$$

(ii) *If for all* $i \in \mathbb{N}$, *there exists* $(E_{n_k})_{0\le k\le l}$, *such that* $0 = n_0$, $i = n_l$ *and* $\operatorname{meas}(E_{n_k} \cap E_{n_{k+1}}) > 0$, $(0 \le k \le l-1)$, *then either* $u_0 = 0$ *a.e. in* Ω *or* $u_0 > 0$ *a.e. in* Ω. *In the latter case,* u_0 *is a solution of* (1).

THEOREM 2. *Let assumption* $(\mathcal{K})$, $(\mathcal{G})$ *be fulfilled and in addition suppose that* $(\mathcal{G}_{\mathrm{c}})$: *there exists* $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ *where* $\varphi > 0$ *a.e. in* Ω, *such that*

$$\lim_{s\to 0} \frac{g(y,s)}{\varphi(y)s} = \infty,$$

uniformly with respect to y *in every* E_i, *then* (2) *has at least one solution.*

2. Proof of Theorem 1(i)

Let us recall certain notation which we shall frequently use here and subsequently. $B_r := B_r(0)$ is the ball of radius $r > 0$ and center 0. $|\ \ |_1$ is the norm of $L^1(\Omega)$. Let $E_r := E \cap B_{1/r}$, where $E \subset \mathbb{R}^N$, and $|E| := \operatorname{meas}(E)$ if E is a measurable set. Given $a, b \in \mathbb{R}$ and $\omega : \Omega \to \mathbb{R}$ we set:

$$a \le \omega \le b := \{x \in \Omega \mid a \le \omega \le b\}.$$

Analogously, we define $a < \omega \le b$, $a < \omega < b$, $a \le \omega$ etc. In a chain of inequalities and in particular equalities, if a term is different from the previous, we indicate only the variation and substitute the previous term with dots.

Let us prove one lemma on the continuity of an integral nonlinear operator, which depends only on the assumption $(\mathcal{K}_{\mathrm{b}})$.

LEMMA 3. *Let* φ, $(\varphi_k)_{k\in\mathbb{N}}$ *be given in* $L^1(\Omega)$, *if* $\varphi_k \to \varphi$ *a.e. in* Ω, *for all* $\eta_0 > 0$ *the following equalities hold:*

$$\lim_k \int\limits_{\eta\le\varphi_k} K(\,\cdot\,,y)|g(y,\varphi_k(y)) - g(y,\varphi(y))|\,dy = 0, \tag{6}$$

and

$$\lim_k \int\limits_{\eta\le\varphi} K(\,\cdot\,,y)|g(y,\varphi_k(y)) - g(y,\varphi(y))|\,dy = 0, \tag{7}$$

in $L^1(\Omega)$, *uniformly with respect to* η *in* $[\eta_0,\infty[$.

Proof. For short, we set

$$\theta(y) := \int\limits_{\Omega} K(x,y)\,dx, \quad \gamma_k(y) := |g(y,\varphi_k(y)) - g(y,\varphi(y))|,\ y \in \Omega.$$

The absolute continuity of the indefinite integral of $\theta g^*(\,\cdot\,,\eta_0/2)$ in Ω implies that for all $\sigma > 0$, there exists $\delta > 0$, such that for every measurable set $E \subset \Omega$ with $|E| < \delta$, it follows that:

$$\int\limits_{E} \theta g^*(\,\cdot\,,\eta_0/2)\,dy < \sigma, \qquad \int\limits_{\Omega\setminus B_{1/\delta}} \theta g^*(\,\cdot\,,\eta_0/2)\,dy < \sigma. \tag{8}$$

According to the Egorov–Severini Theorem there exists a measurable set $E \subset \Omega_\delta$ with $|E| < \delta$ and, given $\tau \in]0,\eta_0/2[$, there exists $\nu \in \mathbb{N}$ such that

$$\nu \leq k,\ x \in \Omega_\delta \setminus E \Rightarrow \varphi_k(x) - \tau < \varphi(x) < \varphi_k(x) + \tau. \tag{9}$$

Thus, for every $\eta, \rho, \eta_0 < \eta < \rho$, we have

$$\nu \leq k \Rightarrow (\eta \leq \varphi_k < \rho)_\delta \setminus E \subset (\eta - \tau \leq \varphi < \rho + \tau)_\delta \setminus E \subset \left(\frac{\eta_0}{2} \leq \varphi < \rho + \frac{\eta_0}{2}\right)_\delta \setminus E.$$

In $(\eta_0/2 \leq \varphi < \rho + \eta_0/2)_\delta \setminus E$ and then in $(\eta \leq \varphi_k < \rho)_\delta \setminus E$, $k \geq \nu$, it results that $\gamma_k \leq 2g^*(\,\cdot\,,\eta_0/2)$. Now the Lebesgue Dominate Convergence Theorem implies that

$$\lim_k \int\limits_{(\eta\leq\varphi_k<\rho)_\delta\setminus E} \theta\gamma_k\,dy \leq \lim_k \int\limits_{(\eta_0/2\leq\varphi<\rho+\eta_0/2)_\delta\setminus E} \ldots = 0, \tag{10}$$

uniformly with respect to η over $[\eta_0,\infty[$.

