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INFINITE PRODUCTS OF RESOLVENTS OF ACCRETIVE OPERATORS

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Dedicated to the memory of Juliusz P. Schauder

Abstract. We study the space $\mathcal{M}_m$ of all m-accretive operators on a Banach space X endowed with an appropriate complete metrizable uniformity and the space $\overline{\mathcal{M}}_m^*$ which is the closure in $\mathcal{M}_m$ of all those operators which have a zero. We show that for a generic operator in $\mathcal{M}_m$ all infinite products of its resolvents become eventually close to each other and that a generic operator in $\overline{\mathcal{M}}_m^*$ has a unique zero and all the infinite products of its resolvents converge uniformly on bounded subsets of X to this zero.

Introduction

Infinite products of operators are of interest in many areas of mathematics and its applications. See, for instance, [1], [3]–[5], [11], [16]–[18], [20] and the references mentioned there. Accretive operators and their resolvents play an important role in nonlinear functional analysis [6], [7], [9], [13]. Infinite products of resolvents of accretive operators and their applications were investigated, for example, in [8], [10], [19], [22], [23], [26], [27].

In the present paper we use Baire's category to study the asymptotic behavior of infinite products of resolvents of a generic m-accretive operator on a general Banach space X. Our first main result is a weak ergodic theorem (Theorem 2.1). Our second main result (Theorem 2.2) provides strong convergence of infinite

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products to the unique zero of such an operator. More precisely, we consider two spaces of m-accretive operators on X. The first space is the space of all m-accretive operators endowed with an appropriate complete metrizable uniformity. The second space is the closure in the first space of all those operators which have a zero. For the first space we construct a subset which is a countable intersection of open everywhere dense sets such that for each operator belonging to this subset all infinite products of resolvents have the same asymptotics. For the second space we again construct a subset which is a countable intersection of open everywhere dense sets such that for each operator belonging to this subset all infinite products of resolvents converge uniformly on bounded subsets of X to the unique zero of the operator. Thus, instead of considering the asymptotic behavior of infinite products of resolvents of a single operator, we investigate it for a space of all such operators, equipped with some natural metric, and show that a certain convergence property holds for most of these operators. This allows us to establish strong convergence without imposing restrictive assumptions on the space or on the operators themselves. Results of this kind for powers of a single (nonexpansive) operator were already established by De Blasi and Myjak [14], while such results for infinite products of (nonlinear) nonexpansive and order-preserving self-mappings of bounded subsets have recently been obtained by the authors [24], [25]. The approach used in these papers and in the present paper is common in global analysis and the theory of dynamical systems [15], [21]. Recently it has also been used in the study of the structure of extremals of variational and optimal control problems [28], [29].

The paper is organized as follows. In the first section we recall several properties of accretive operators and define the spaces of m-accretive operators which we are going to study. We state our two main results (Theorems 2.1 and 2.2) in the second section. Section 3 contains three auxiliary results. We establish Theorems 2.1 and 2.2 in Sections 4 and 5, respectively.

1. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. We denote by $I : X \to X$ the identity operator on X (that is, $Ix = x$, $x \in X$). Recall that a set-valued operator $A : X \to 2^X$ with a nonempty domain

$$D(A) = \{x \in X : Ax \neq \emptyset\}$$

and range

$$R(A) = \{y \in X : y \in Ax \text{ for some } x \in D(A)\}$$

is said to be *accretive* if

$$\|x - y\| \leq \|x - y + r(u - v)\| \tag{1.1}$$

for all $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ and $r > 0$. When the operator A is accretive, then it follows from (1.1) that its resolvents

$$J_r^A = (I + rA)^{-1} : R(I + rA) \to D(A) \tag{1.2}$$

are single-valued nonexpansive operators for all positive r. In other words,

$$\|J_r^A x - J_r^A y\| \le \|x - y\| \tag{1.3}$$

for all x and y in $D(J_r^A) = R(I + rA)$. As usual, the graph of the operator A is defined by

$$\mathrm{graph}(A) = \{(x, y) \in X \times X : y \in Ax\}.$$

Note that if A is accretive, then the operator $\overline{A} : X \to 2^X$, the graph of which is the closure of graph(A) in the norm topology of $X \times X$, is also accretive. We will say that the operator A is closed if its graph is closed in $X \times X$.

An accretive operator $A : X \to 2^X$ is said to be *m-accretive* if

$$R(I + rA) = X \quad \text{for all } r > 0.$$

Note that if X is a Hilbert space $(H, \langle \cdot\,, \cdot \rangle)$, then an operator A is accretive if and only if it is monotone; that is, if and only if

$$\langle u - v, x - y \rangle \ge 0 \quad \text{for all } (x, u), (y, v) \in \mathrm{graph}(A).$$

It is well-known that in a Hilbert space an operator A is m-accretive if and only if it is maximal monotone. It is not difficult to see that in any Banach space an m-accretive operator is maximal accretive; that is, if $\widetilde{A} : X \to 2^X$ is accretive and graph(A) $\subset$ graph($\widetilde{A}$), then $\widetilde{A} = A$. However the converse is not true in general.

