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ON REPRESENTATION FORMULAS FOR HAMILTON JACOBI'S EQUATIONS RELATED TO CALCULUS OF VARIATIONS PROBLEMS

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Abstract. In this paper, existence and uniqueness of generalized solutions of some first order Hamilton Jacobi equations are proved. This task is accomplished by showing that the value function for a certain problem of the calculus of variations is the unique solution of the PDE. This can be viewed as a representation formula of the solution.

1. Introduction

The question of existence and uniqueness of solutions of the following Hamilton–Jacobi equation

$$(1)\qquad \begin{cases} \dfrac{\partial U}{\partial t} + H\bigg(t, x, U, \dfrac{\partial U}{\partial x}\bigg) = 0 & \text{for } (t,x) \in \,]0,T[\, \times \mathbb{R}^n, \\ U(T,x) = g(x) & \text{for } x \in \mathbb{R}^n, \end{cases}$$

needs an appropriate definition of solutions. So this problem could be reformulated as follows: which concept of solutions does provide existence and uniqueness for (1)?

Crandall and Lions defined (bounded uniformely) continuous viscosity solutions (see for instance [17]). In control theory, an area where such Hamilton–Jacobi's equation appears, when the value function is smooth, it is the unique

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solution of the PDE. But generally, for Mayer's Problem for instance, the value functions are only semicontinuous. So there were constant effort to extend the concept of viscosity solution to the semicontinous case (see extended bibliography in [5]). Another notion of semicontinuous solutions introduced by Barron–Jensen and Frankowska allows to solve the problem of existence and uniqueness when (1) comes from control theory ([6], [14], [5], see also [22] for another concept of solutions, [18] for extensions to fully discontinuous solutions).

Our approach involves relations between (1), when H does not depend on u, and the following problem of the calculus of variations:

$$\min_{x(\,\cdot\,)\in W^{1,1}[t_0,T]} g(x(T)) + \int_{t_0}^{T} L(s, x(s), x'(s))\, ds, \tag{2}$$

where L is deduced from H by the Legendre–Fenchel transform (when H depends on u the relation will be explained later on).

Our aim consists in proving the existence and uniqueness of the solution of (1), even in the semicontinuous case. We split this problem in two questions:

Existence. We consider the value function associated with (2) and we prove that it is a solution of (1).

Uniqueness. The common method in the theory of viscosity solutions is to prove that any supersolution is greater or equal than a subsolution, which is called a comparison result. We proceed in a slightly different way. We prove that any supersolution is greater or equal than the value function, and that the value function is greater or equal than all subsolutions. So we obtain a comparison result between sub/super solutions, but the introduction of the value function is crucial in this approach. It has been exploited by Frankowska for Bellman's equations arising in control theory by using semicontinuous solutions, and by Subbotin for the Isaacs equations by using minimax solutions. But in these authors' works the Hamiltonian was of linear growth with respect to the last variable; this assumption does not cover the case of calculus of variations, which is our interest. We consider Hamiltonians that are convex with respect to the last variable but we do not assume the linear growth. This covers the case of problems appearing in calculus of variations. At the moment we mean that in calculus of variations problems the Lagrangian $L(t, x, u, v)$ corresponding to the Hamitonian $H(t, x, u, p)$ is estimated from below by a superlinear function ϕ

$$L(t, x, u, v) \geq \phi(|v|)$$

and in optimal control problems $L(t, x, u, v) = \infty$ outside some bounded domain of v.

Before explaining the key point of representation formulas, let us recall the ideas of semicontinuous solutions for Hamiltonians coming from control prolems.

A lower semicontinuous solution is a function U satisfying some boundary conditions and such that for every $(t,x) \in \mathrm{Dom}(U)$, $t < T$,

$$p_t + H(t,x,U(t,x),p_x) = 0 \quad \text{for all } (p_t,p_x) \in \partial_- U(t,x).$$

Barron–Jensen and Frankowska ([6], [14]) proved the existence and uniqueness of such solutions for equations arising in control problems. Our approach is more related to Frankowska's one who noticed that definitions of sub and super solutions are equivalent to some monotony properties of the function U along trajectories of the control system (more precisely that the epigraph of U is forward viable and backward invariant, cf. [2]).