From (9) we have

$$\begin{aligned}
\nu \leq k \Rightarrow &(\rho \leq \varphi_k)_\delta \setminus E \subset (\rho - \eta_0/2 \leq \varphi)_\delta \setminus E \\
&\Rightarrow \int\limits_{(\rho\leq\varphi_k)_\delta\setminus E} \theta\gamma_k\,dy \leq \int\limits_{(\rho\leq\varphi_k)_\delta\setminus E} \theta g^*(\,\cdot\,,\varphi_k)\,dy + \int\limits_{(\rho-\eta_0/2\leq\varphi)_\delta\setminus E} \theta g^*(\,\cdot\,,\varphi)\,dy \\
&\leq \int\limits_{(\rho\leq\varphi_k)_\delta\setminus E} \theta g^*(\,\cdot\,,\rho)\,dy + \int\limits_{(\rho-\eta_0/2\leq\varphi)_\delta\setminus E} \theta g^*(\,\cdot\,,\rho-\eta_0/2)\,dy \\
&\leq 2 \int\limits_{(\rho-\eta_0/2\leq\varphi)_\delta\setminus E} \theta g^*(\,\cdot\,,\rho-\eta_0/2)\,dy.
\end{aligned}$$

Since $\rho \mapsto (\rho - \eta_0/2 \leq \varphi)$ is decreasing, it follows that $\lim_{\rho\to\infty}(\rho - \eta_0/2 \leq \varphi) = (\infty = \varphi)$, and the summability of φ implies that $|\infty = \varphi| = 0$. Then:

$$\lim_{\rho\to\infty} \int\limits_{(\rho\leq\varphi_k)_\delta\setminus E} \theta\gamma_k\,dy = 0,$$

uniformly with respect to $k \geq \nu$. Hence from (10) we obtain

$$\lim_{k} \int_{(\eta \leq \varphi_k)_\delta \setminus E} \theta \gamma_k \, dy = 0, \tag{11}$$

uniformly with respect to η over $[\eta_0, \infty[$. Finally, by (8), we obtain:

$$\begin{aligned} \int_{\eta \leq \varphi_k} \theta \gamma_k \, dy &\leq \int_{(\eta \leq \varphi_k)_\delta \setminus E} \ldots + \int_{E \cap (\eta \leq \varphi_k)_\delta} \ldots + \int_{(\eta \leq \varphi_k) \setminus B_{1/\delta}} \ldots \\ &\leq \int_{(\eta \leq \varphi_k)_\delta \setminus E} \ldots + 2 \int_{E \cap (\eta \leq \varphi_k)_\delta} \theta g^* \left(\cdot \,, \frac{\eta_0}{2} \right) dy + 2 \int_{(\eta \leq \varphi_k) \setminus B_{1/\delta}} \theta g^* \left(\cdot \,, \frac{\eta_0}{2} \right) dy \\ &\leq \int_{(\eta \leq \varphi_k)_\delta \setminus E} \ldots + 4\sigma. \end{aligned}$$

This together with (11) implies (6).

The same reasoning yields

$$\int_{(\eta \leq \varphi)} \theta \gamma_k \, dy \leq \int_{(\eta \leq \varphi)_\delta \setminus E} \ldots + 4\sigma.$$

By (9) we have

$$\int_{(\eta \leq \varphi)} \theta \gamma_k \, dy \leq \int_{(\eta - \eta_0/2 \leq \varphi_k)_\delta \setminus E} \ldots + 4\sigma.$$

Thus, (7) follows from (11). □

As stated in the Introduction, we have considered the solutions of the approximate equations $(2)_\varepsilon$ before studying (2). We remark that the integral which compares in $(2)_\varepsilon$ is finite because the assumption $(\mathcal{K}_\mathrm{b})$ holds and the integrand is positive.