In the sequel we are going to use a certain topology on the space of nonempty closed subsets of $Y = X \times X$. We will now define this topology in a more general setting (cf. [2]). Let (Y, ρ) be a complete metric space. Fix $\theta \in Y$. For each positive $r > 0$ define

$$Y_r = \{y \in Y : \rho(y, \theta) \le r\}.$$

For each $y \in Y$ and each $E \subset Y$ define

$$\rho(y, E) = \inf\{\rho(y, z) : z \in E\}.$$

Denote by $S(Y)$ the set of all nonempty closed subsets of Y. For $F, G \in S(Y)$ and an integer $n \ge 1$ define

$$h_n(F, G) = \sup_{y \in Y_n} |\rho(y, F) - \rho(y, G)|.$$

Clearly $h_n(F, G) < \infty$ for each integer $n \ge 1$ and each pair of sets $F, G \in S(Y)$. For the set $S(Y)$ we consider the uniformity generated by the following base:

$$\widetilde{E}(n) = \{(F, G) \in S(Y) \times S(Y) : h_n(F, G) < n^{-1}\}, \quad n = 1, 2, \ldots \tag{1.4}$$

This uniform space is metrizable by the metric

$$h(F,G)=\sum_{n=1}^{\infty}2^{-n}[h_n(F,G)/(1+h_n(F,G))]. \tag{1.5}$$

The metric space $(S(Y),h)$ is complete.

From now on we apply the above to the space $Y=X\times X$ with the metric

$$\rho((x_1,x_2),(z_1,z_2))=\|x_1-z_1\|+\|x_2-z_2\|,\quad x_i,z_i\in X,\ i=1,2,$$

and with $\theta=(0,0)$.

Denote by $\mathcal{M}_a$ the set of all closed accretive operators $A:X\to 2^X$. For each $A,B\in\mathcal{M}_a$ define

$$h_a(A,B)=h(\operatorname{graph}(A),\operatorname{graph}(B)). \tag{1.6}$$

Clearly $(\mathcal{M}_a,h_a)$ is a metric space and the set $\{\operatorname{graph}(A):A\in\mathcal{M}_a\}$ is a closed subset of $S(X\times X)$. Therefore $(\mathcal{M}_a,h_a)$ is a complete metric space. Denote by $\mathcal{M}_m$ the set of all m-accretive operators $A\in\mathcal{M}_a$.

Proposition 1.1. *$\mathcal{M}_m$ is a closed subset of $\mathcal{M}_a$.*

Proof. Suppose that $\{A_i\}_{i=1}^{\infty}\subset\mathcal{M}_m$, $A\in\mathcal{M}_a$, and that $A_i\to A$ as $i\to\infty$ in $\mathcal{M}_a$. Assume that r is a positive number. We have to show that $R(I+rA)=X$. To this end, let $z\in X$. For each integer $n\geq 1$ there exists $y_n\in X$ for which

$$z\in(I+rA_n)y_n \text{ or, equivalently, } y_n=(I+rA_n)^{-1}z. \tag{1.7}$$

We will show that the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded. Fix $(x,u)\in\operatorname{graph}(A)$. There is a sequence $\{(x_n,u_n)\}_{n=1}^{\infty}\subset X\times X$ such that

$$(x_n,u_n)\in\operatorname{graph}(A_n),\quad n=1,2,\ldots,\quad\text{and}\quad\lim_{n\to\infty}(x_n,u_n)=(x,u). \tag{1.8}$$

For each integer $n\geq 1$,

$$x_n=(I+rA_n)^{-1}(x_n+ru_n)\quad\text{and}\quad\|x_n-y_n\|\leq\|x_n+ru_n-z\|. \tag{1.9}$$

By (1.8) and (1.9) the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded. By (1.7), for each integer $n\geq 1$ there exists v_n for which

$$v_n\in A_n(y_n)\quad\text{and}\quad z=y_n+rv_n. \tag{1.10}$$

Clearly the sequence $\{(y_n,v_n)\}_{n=1}^{\infty}$ is bounded. There exists a sequence

$$\{(\widetilde{y}_n,\widetilde{v}_n)\}_{n=1}^{\infty}\subset\ \operatorname{graph}(A)$$

such that

$$\|\widetilde{y}_n-y_n\|+\|\widetilde{v}_n-v_n\|\to 0\quad\text{as } n\to\infty. \tag{1.11}$$

Set, for all integers $n \geq 1$,

$$z_n = \widetilde{y}_n + r\widetilde{v}_n \in (I + rA)\widetilde{y}_n. \tag{1.12}$$

By (1.10)–(1.12),

$$\lim_{n\to\infty} z_n = z \quad \text{and} \quad \|z_n - z_k\| \geq \|\widetilde{y}_n - \widetilde{y}_k\| \quad \text{for all integers } n, k.$$

Therefore the sequence $\{(\widetilde{y}_n, \widetilde{v}_n)\}_{n=1}^{\infty}$ converges to $(y, v) \in \operatorname{graph}(A)$. Clearly $z = y + rv$. Proposition 1.1 is proved. □

Denote by $\mathcal{M}_m^*$ the set of all $A \in \mathcal{M}_m$ such that there exists x_A for which $0 \in A(x_A)$ and denote by $\overline{\mathcal{M}_m^*}$ the closure of $\mathcal{M}_m^*$ in $\mathcal{M}_m$. The two complete metric spaces $(\mathcal{M}_m, h_a)$ and $(\overline{\mathcal{M}_m^*}, h_a)$ are the focal points of our investigations. Finally, we denote by $\mathcal{M}_0^*$ the set of all $A \in \mathcal{M}_m^*$ for which there exists $x_A \in X$ such that $0 \in A(x_A)$ and $(J_1^A)^n(x) \to x_A$ as $n \to \infty$ for all $x \in X$.