Since $K(x,y)$ satisfies $(\mathcal{K}_\mathrm{a})$, $(\mathcal{K}_\mathrm{b})$ and $g(y,s)$ satisfies $(\mathcal{G})$, there exists $u_\varepsilon \in L^1(\Omega)$ positive a.e. in Ω which is a solution of $(2)_\varepsilon$. For completeness, the proof of the existence of these solutions is sketched in Appendix 2, (see [4, Theorem 5]). The proof of Theorem 1_i consists in a suitable analysis of u_ε as $\varepsilon \to 0^+$.

LEMMA 4. *There exists* $(\varepsilon_k)_{k \in \mathbb{N}}$, $\varepsilon_k \to 0$, *such that*

$$\left(\int_{\Omega_n} K(\cdot\,, y) g(y, \varepsilon_k + u_{\varepsilon_k}) \, dy \right)_{k \in \mathbb{N}}$$

converges in $L^1(\Omega_n)$, *for all* $n \in \mathbb{N}$.

PROOF. For abbreviation, we set $g_\varepsilon := g(\cdot\,, \varepsilon + u_\varepsilon)$. Given $n \in \mathbb{N}$, let I be the set of $i = 0, \ldots, n$, such that $T_i := E_i \cap (u_\varepsilon < 1)$ has no zero measure. Let

$i \in I$, $r > 0$, such that $|T_i \cap B_r| > 0$, since

$$\int_{T_i \cap B_r} u_\varepsilon(x)\,dx = \int_\Omega g_\varepsilon(y)\,dy \int_{T_i\cap B_r} K(x,y)\,dx$$
$$\geq \int_{E_i} \dots \int_{T_i\cap B_r} \dots \geq R_i |T_i \cap B_r| \int_{E_i} g_\varepsilon(y)\,dy,$$

then we obtain

$$\int_{E_i} g_\varepsilon(y)\,dy \leq \frac{1}{R_i}, \quad i \in I.$$

Successively, if we consider $i \notin I$, since $1 \leq u_\varepsilon$ a.e. in E_i, it follows that

$$\int_{E_i} g_\varepsilon(y)\,dy \leq \int_{E_i} g^*(y,1)\,dy \leq |g^*(\,\cdot\,,1)|_1, \quad i \notin I.$$

Seeing that $(E_i)_{0\leq i\leq n}$ is a covering of Ω_n, we conclude that $(g_\varepsilon)_{0<\varepsilon}$ is bounded in $L^1(\Omega_n)$. According to the compactness of K_0 from $L^1(\Omega_0)$ into itself (see $(\mathcal{K}_\mathrm{d})$), there exists $(\varepsilon_k^0)_{k\in\mathbb{N}}$, $\varepsilon_k^0 \to 0$, such that

$$(K_0(g_{\varepsilon_k^0}))_{k\in\mathbb{N}}$$

converges in $L^1(\Omega_0)$.

We now proceed by induction. We can say that there exists $(\varepsilon_k^n)_{k\in\mathbb{N}}$, $\varepsilon_k^n \to 0$, which is a subsequence of $(\varepsilon_k^i)_{k\in\mathbb{N}}$, where $\varepsilon_k^i \to 0$, $0 \leq i \leq n-1$, such that

$$(K_0(g_{\varepsilon_k^n}))_{k\in\mathbb{N}}, \dots, (K_n(g_{\varepsilon_k^n}))_{k\in\mathbb{N}}$$

converge in $L^1(\Omega_0), \dots, L^1(\Omega_n)$, respectively. Now, if we consider the diagonal subsequence $(\varepsilon_k^k)_{k\in\mathbb{N}}$, we conclude that

$$(K_n(g_{\varepsilon_k^k}))_{k\in\mathbb{N}}$$

converges in $L^1(\Omega_n)$, for all $n \in \mathbb{N}$. □

Let

$$u_k := u_{\varepsilon_k}, \quad g_k := g(\,\cdot\,, \varepsilon_k + u_{\varepsilon_k}),$$
$$u'_{k,n} := \int_{\Omega_n} K(\,\cdot\,,y) g_k(y)\,dy, \quad u''_{k,n} := u_k - u'_{k,n},$$

and set

$$v_n(x) := \begin{cases} \lim\limits_k \displaystyle\int_{\Omega_n} K(x,y) g_k(y)\,dy & \text{for } x \in \Omega_n, \\ 0 & \text{for } x \in \Omega \setminus \Omega_n. \end{cases}$$