2. Statements of the main results

Let $\{\overline{r}_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that

$$\overline{r}_n < 1, \quad n = 1, 2, \ldots, \quad \lim_{n\to\infty} \overline{r}_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \overline{r}_n = \infty \tag{2.1}$$

and let $\widetilde{r} > 1$. We now state our two main results.

THEOREM 2.1. *There exists a set $\mathcal{F} \subset \mathcal{M}_m$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}_m$ such that for each $A \in \mathcal{F}$, each $\delta > 0$ and each $K > 0$ the following assertion holds:*

There exist a neighbourhood U of A in $\mathcal{M}_m$ and an integer $n_0 \geq 1$ such that for each sequence of positive numbers $\{r_n\}_{n=1}^{\infty}$ satisfying $\widetilde{r} > r_n \geq \overline{r}_n$, $n = 1, 2, \ldots$, each $B \in U$ and each $x, y \in X$ satisfying $\|x\|, \|y\| \leq K$, we have

$$\|J_{r_n}^B \cdot J_{r_{n-1}}^B \cdot \ldots \cdot J_{r_1}^B x - J_{r_n}^B \cdot J_{r_{n-1}}^B \cdot \ldots \cdot J_{r_1}^B y\| \leq \delta$$

for all integers $n \geq n_0$.

We remark in passing that such a result is called a weak ergodic theorem in population biology [12]. It means that for a generic operator in $\mathcal{M}_m$ all infinite products of its resolvents become eventually close to each other.

THEOREM 2.2. *There exists a set $\mathcal{F} \subset \mathcal{M}_0^* \cap \overline{\mathcal{M}_m^*}$ which is a countable intersection of open everywhere dense sets in $\overline{\mathcal{M}_m^*}$ such that for each $A \in \mathcal{F}$ the following assertions hold:*

(i) *There exists a unique $x_A \in X$ such that $0 \in A(x_A)$.*
(ii) *For each $\delta > 0$ and each $K > 0$ there exist a neighbourhood U of A in $\mathcal{M}_m$ and an integer $n_0 \geq 1$ such that for each sequence of positive*

numbers $\{r_n\}_{n=1}^{\infty}$ *satisfying* $\widetilde{r} > r_n \geq \overline{r}_n$, $n = 1, 2, \ldots$, *each* $B \in U \cap \mathcal{M}_0^*$ *and each* $x \in X$ *satisfying* $\|x\| \leq K$, *we have*

$$\|J_{r_n}^B \cdot J_{r_{n-1}}^B \ldots J_{r_1}^B x - x_A\| \leq \delta$$

for all integers $n \geq n_0$.

This result means that a generic operator in $\overline{\mathcal{M}}_m^*$ has a unique zero and all the infinite products of its resolvents converge uniformly on bounded subsets of X to this zero.

3. Auxiliary results

Let $\{\overline{r}_n\}_{n=1}^{\infty} \subset (0,1)$ satisfy (2.1) and let $\widetilde{r} > 1$.

Lemma 3.1. *Let* $A \in \mathcal{M}_m$, $K_0 > 0$ *and let* $n_0 \geq 2$ *be an integer. Then there exist a neighbourhood* U *of* A *in* $\mathcal{M}_m$ *and a number* $c_0 > 0$ *such that for each* $B \in U$, *each sequence* $\{r_i\}_{i=1}^{n_0-1} \subset (0, \widetilde{r})$ *and each sequence* $\{x_i\}_{i=1}^{n_0} \subset X$ *satisfying* $\|x_1\| \leq K_0$, $x_{i+1} = J_{r_i}^B(x_i)$, $i = 1, \ldots, n_0 - 1$, *we have* $\|x_i\| \leq c_0$ *for all* $i = 1, \ldots, n_0$.

Proof. Choose $(x_A, u_A) \in \operatorname{graph}(A)$. There exists a neighbourhood U of A in $\mathcal{M}_m$ such that for each $B \in U$ there exists $(x_B, u_B) \in \operatorname{graph}(B)$ satisfying

$$\|x_B - x_A\| + \|u_A - u_B\| < 1. \tag{3.1}$$

Assume that $B \in U$,

$$\{r_i\}_{i=1}^{n_0-1} \subset (0, \widetilde{r}), \quad x_1 \in X, \quad \|x_1\| \leq K_0 \quad \text{and } x_{i+1} = J_{r_i}^B(x_i),\ i = 1, \ldots, n_0 - 1. \tag{3.2}$$

We will estimate $\|x_i\|$ for $i = 1, \ldots, n_0$. To this end, set

$$z_i = x_B + r_i u_B, \quad i = 1, \ldots, n_0 - 1. \tag{3.3}$$

For such i we clearly have, by (3.1)–(3.3), $x_B = J_{r_i}^B(z_i)$, $\|x_B - x_{i+1}\| \leq \|z_i - x_i\|$ and

$$\begin{aligned}
\|x_{i+1}\| &\leq \|x_B\| + \|x_i\| + \|z_i\| \leq \|x_i\| + \|x_A\| + 1 + \|x_B + r_i u_B\| \\
&\leq \|x_i\| + 1 + \|x_A\| + \|x_B\| + \widetilde{r}\|u_B\| \\
&\leq \|x_i\| + 1 + 2\|x_A\| + 1 + \widetilde{r}(\|u_A\| + 1).
\end{aligned}$$

This implies that for $i = 1, \ldots, n_0 - 1$,

$$\|x_{i+1}\| \leq i(2\|x_A\| + 2 + \widetilde{r}(\|u_A\| + 1)) + K_0. \qquad \square$$

Assumption (2.1) and Lemma 3.1 imply the following result.