From the above statement, it follows that $v_n \in L^1(\Omega)$ and that $(v_n)_{n\in\mathbb{N}}$ is increasing. Then, there exists u_0 measurable in Ω, $0 \leq u_0$ a.e. in Ω, such that

$$u_0 = \lim_n v_n = \sup_n v_n, \quad \text{a.e. in } \Omega.$$

Lemma 5. $(u_k)_{k\in\mathbb{N}}$ *converges a.e. in* Ω *to* u_0. *In particular, for every* $n \in \mathbb{N}$, *it results that*

$$\lim_k u_k = v_n = u_0, \quad \text{a.e. in } \Omega'_n. \tag{12}$$

Moreover, it results that

$$\int\limits_{0<u_0} K(x,y)g(y,u_0)\,dy \le u_0(x). \tag{13}$$

Proof. By $(\mathcal{K}_\mathrm{e})$, it follows that, for every n, $K(x,y) = 0$ in $\Omega'_n \times (\Omega \setminus \Omega_n)$. Thus,

$$u''_{k,n}(x) = \int\limits_{\Omega\setminus\Omega_n} K(x,y)g_k(y)\,dy = 0, \quad \text{a.e. in } \Omega'_n.$$

Then, $u_k = u'_{k,n}$ a.e. in Ω'_n, and

$$\lim_k u_k = v_n \quad \text{a.e. in } \Omega'_n.$$

Since $\lim_k u_k$ does not depend on $n \in \mathbb{N}$, (12) holds and consequently $u_k \to u_0$ a.e. in Ω.

Having observed that $\lim_k g_k = g(\,\cdot\,, u_0)$ a.e. in $(0 < u_0)$, and by the Fatou Lemma and the definition of u_k we get (13). □

Lemma 6. *For all* $n \in \mathbb{N}$ *and* $i \in \{0, \dots, n\}$,

$$\operatorname*{essinf}_{E_i} v_n = 0 \;\Rightarrow\; \lim_k \int\limits_{E_i} g_k(y)\,dy = 0.$$

Proof. If $\operatorname{essinf}_{E_i} v_n = 0$, there exists $(X_l)_{l\in\mathbb{N}}$ decreasing, with measurable set $X_l \subset E_i$, $0 < |X_l| < \infty$, such that $v_n < 1/(l+1)$ a.e. in X_l. Then from the definition of $u'_{k,n}$,

$$\begin{aligned} |u'_{k,n}|_{L^1(X_l)} &= \int\limits_{X_l} dx \int\limits_{\Omega_n} K(x,y)g_k(y)\,dy \\ &\ge \int\limits_{E_i} g_k(y)\,dy \int\limits_{X_l} K(x,y)\,dx \ge R_i|X_l| \int\limits_{E_i} g_k(y)\,dy. \end{aligned}$$

Consequently,

$$R_i|X_l|\,\overline{\lim_k} \int\limits_{E_i} g_k(y)\,dy \le |v_n|_{L^1(X_l)} \le \frac{|X_l|}{l+1},$$

from which

$$\overline{\lim_k} \int\limits_{E_i} g_k(y)\,dy \le \frac{1}{R_i(l+1)}.$$

Since this last estimate holds for all $l \in \mathbb{N}$ the statement holds. □

Let I_n be the set of the $i \in \{0, \dots, n\}$, such that $\operatorname{essinf}_{E_i} v_n = 0$. We set

$$N_n := \bigcup_{i \in I_n} E_i; \quad \Omega_n^* := \Omega_n \setminus N_n.$$

Corollary 7. *For every $n \in \mathbb{N}$, the following assertions are valid:*

$$\lim_k \int_{N_n} g_k(y)\, dy = 0, \tag{14}$$

$$\eta_n = \operatorname*{essinf}_{\Omega_n^*} v_n > 0, \tag{15}$$

$$v_n := \begin{cases} \lim_k \displaystyle\int_{\Omega_n^*} K(\,\cdot\,, y) g_k(y)\, dy & \text{in } L^1(\Omega_n), \\ 0 & \text{in } L^1(\Omega \setminus \Omega_n). \end{cases} \tag{16}$$

Proof of Theorem 1(i). For (2), (14), (15), and (6) of Lemma 3, for each $n \in \mathbb{N}$, since $u_k \to v_n = u_0$ a.e. in Ω_n' and $u_k \to u_0$ a.e. in Ω, it follows that:

$$\begin{aligned} u_0(x) = v_n(x) = \lim_k \int_{\eta_n \le v_n} K(x,y) g_k(y)\, dy &\le \lim_k \int_{\eta_n \le u_0} K(x,y) g_k(y)\, dy \\ = \int_{\eta_n \le u_0} K(x,y) g(y, u_0(y))\, dy &\le \int_{0 < u_0} K(x,y) g(y, u_0(y))\, dy, \end{aligned}$$

for $x \in \Omega_n'$. Since $(\Omega_n')_{n \in \mathbb{N}}$ is increasing, we obtain:

$$u_0(x) \le \int_{0 < u_0} K(x,y) g(y, u_0(y))\, dy, \quad x \in \Omega \text{ a.e.}$$