LEMMA 3.2. *Let $A \in \mathcal{M}_m$, $K_0 > 0$ and let $n_0 \geq 2$ be an integer. Then there exist a neighbourhood U of A in $\mathcal{M}_m$ and a number $c_1 > 0$ such that for each $B \in U$, each sequence $r_i \in [\overline{r}_i, \widetilde{r})$, $i = 1, \ldots, n_0 - 1$, and each two sequences $\{x_i\}_{i=1}^{n_0} \subset X$, $\{y_i\}_{i=2}^{n_0} \subset X$ satisfying*

$$\|x_1\| \leq K_0, \quad x_{i+1} = J_{r_i}^B(x_i), \quad x_i = x_{i+1} + r_i y_{i+1}, \quad y_{i+1} \in B(x_{i+1}),$$

$i = 1, \ldots, n_0 - 1$, the following two estimates hold:

$$\|x_i\| \leq c_1, \quad i = 1, \ldots, n_0 \quad \textit{and} \quad \|y_i\| \leq c_1, \quad i = 2, \ldots, n_0.$$

LEMMA 3.3. *Let $A \in \mathcal{M}_m$, $x_* \in X$, $0 \in A(x_*)$, $\varepsilon > 0$ and let $n_0 \geq 2$ be an integer. Then there exists a neighbourhood U of A in $\mathcal{M}_m$ such that for each $B \in U$ and each sequence $r_i \in (0, \widetilde{r})$, $i = 1, \ldots, n_0 - 1$, there exists a sequence $\{x_i\}_{i=1}^{n_0} \subset X$ such that*

$$x_{i+1} = J_{r_i}^B(x_i), \quad i = 1, \ldots, n_0 - 1, \quad \textit{and} \quad \|x_i - x_*\| \leq \varepsilon, \quad i = 1, \ldots, n_0.$$

PROOF. Choose a natural number p such that

$$p > 4 + n_0 + \|x_*\| \quad \text{and} \quad p > \widetilde{r}(n_0 + 1)(\inf\{1, \varepsilon\})^{-1} \tag{3.4}$$

and define

$$U = \{B \in \mathcal{M}_m : \ h_p(\operatorname{graph}(A), \operatorname{graph}(B)) < p^{-1}\}. \tag{3.5}$$

Assume that $B \in U$ and $r_i \in (0, \widetilde{r})$, $i = 1, \ldots, n_0 - 1$. By (3.4) and (3.5) there exists $(x_1, y_1) \in \operatorname{graph}(B)$ such that

$$\|x_1 - x_*\| + \|y_1\| < p^{-1}. \tag{3.6}$$

Set

$$\xi_i = x_1 + r_i y_1, \quad i = 1, \ldots, n_0 - 1. \tag{3.7}$$

Then

$$x_1 = J_{r_i}^B(\xi_i) \quad \text{and} \quad \|x_1 - \xi_i\| < \widetilde{r}/p, \quad i = 1, \ldots, n_0 - 1. \tag{3.8}$$

Set

$$x_{i+1} = J_{r_i}^B(x_i), \quad i = 1, \ldots, n_0 - 1. \tag{3.9}$$

Since for $i = 1, \ldots, n_0 - 1$, $J_{r_i}^B$ is a nonexpansive operator it follows from (3.6)–(3.9) that for each integer $k \in [2, n_0]$ we have

$$\begin{aligned} \|x_k - x_1\| &\leq \|x_{k-1} - \xi_{k-1}\| \leq \|x_{k-1} - x_1\| + \widetilde{r}\|y_1\| < \|x_{k-1} - x_1\| + \widetilde{r}/p, \\ \|x_k - x_1\| &\leq k\widetilde{r}/p, \end{aligned}$$

and $\|x_k - x_*\| < \|x_k - x_1\| + \|x_1 - x_*\| < (k+1)\widetilde{r}/p \leq (n_0 + 1)\widetilde{r}/p < \varepsilon.$ □

4. Proof of Theorem 2.1

For each $A \in \mathcal{M}_m$, $\xi \in X$ and each positive number γ let the operator $A_{\gamma,\xi}$ be defined by

$$A_{\gamma,\xi}x = Ax + \gamma(x - \xi), \quad x \in X.$$

We begin the proof with the following three observations.

Lemma 4.1. *If $A \in \mathcal{M}_m$, $\xi \in X$ and $\gamma > 0$, then $A_{\gamma,\xi} \in \mathcal{M}_m$.*

Lemma 4.2. *Let $A \in \mathcal{M}_m$, $\xi \in X$, $\gamma, r > 0$ and let $x, y \in X$. Then*

$$\|J_r^{A_{\gamma,\xi}}(x) - J_r^{A_{\gamma,\xi}}(y)\| \le (1 + \gamma r)^{-1}\|x - y\|.$$

Lemma 4.3. *For each fixed $\xi \in X$, the set $\{A_{\gamma,\xi} : A \in \mathcal{M}_m,\ \gamma \in (0,1)\}$ is everywhere dense in $\mathcal{M}_m$.*

In the rest of the proof we assume that (cf. (2.1))

$$\widetilde{r} > 1, \quad \{\overline{r}_n\}_{n=1}^{\infty} \subset (0,1), \quad \lim_{n\to\infty} \overline{r}_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \overline{r}_n = \infty. \tag{4.1}$$

Lemma 4.4. *Let $A \in \mathcal{M}_m$, $\xi \in X$, $\gamma \in (0,1)$ and $\delta, K > 0$. Then there exist a neighbourhood U of $A_{\gamma,\xi}$ in $\mathcal{M}_m$ and an integer $n_0 \ge 4$ such that for each $B \in U$, each sequence of numbers $r_i \in [\overline{r}_i, \widetilde{r})$, $i = 1, \ldots, n_0 - 1$, and each $x, y \in X$ satisfying $\|x\|, \|y\| \le K$, the following estimate holds:*

$$\|J^B_{r_{n_0-1}} \cdot J^B_{r_{n_0-2}} \cdot \ldots \cdot J^B_{r_1}x - J^B_{r_{n_0-1}} \cdot J^B_{r_{n_0-2}} \cdot \ldots \cdot J^B_{r_1}y\| \le \delta. \tag{4.2}$$

Proof. Choose a number γ_0 such that

$$\gamma_0 \in (0, \gamma). \tag{4.3}$$

Clearly

$$\prod_{i=1}^{n}(1 + \gamma_0\overline{r}_i) \to \infty \quad \text{as } n \to \infty. \tag{4.4}$$

Therefore there exists an integer $n_0 \ge 4$ such that

$$(2K + 2)\prod_{i=1}^{n_0-1}(1 + \gamma_0\overline{r}_i)^{-1} < \delta/2. \tag{4.5}$$

By Lemma 3.2 there exist a neighbourhood U_1 of $A_{\gamma,\xi}$ in $\mathcal{M}_m$ and a number $c_1 > 0$ such that for each $B \in U_1$, each sequence $r_i \in [\overline{r}_i, \widetilde{r}_i)$, $i = 1, \ldots, n_0 - 1$, and each pair of sequences $\{x_i\}_{i=1}^{n_0} \subset X$ and $\{u_i\}_{i=2}^{n_0} \subset X$ satisfying

$$\|x_1\| \le K, \quad x_{i+1} = J^B_{r_i}(x_i), \quad x_i = x_{i+1} + r_iu_{i+1}, \quad u_{i+1} \in B(x_{i+1}) \tag{4.6}$$

for $i = 1, \dots, n_0 - 1$, the following estimates hold:

$$\begin{aligned} \|x_i\| &\le c_1, \quad i = 1, \dots, n_0, \\ \|u_i\| &\le c_1, \quad i = 2, \dots, n_0. \end{aligned} \tag{4.7}$$

Choose a natural number m_1 such that

$$\begin{gathered} m_1 > 4(n_0 + 8(c_1 + 1)), \\ [(1+\gamma_0 \overline{r}_i)^{-1} - (1+\gamma \overline{r}_i)^{-1}]\delta > 2(2+\widetilde{r})m_1^{-1}, \quad i = 1, \dots, n_0, \end{gathered} \tag{4.8}$$

and set

$$U = \{B \in U_1 : h_{m_1}(\mathrm{graph}(A_{\gamma,\xi}), \mathrm{graph}(B)) < m_1^{-1}\}. \tag{4.9}$$

Assume that $B \in U$, $r_i \in [\overline{r}_i, \widetilde{r})$, $i = 1, \dots, n_0 - 1$, and

$$x, y \in X \quad \text{and} \quad \|x\|, \|y\| \le K. \tag{4.10}$$

Set

$$x_1 = x, \quad y_1 = y, \quad x_{i+1} = J^B_{r_i}(x_i) \quad \text{and} \quad y_{i+1} = J^B_{r_i}(y_i),\ i = 1, \dots, n_0 - 1. \tag{4.11}$$

For each $i = 1, \dots, n_0 - 1$ there exist u_{i+1} and $v_{i+1} \in X$ such that

$$\begin{aligned} u_{i+1} &\in B(x_{i+1}), \quad x_i = x_{i+1} + r_i u_{i+1}, \\ v_{i+1} &\in B(y_{i+1}), \quad y_i = y_{i+1} + r_i v_{i+1}. \end{aligned} \tag{4.12}$$

It follows from the definition of U_1 (see (4.6)) and (4.12) that

$$\begin{aligned} \|x_i\|, \|y_i\| &\le c_1, \quad i = 1, \dots, n_0, \\ \|u_i\|, \|v_i\| &\le c_1, \quad i = 2, \dots, n_0. \end{aligned} \tag{4.13}$$