Combining this estimate and (3), we obtain (5). □

3. Proof of Theorem 1(ii)

According to $(\mathcal{K}_c)$, we deduce:

Lemma 8. *$K(x,y) \neq 0$ a.e. in $\bigcup_{i=0}^\infty (E_i \times E_i)$.*

Proof. On the contrary, if we assume that the above statement is not true, there exists i and $X_i \times Y_i \subset E_i \times E_i$, $|X_i \times Y_i| > 0$, such that

$$K(x,y) = 0, \quad (x,y) \in X_i \times Y_i \text{ a.e.}$$

By $(\mathcal{K}_c)$ we obtain

$$R_i |E_i \cap X_i| \le \int_{E_i \cap X_i} K(x,y)\, dx, \quad y \in E_i \text{ a.e.}$$

Since $|E_i \cap X_i| = |X_i| > 0$ and $Y_i \subset E_i$, $|Y_i| > 0$, it follows that $R_i |E_i \cap X_i| = 0$. Again, given $(\mathcal{K}_c)$, this is a contradiction. □

PROOF OF THEOREM 1(ii). First we will observe that, given $i \in \mathbb{N}$, by (5) and $(\mathcal{K}_e)$ it follows that:

$$u_0(x) = \int_{(0<u_0)\cap\Omega_n} K(x,y)g(y,u_0(y))\,dy, \quad x \in E_i \cap \Omega'_n,\ n \geq i. \tag{17}$$

By virtue of the previous lemma, it results

$$u_0|_{E_i} = 0, \text{ or } u_0|_{E_i} > 0, \quad \text{a.e. in } E_i,\ i \in \mathbb{N}. \tag{18}$$

In fact, if $0 < |E_i \cap (u_0 = 0)| < |E_i|$, as $(\Omega'_n)_{n\in\mathbb{N}}$ is an increasing covering of Ω, there exists $\nu \in \mathbb{N}$, such that $0 < |E_i \cap \Omega'_n \cap (u_0 = 0)|$, $n \geq \nu$. From (17), we have

$$K(x,y)g(y,u_0(y)) = 0, \quad x \in E_i \cap \Omega'_n \cap (u_0 = 0), \quad y \in \Omega_n \cap (u_0 > 0).$$

Proceeding from the previous lemma, it follows that $g(\,\cdot\,, u_0) = 0$ in $\Omega_n \cap (u_0 > 0)$. Consequently, from (17), we have $0 < |E_i \cap \Omega'_n \cap (u_0 = 0)| = |E_i \cap \Omega'_n|$. If $n \to \infty$, we deduce that $0 < |E_i \cap (u_0 = 0)| = |E_i|$. This contradicts our assumption, therefore (18) is true.

If $u_0 = 0$ a.e. in E_0, given $i \in \mathbb{N}$, let $(E_{n_k})_{0\leq k\leq l}$ satisfy the hypothesis. Since $|E_0 \cap E_{n_1}| > 0$ and $u_0 = 0$ in $E_0 \cap E_{n_1}$, for (18) it results that $u_0 = 0$ in E_{n_1}. By finite induction we obtain $u_0 = 0$ in E_i. The arbitrariness of i permits us to conclude that $u_0 = 0$ a.e. in Ω.

If $u_0 > 0$ a.e. in E_0, repeating the above reasoning we obtain that $u_0 > 0$ a.e. in Ω. □

4. Proof of Theorem 2

We begin with a lemma:

LEMMA 9. *For all $i \in \mathbb{N}$ and $A \subset E_i$ compact with $|A| > 0$, it results that*

$$0 < \operatorname*{essinf}_{x\in A} \int_A K(x,y)\varphi(y)\,dy.$$

PROOF. On the contrary, we assume that the thesis does not hold, since there exists $i \in \mathbb{N}$ and $A \subset E_i$ measurable with $|A| > 0$, such that, for all $\varepsilon > 0$, there is a set $X \subset A$ measurable with $0 < |X| < \infty$, therefore, we have

$$\int_A K(x,y)\varphi(y)\,dy < \varepsilon, \quad x \in X \text{ a.e.}$$

Integrating on X, according to $(\mathcal{K}_c)$, we obtain

$$\varepsilon|X| \geq \int_X dx \int_A K(x,y)\varphi(y)\,dy = \int_A \varphi(y)\,dy \int_X K(x,y)\,dx \geq R_i|X| \int_A \varphi(y)\,dy.$$

Consequently,

$$R_i \int_A \varphi(y)\,dy < \varepsilon.$$

Thus, we obtain $\int_A \varphi(y)\,dy = 0$ and $\varphi = 0$ a.e. in A. This conclusion is not true, therefore the thesis is proved. □

Proof of Theorem 2. Let us consider the solution used to prove Theorem 1(i). On the contrary, if $|u_0 = 0| > 0$, setting $N = (u_0 = 0)$, from Lemma 5 we obtain

$$\lim_k u_k = 0, \quad \text{a.e. in } N.$$

Let $i \in \mathbb{N}$, such that $|E_i \cap N| > 0$. By the Egorov–Severini Theorem, there exists a compact $A \subset E_i \cap N$ with $|A| > 0$, such that $u_k \to 0$ uniformly in A. On the other hand, setting

$$b = \operatorname*{essinf}_{x \in A} \int_A K(x,y)\varphi(y)\,dy,$$

to the previous lemma it follows that $b > 0$. For $(\mathcal{G}_c)$, there exists $s_0 > 0$ such that

$$0 < s < s_0 \Rightarrow \frac{2}{b}\varphi(y)s < g(y,s), \quad y \in E_i \text{ a.e.}$$

Let $k_0 \in \mathbb{N}$ such that:

$$k_0 \leq k \Rightarrow \varepsilon_k + u_k(y) < s_0, \quad y \in A \text{ a.e.}$$

Consequently,

$$\begin{aligned}
k_0 \leq k \Rightarrow{}& \frac{2}{b}\varphi(y)(\varepsilon_k + u_k(y)) \leq g(y, \varepsilon_k + u_k(y)) = g_k(y), \quad y \in A \text{ a.e.} \\
&\Rightarrow u_k(x) \geq \int_A K(x,y) g_k(y)\,dy \geq \frac{2}{b}\int_A K(x,y)\varphi(y)(\varepsilon_k + u_k(y))\,dy \\
&\geq \frac{2}{b}\operatorname*{essinf}_A(\varepsilon_k + u_k)\int_A K(x,y)\varphi(y)\,dy \geq 2\operatorname*{essinf}_A(\varepsilon_k + u_k).
\end{aligned}$$

This is not true, so Theorem 2 is proved. □

5. Appendix 1

The kernel mentioned in Section 1 evidently satisfies the hypotheses $(\mathcal{K}_a)$, $(\mathcal{K}_b)$ and $(\mathcal{K}_c)$ with

$$R_i := (\inf_{E_i} \varphi)(\inf_{E_i} \psi), \quad i \in \mathbb{N}.$$

We proceed to show that also $(\mathcal{K}_{\mathrm{d}})$ is satisfied.

We begin by proving that for all pairs of bounded intervals $I, J \subset \mathbb{R}_+^*$, the operator:

$$\omega \mapsto \int_I \varphi(\,\cdot\,)\psi(y)H(\,\cdot\,,y)\omega(y)\,dy, \tag{19}$$

is compact by $L^1(I)$ into $L^1(J)$.

Let $\mathcal{E}$ be a bounded non-empty subset of $L^1(I)$, we observe that:

(i) $$\left|\int_I \varphi\psi(y)H(\,\cdot\,,y)\omega(y)\,dy\right|_{L^1(J)} \le 2\sup_I \psi|\varphi|_{L^1(J)} \sup_{\omega\in\mathcal{E}}|\omega|_{L^1(I)}$$

and

(ii) $$\begin{aligned}\delta(h) &:= \int_J dx\left|\int_I \varphi(x+h)\psi(y)H(x+h,y)\omega(y)\,dy\right.\\ &\qquad\left.-\int_I \varphi(x)\psi(y)H(x,y)\omega(y)\,dy\right|\\ &\le \int_J |\varphi(x+h)-\varphi(x)|\,dx\int_I \psi(y)H(x+h,y)|\omega(y)|\,dy\\ &\qquad+\int_J \varphi(x)\,dx\left|\int_I \psi(y)(H(x+h,y)-H(x,y))\omega(y)\,dy\right|\\ &\le 2\sup_I\psi\sup_{\omega\in\mathcal{E}}|\omega|_{L^1(I)}\int_J|\varphi(x+h)-\varphi(x)|\,dx\\ &\qquad+\sup_I\psi\sup_J\varphi\int_I|\omega(y)|\,dy\int_J|H(x+h,y)-H(x,y)|\,dx.\end{aligned}$$