To prove the lemma it is sufficient to show that

$$\|x_{n_0} - y_{n_0}\| \le \delta. \tag{4.14}$$

Assume the contrary. Then

$$\|x_i - y_i\| > \delta, \quad i = 1, \dots, n_0. \tag{4.15}$$

Let $i \in \{1, \dots, n_0 - 1\}$. It follows from (4.12), (4.13), (4.9) and (4.8) that there exist

$$(\overline{x}_{i+1}, \overline{u}_{i+1}) \in \mathrm{graph}(A_{\gamma,\xi}) \quad \text{and} \quad (\overline{y}_{i+1}, \overline{v}_{i+1}) \in \mathrm{graph}(A_{\gamma,\xi}) \tag{4.16}$$

such that

$$\begin{aligned} \|\overline{x}_{i+1} - x_{i+1}\| + \|\overline{u}_{i+1} - u_{i+1}\| &< m_1^{-1}, \\ \|\overline{y}_{i+1} - y_{i+1}\| + \|\overline{v}_{i+1} - v_{i+1}\| &< m_1^{-1}. \end{aligned} \tag{4.17}$$

Set

$$\overline{x}_i = \overline{x}_{i+1} + r_i\overline{u}_{i+1} \quad \text{and} \quad \overline{y}_i = \overline{y}_{i+1} + r_i\overline{v}_{i+1}. \tag{4.18}$$

By Lemma 4.2, (4.16) and (4.18),

$$\begin{aligned} \|\overline{x}_{i+1} - \overline{y}_{i+1}\| &= \|J_{r_i}^{A_{\gamma,\xi}}\overline{x}_i - J_{r_i}^{A_{\gamma,\xi}}\overline{y}_i\| \\ &\le (1+\gamma r_i)^{-1}\|\overline{x}_i - \overline{y}_i\| \le (1+\gamma\overline{r}_i)^{-1}\|\overline{x}_i - \overline{y}_i\|. \end{aligned} \tag{4.19}$$

It follow from (4.18), (4.12) and (4.17) that

$$\|\overline{x}_i - x_i\| \le \|\overline{x}_{i+1} - x_{i+1}\| + r_i\|\overline{u}_{i+1} - u_{i+1}\| \le m_1^{-1}(1+\widetilde{r}) \tag{4.20}$$

and

$$\|\overline{y}_i - y_i\| \le \|\overline{y}_{i+1} - y_{i+1}\| + r_i\|\overline{v}_{i+1} - v_{i+1}\| \le m_1^{-1}(1+\widetilde{r}).$$

By (4.17), (4.19) and (4.20),

$$\begin{aligned} \|x_{i+1} - y_{i+1}\| &\le \|\overline{x}_{i+1} - \overline{y}_{i+1}\| + 2m_1^{-1} \\ &\le 2m_1^{-1} + (1+\gamma\overline{r}_i)^{-1}\|\overline{x}_i - \overline{y}_i\| \\ &\le 2m_1^{-1} + (1+\gamma\overline{r}_i)^{-1}(\|x_i - y_i\| + 2m_1^{-1}(1+\widetilde{r})) \\ &\le (1+\gamma\overline{r}_i)^{-1}\|x_i - y_i\| + 2m_1^{-1}(1 + (1+\gamma\overline{r}_i)^{-1}(1+\widetilde{r})) \\ &\le (1+\gamma\overline{r}_i)^{-1}\|x_i - y_i\| + 2m_1^{-1}(2+\widetilde{r}). \end{aligned} \tag{4.21}$$

Now (4.21), (4.8) and (4.15) imply that

$$\|x_{i+1} - y_{i+1}\| \le (1+\gamma_0\overline{r}_i)^{-1}\|x_i - y_i\|$$

and since these inequalities are valid for all $i \in \{1,\dots,n_0-1\}$, it follows from (4.10), (4.11) and (4.5) that

$$\|x_{n_0} - y_{n_0}\| \le 2K\prod_{i=1}^{n_0-1}(1+\gamma_0\overline{r}_i)^{-1} < \delta/2.$$

This contradicts (4.15). Therefore (4.14) is true and Lemma 4.4 is proved. □

Completion of the proof of Theorem 2.1.. Let $A \in \mathcal{M}_m$, $\xi = 0$, $\gamma \in (0,1)$ and let $i \ge 1$ be an integer. By Lemma 4.4 there exist an open neighbourhood $U(A,\gamma,i)$ of $A_{\gamma,0}$ in $\mathcal{M}_m$ and an integer $q(A,\gamma,i) \ge 4$ such that for each $B \in U(A,\gamma,i)$, each sequence of numbers $r_i \in [\overline{r}_i,\widetilde{r})$, $i = 1,\dots,q(A,\gamma,i)-1$, and each $x,y \in X$ satisfying $\|x\|,\|y\| \le 2^{i+1}$, the following estimate holds:

$$\|J^B_{r_{q(A,\gamma,i)-1}} \cdot \ldots \cdot J^B_{r_1}x - J^B_{r_{q(A,\gamma,i)-1}} \cdot \ldots \cdot J^B_{r_1}y\| \le 2^{-i-1}.$$

Define

$$\mathcal{F} = \bigcap_{n=1}^{\infty}\bigcup\{U(A,\gamma,i) : A \in \mathcal{M}_m,\ \gamma \in (0,1),\ i \ge n\}.$$

Clearly (see Lemma 4.3) $\mathcal{F}$ is a countable intersection of open everywhere dense sets in $\mathcal{M}_m$. Let $A \in \mathcal{F}$, $\delta > 0$ and $K > 0$. Choose an integer $n > 2K+2+8\delta^{-1}$. There exist $C \in \mathcal{M}_m$, $\gamma \in (0,1)$ and $i \geq n$ such that $A \in U(C,\gamma,i)$. The validity of Theorem 2.1 now follows from the definitions of $U(C,\gamma,i)$ and $q(C,\gamma,i)$. $\square$