Since

$$H(x+h,y)-H(x,y)=\begin{cases}0 & (y<x \text{ and } y<x+h)\vee(x\le y \text{ and } x+h\le y),\\ 1 & x+h\le y<x,\ h<0,\\ -1 & x\le y<x+h,\ h>0,\end{cases}$$

it follows that

$$\begin{aligned}\delta(h) &\le 2\sup_I\psi\sup_{\omega\in\mathcal{E}}|\omega|_{L^1(I)}\int_J|\varphi(x+h)-\varphi(x)|\,dx\\ &\qquad+\sup_I\psi\sup_J\varphi\int_I|\omega(y)|\,|J\cap[y\wedge(y-h),y\vee(y-h)]|\,dy\\ &\le \sup_I\psi\sup_{\omega\in\mathcal{E}}|\omega|_{L^1(I)}\left\{2\int_J|\varphi(x+h)-\varphi(x)|\,dx+\sup_J\varphi\cdot|h|\right\}.\end{aligned}$$

According to the Fréchet–Kolmogorov Theorem operator (19) is compact.

Given a sequence $(\omega_k)_{k\in\mathbb{N}}$ bounded in $L^1(\Omega_n)$, using finite induction arguments, we find a subsequence that we still denote with the same symbol, such that for all $i,j=1,\dots,n$, $(\int_{E_i}\varphi(\cdot)\psi(y)H(\cdot,y)\omega_k(y)\,dy)_{k\in\mathbb{N}}$ converges in $L^1(E_j)$. Since

$$\begin{aligned}\sigma(k,h) &:= \left|\int_{\Omega_n} K(\cdot,y)(\omega_k-\omega_h)(y)\,dy\right|_{L^1(\Omega_n)}\\ &\le \sum_{i,j}^{1,\dots,n}\int_{E_j}dx\left|\int_{E_i}\varphi(x)\psi(y)H(x,y)(\omega_k-\omega_h)(y)\,dy\right|,\end{aligned}$$

then $\lim_{k,h}\sigma(k,h)=0$.

Finally, we may conclude that

$$\omega\mapsto\int_{\Omega_n}K(\cdot,y)\omega(y)\,dy,$$

is compact from $L^1(\Omega_n)$ into itself. Then the kernel satisfies $(\mathcal{K}_{\mathrm{d}})$.

As for $(\mathcal{K}_{\mathrm{e}})$, first we will observe that

$$\Omega_n=\begin{cases}[1,3] & n=0,\\ [1,4] & n=1,\\ \left[\dfrac{1}{n-1},\dfrac{n+6}{2}\right] & n\ge 2,\ n \text{ even},\\ \left[\dfrac{1}{n},\dfrac{n+5}{2}\right] & n\ge 3,\ n \text{ odd}.\end{cases}$$

Setting

$$\Omega'_n=\begin{cases}\emptyset & n=0,\\ \left[\dfrac{4}{n+1},\dfrac{n+4}{2}\right] & n \text{ even},\\ \left[\dfrac{4}{n+2},\dfrac{n+3}{2}\right] & n \text{ odd},\end{cases}$$

we get that $(\Omega'_n)_{n\in\mathbb{N}}$ is an increasing covering of $\mathbb{R}_+^*$ and $\Omega'_n\subset\Omega_n$. Moreover, since

$$(\Omega'_n\times(\Omega\setminus\Omega_n))\cap E=\emptyset,\quad n\ge 1,$$

it follows that $K(x,y)=0$ in every $\Omega'_n\times(\Omega\setminus\Omega_n)$, $n\ge 1$. Thus also $(\mathcal{K}_{\mathrm{e}})$ is satisfied.

6. Appendix 2

We now sketch the proof of the existence of solutions of $(2)_\varepsilon$.

Let $L^1_+(\Omega)$ be the cone of the non-negative (a.e.) functions of $L^1(\Omega)$ and let $\varepsilon > 0$. From the definition of g^*, together with the assumption $(\mathcal{G})$, it follows that $g^*(\,\cdot\,,\varepsilon) \in L^1_+(\Omega)$ and

$$u \in L^1_+(\Omega) \Rightarrow g(\,\cdot\,,\varepsilon + u) \in L^1_+(\Omega) \text{ and } g(\,\cdot\,,\varepsilon + u) \le g^*(\,\cdot\,,\varepsilon).$$