5. Proof of Theorem 2.2

As in (4.1) let

$$\widetilde{r} > 1, \quad \{\overline{r}_n\}_{n=1}^{\infty} \subset (0,1), \quad \lim_{n\to\infty} \overline{r}_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \overline{r}_n = \infty. \tag{5.1}$$

By definition, for each $A \in \mathcal{M}_m^*$ there exists $x_A \in X$ such that

$$0 \in A(x_A). \tag{5.2}$$

Recalling the definition of $A_{\gamma,\xi}$ at the beginning of Section 4, we will use in this section the operator A_{γ,x_A}. In other words,

$$A_{\gamma,x_A} x = Ax + \gamma(x - x_A), \quad x \in X. \tag{5.3}$$

By Lemma 4.1 and (5.2), for each $A \in \mathcal{M}_m^*$ and each $\gamma \in (0,1)$,

$$A_{\gamma,x_A} \in \mathcal{M}_m^* \quad \text{and} \quad 0 \in A_{\gamma,x_A}(x_A). \tag{5.4}$$

The following observation is also clear.

Lemma 5.1. *The set $\{A_{\gamma,x_A} : A \in \mathcal{M}_m^*,\ \gamma \in (0,1)\}$ is everywhere dense in $\overline{\mathcal{M}}_m^*$.*

Let $A \in \mathcal{M}_m^*$, $\gamma \in (0,1)$ and let $i \geq 1$ be an integer. By Lemma 4.4 with $\xi = x_A$ there exist an open neighbourhood $U_1(A,\gamma,i)$ of A_{γ,x_A} in $\mathcal{M}_m$ and an integer $n(A,\gamma,i) \geq 4$ such that the following property holds:

(a) For each $B \in U_1(A,\gamma,i)$, each sequence

$$r_j \in [\overline{r}_j, \widetilde{r}), \quad j = 1,\dots,n(A,\gamma,i)-1,$$

and each $x, y \in X$ satisfying

$$\|x\|, \|y\| \leq 8^{i+1}(4 + 4\|x_A\|), \tag{5.5}$$

the following estimate holds:

$$\|J^B_{r_{n(A,\gamma,i)-1}} \cdot \dots \cdot J^B_{r_1} x - J^B_{r_{n(A,\gamma,i)-1}} \cdot \dots \cdot J^B_{r_1} y\| \leq 8^{-i-1}. \tag{5.6}$$

By Lemma 3.3 there exists an open neighbourhood $U(A,\gamma,i)$ of A_{γ,x_A} in $\mathcal{M}_m$ such that

$$U(A,\gamma,i) \subset U_1(A,\gamma,i) \tag{5.7}$$

and the following property holds:

(b) For each $B \in U(A,\gamma,i)$ and each sequence

$$r_j \in (0,\widetilde{r}), \quad j = 1,\dots,8n(A,\gamma,i)-1,$$

there exists a sequence $\{x_j : j = 1,\dots,8n(A,\gamma,j)\} \subset X$ such that

$$x_{j+1} = J^B_{r_j}(x_j), \quad j = 1,\dots,8n(A,\gamma,i)-1, \tag{5.8}$$

and

$$\|x_j - x_A\| \le 8^{-i-1}, \quad j = 1,\dots,8n(A,\gamma,i).$$

We will now show that the following property also holds:

(c) For each $B \in U(A,\gamma,i)$, each $x \in X$ satisfying $\|x\| \le 8^{i+1}(2+2\|x_A\|)$ and each integer $m \ge n(A,\gamma,i)-1$,

$$\|(J^B_1)^m(x) - x_A\| \le 2\cdot 8^{-i-1}. \tag{5.9}$$

Indeed, let $B \in U(A,\gamma,i)$. By property (b) there exists a sequence

$$\{\overline{x}_j :\ j = 1,\dots,8n(A,\gamma,i)\} \subset X \tag{5.10}$$

such that

$$\overline{x}_{j+1} = J^B_1(\overline{x}_j), \quad j = 1,\dots,8n(A,\gamma,i)-1, \tag{5.11}$$

and

$$\|\overline{x}_j - x_A\| < 8^{-i-1}, \quad j = 1,\dots,8n(A,\gamma,i).$$

Let $x \in X$ with

$$\|x\| \le 8^{i+1}(2+2\|x_A\|) \tag{5.12}$$

and consider the sequence $\{(J^B_1)^j(x)\}_{j=1}^{\infty}$. Since the operator J^B_1 is nonexpansive it follows from (5.11) and (5.12) that for $j = 1,\dots,8n(A,\gamma,i)-1$,

$$\begin{aligned}\|(J^B_1)^j x\| &\le \|\overline{x}_{j+1}\| + \|(J^B_1)^j x - \overline{x}_{j+1}\| \\ &\le \|x_A\| + \|\overline{x}_{j+1} - x_A\| + \|(J^B_1)^j x - (J^B_1)^j(\overline{x}_1)\| \\ &\le \|x_A\| + 8^{-i-1} + \|x - \overline{x}_1\| \\ &\le 2(\|x_A\| + 8^{-i-1}) + \|x\| \\ &\le 8^{i+1}(2+2\|x_A\|) + 2(\|x_A\| + 2^{-1}) < 8^{i+1}(4+4\|x_A\|).\end{aligned} \tag{5.13}$$