Setting

$$\overline{u}(x) = K[g^*(\,\cdot\,,\varepsilon)](x) := \int_\Omega K(x,y) g^*(y,\varepsilon)\,dy,$$

by $(\mathcal{K}_\mathrm{b})$ we observe that $\overline{u} \in L^1_+(\Omega)$ and

$$0 \le u \le \overline{u} \text{ a.e. } \Rightarrow 0 \le K[g(\,\cdot\,,\varepsilon + u)] \le \overline{u} \text{ a.e.}$$

Since, for all $u \in L^1_+(\Omega)$, it results

(i) $$|K[g(\,\cdot\,,\varepsilon+u)](\,\cdot\,+h) - K[g(\,\cdot\,,\varepsilon+u)]|_1 \le \int_{\Omega\times\Omega} |K(x+h,y) - K(x,y)| g^*(y,\varepsilon)\,dx\,dy,$$

(ii) $$\int_{\Omega\setminus B_\rho} K[g(\,\cdot\,,\varepsilon+u)]\,dx \le \int_{\Omega\setminus B_\rho} \overline{u}\,dx.$$

The Fréchet–Kolmogorov Theorem gives that $\{K[g(\,\cdot\,,\varepsilon+u)] \mid u \in L^1_+(\Omega)\}$ is compact in $L^1(\Omega)$.

Finally, since g is a Carathéodory function, by using the Lebesgue Dominated Convergence Theorem it is easily seen that $u \mapsto K[g(\,\cdot\,,\varepsilon+u)]$ is continuous in $L^1_+(\Omega)$. Consequently, the Shauder Theorem implies that there exists $u_\varepsilon \in L^1_+(\Omega)$, solution of $(2)_\varepsilon$.

References

[1] M. M. Coclite, *On a singular nonlinear Dirichlet problem II*, Boll. Un. Mat. Ital. B(7) **5** (1991), 955–975.

[2] ———, *On a singular nonlinear Dirichlet problem III*, Nonlinear Anal. **21** (1993), 547–564.

[3] ———, *On a singular nonlinear Dirichlet problem IV*, Nonlinear Anal. **23** (1994), 925–936.

[4] ———, *Summable positive solutions of a Hammerstein integral equation with sigular nonlinear term*, Atti Sem. Mat. Fis. Univ. Modena **XLVI** (1998), 625–639.

[5] L. Erbe, D. Guo and X. Liu, *Positive solutions of a class of nonlinear integral equations and applications*, J. Integral Equations Appl. **4** (1992), 179–195.

[6] S. Gomes, *On a singular nonlinear elliptic problem*, SIAM J. Math. Anal. **17** (1986), 1359–1369.

[7] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Inc., San Diego, 1988.

[8] R. K. Juberg, *The measure of non-compactness in L^p for a class of integral operator*, Indiana Univ. Math. J. **23** (1974), 925–936.

[9] N. K.Karapetyants and A. Ya. Yakubov, *A convolution equation with a power nonlinearity of negative order*, Dokl. Akad. Nauk SSSR **320** (1991), 777–780.

[10] S. Karlin and L. Nirenberg, *On a theorem of Nowosad*, J. Math. Anal. Appl. **17** (1967), 61–67.

[11] M. A. Krasnosel'skiĭ, *Positive Solutions of Operator Equations*, P. Noordhoff, Gröningen, The Netherlands, 1964.

[12] L. Kurz, P. Nowosad and B. R. Saltzberg, *On the solution of a quadratic integral equation arising in signal design*, J. Franklin Inst. B **281** (1966), 437–454.

[13] P. Nowosad, *On the integral equation $Kf = 1/f$ arising in a problem in communication*, J. Math. Anal. Appl. **14** (1966), 484-492.

[14] G. E. Parker and T. J. Walters, *Constructing solutions to $\int_0^1 f(j)A(j,x)/f(x)dj = \int_0^1 f(x)A(x,j)/f(j)\,dj$*, SIAM J. Math. Anal. **13** (1982), 856–865.

[15] ———, *Positive nonlinear integral equations with reciprocals of the solution in the integrand*, Nonlinear Anal. **5** (1981), 1163–1172.

[16] C. A. Stuart, *The measure of noncompactness of some linear integral operator*, Proc. Roy. Soc. Edinburgh **71** (1973), 167–179.

[17] P. P. Zabreĭko, A. I. Koshelev, M. A. Krasnosel'skiĭ, S. G. Mikhlin, L. S. Rakovshchik and V. J. Stecenko, *Integral Equations — a Reference Text*, Noordhoff International Publishing, Leyden, 1975.

Manuscript received June 8, 1999

Mario Michele Coclite
Dipartimento di Matematica
Università di Bari
Orabona 4
70125 Bari, ITALY

E-mail address: coclite@dm.uniba.it