We will show by induction that (5.9) is valid for all integers $m \geq n(A,\gamma,i)-1$. Let $m = n(A,\gamma,i)-1$. Then by property (a) and (5.11),

$$\begin{aligned}\|(J_1^B)^m(x) - x_A\| &\leq \|(J_1^B)(x) - (J_1^B)^m(\overline{x}_1)\| + \|(J_1^B)^m(\overline{x}_1) - x_A\| \\ &\leq 8^{-i-1} + \|\overline{x}_{m+1} - x_A\| \leq 2\cdot 8^{-i-1}.\end{aligned}$$

Therefore for $m = n(A,\gamma,i)-1$ (5.9) is valid. Assume that $q \geq n(A,\gamma,i)-1$ and that (5.9) is valid for all integers $m \in [n(A,\gamma,i)-1, q]$. Consider

$$y = (J_1^B)^p(x) \quad \text{with } p = q - (n(A,\gamma,i)-1)+1. \tag{5.14}$$

It follows from (5.9), which is valid by our inductive assumption for all integers $m \in [n(A,\gamma,i)-1,q]$, and (5.13), which holds for all $j = 1,\dots,8n(A,\gamma,i)-1$, that

$$\|y\| \leq 8^{i+1}(4+4\|x_A\|).$$

By this estimate, (5.14), (5.11) and property (a),

$$\begin{aligned}\|(J_1^B)^{q+1}(x) - x_A\| &= \|(J_1^B)^{n(A,\gamma,i)-1}(y) - x_A\| \\ &\leq \|(J_1^B)^{n(A,\gamma,i)-1}y - (J_1^B)^{n(A,\gamma,i)-1}(\overline{x}_1)\| \\ &\quad + \|\overline{x}_{n(A,\gamma,i)} - x_A\| \leq 2\cdot 8^{-i-1}.\end{aligned}$$

Therefore (5.9) is valid for all integers $m \geq n(A,\gamma,i)-1$ and property (c) holds. Next we define

$$\mathcal{F} = \left[\bigcap_{k=1}^{\infty} \bigcup \{U(A,\gamma,i) : A \in \mathcal{M}_m^*,\ \gamma \in (0,1),\ i \geq k\}\right] \cap \overline{\mathcal{M}}_m^*.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense sets in $\overline{\mathcal{M}}_m^*$. We will show that $\mathcal{F} \subset \mathcal{M}_0^*$.

Let $A \in \mathcal{F}$. Then there exist sequences $\{A_k\}_{k=1}^{\infty} \subset \mathcal{M}_m^*$, $\{\gamma_k\}_{k=1}^{\infty} \subset (0,1)$ and a strictly increasing sequence of natural numbers $\{i_k\}_{k=1}^{\infty}$ such that $A \in U(A_k,\gamma,i_k)$ for all natural numbers k. Property (c) implies that there exists $x_A \in X$ such that

$$\lim_{j\to\infty} (J_1^A)^j(x) = x_A \quad \text{for all } x \in X.$$

Clearly $0 \in A(x_A)$ and if $y \in X$ satisfies $0 \in A(y)$, then $y = x_A$. Therefore $\mathcal{F} \subset \mathcal{M}_0^*$.

Let $\delta, K > 0$. Choose a natural number q such that

$$4^q > 4K+4 \quad \text{and} \quad 4^q > \delta^{-1}, \tag{5.15}$$

and consider the open set $U(A_q,\gamma_q,i_q)$. Let $r_i \in [\overline{r}_i, \widetilde{r})$, $i = 1,2,\dots$, and let

$$B \in \mathcal{M}_0^* \cap U(A_g,\gamma_q,i_q). \tag{5.16}$$

There exists a unique $x_B \in X$ such that

$$0 \in B(x_B) \tag{5.17}$$

and

$$(J_1^B)^n y \to x_B \quad \text{as } n \to \infty \text{ for all } y \in X. \tag{5.18}$$

It follows from (5.18) and property (c) that

$$\|x_A - x_{A_q}\|,\ \|x_B - x_{A_q}\| \le 2 \cdot 8^{-i_q-1}. \tag{5.19}$$

Let $x \in X$ with

$$\|x\| \le K. \tag{5.20}$$

Set $\overline{n} = n(A_q, \gamma_q, i_q)$. It follows from (5.16), (5.19), (5.20), (5.15) and property (a) that

$$\|J^B_{r_{\overline{n}-1}} \cdot \ldots \cdot J^B_{r_1} x - J^B_{r_{\overline{n}-1}} \cdot \ldots \cdot J^B_{r_1} x_B\| \le 8^{-i_q-1}. \tag{5.21}$$

By (5.17), (5.21) and (5.19) we now have, for each integer $n \ge \overline{n}$,

$$\|J^B_{r_{n-1}} \cdot \ldots \cdot J^B_{r_1} x - x_B\| \le \|J^B_{r_{\overline{n}-1}} \cdot \ldots \cdot J^B_{r_1} x - x_B\| \le 8^{-i_q-1}$$

and

$$\|J^B_{r_{n-1}} \cdot \ldots \cdot J^B_{r_1} x - x_A\| \le 5 \cdot 8^{-i_q-1} < \delta.$$

This completes the proof of Theorem 2.2. □

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