In the present paper we consider the following dynamical system

$$\text{(3)} \qquad \begin{cases} u'(t) \le -L(t,x(t),u(t),x'(t)) \quad \text{a.e. } t \in [t_0,T], \\ x(t_0) = x_0,\ u(t_0) = u_0, \end{cases}$$

whose solutions are pairs of absolutely continuous functions $(x(\,\cdot\,),u(\,\cdot\,))$. Remark that system (3) is in an implicit form and it is quite natural to interpret it as a differential inclusion. This interpretation will be of constant use throughout the paper. With the dynamical system (3) we associate the value function

$$\text{(4)} \qquad V(t_0,x_0) := \inf\{u_0 : \text{there exists a solution of (3) such that } u(T) \ge g(x(T))\}.$$

Remark that when H does not depend on u, (4) becomes

$$\text{(5)} \qquad V(t_0,x_0) = \min_{x(\,\cdot\,)\in W^{1,1}[t_0,T],\ x(t_0)=x_0} g(x(T)) + \int_{t_0}^{T} L(s,x(s),x'(s))\,ds.$$

We prove that value function (4) is the unique l.s.c. solution of (1). This task is based on the invariance and viability properties of the epigraph of V. The crucial point lies in proving new invariance and viability theorems for equations of type (3) viewed as unbounded differential inclusions. Since sets of solutions of such unbounded differential inclusions are not compact[1] we need another way to tackle this difficulty. We wish to thank Halina Frankowska, who brought our attention to Tonelli's Theorem, for overcoming the difficulty of noncompactness of solutions of (3).

Finally, we obtain a new result on the existence and uniqueness of lower semicontinuous solutions of (1) using the crucial interpretation (4) through the system (3) which reduces to the classical calculus of variations problem (5) when H is u-independent. Let us point out that our result covers the classical case of control problems with lower semicontinuous cost function

$$\text{(6)} \qquad V(t_0,x_0) = \min_{x'(t)=f(x(t),u(t)),\ u:[t_0,T]\mapsto U,\ x(t_0)=x_0} g(x(T))$$

[1] This fact is crucial in control theory.

with bounded set of controls U(cf. for instance [14] for detailed hypothesis).

Recently in [16] , [12] and [13] some results where obtained for the calculus of variations case *i.e. when H does not depend on u.* The existence and uniqueness for (1) were proved in [12] and [13] in the sense of epiderivative solution, and in [16] in the sense of subgradient solutions, whilst our method use proximal solutions. Note that neither [12] nor [13] cover the control case (6) because their assumptions imply that the value function is locally Lipschitz (except possibly at $t = 0$) which is no longer true in the control case (cf. [14]).

The results of the present paper were announced in [19].

Let us explain how the paper is organized. First section contains preliminaries and statements of regularity results for the value function in the calculus of variations.

The second section is devoted to the existence and uniqueness to the PDE through a representation formula based on some new results of viability theory.

To facilitate access to the paper, most technical proofs are postponed to the Appendix devoted to the viability and invariance results.

2. A calculus of variations problem

2.1. Assumptions. Having a Hamiltonian $H\colon [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ we can define a Lagrangian $L\colon [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. These function are related by the following formulas

$$H(t,x,u,p) = \inf_{f\in\mathbb{R}^n} \langle f,p\rangle + L(t,x,u,f), \tag{7}$$

$$L(t,x,u,f) = \sup_{p\in\mathbb{R}^n} \langle p,f\rangle + H(t,x,u,-p), \tag{8}$$

which can be viewed as Legendre–Fenchel's transform. Indeed, we have

$$H(t,x,u,p) = -\sup_{f\in\mathbb{R}^n} \langle f,-p\rangle - L(t,x,u,f)$$

which means that $H(t,x,u,p) = -L^*(t,x,u,-p)$, where * denotes the Legendre–Fenchel transform (cf. [21]) with respect to the last variable. Having a concave Hamiltonian by the inverse procedure we can obtain the starting Lagrangian, i.e. $L(t,x,u,f) = (-h)^*(t,x,u,-f)$.

Let us describe the assumptions needed in this paper.

- (A1) $L(t,x,u,f)$ is lower semicontinuous.
- (A2) $L(t,x,u,f) \geq 0$ for all t, x, u, f.
- (A3) $L(t,x,u,f)$ is convex and proper with respect to f for every t, x, u.
- (A4) $L(t,x,u,f)$ is nonincreasing with respect to u.
- (A5) For every $u \in \mathbb{R}$, there exists a convex function $\phi\colon [0,\infty) \to (-\infty,\infty]$, such that $\lim_{r\to\infty} \phi(r)/r = \infty$ and there exists $C > 0$ such that for all

$t \in [0,T]$, for all $x \in \mathbb{R}^n$, and for all $|f| > C(1+|x|)$,

$$L(t,x,u,f) \geq \phi(|f|).$$

(A6) For every $r > 0$ there exists $C > 0$ such that for all $(t_0, x_0, u_0) \in [0,T) \times rB \times [-r,r]$ and $v_0 \in \operatorname{Dom} L(t_0, x_0, u_0, \cdot\,)$ we have for all $(t,x,u) \in [0,T) \times B^n(r) \times [-r,r]$, there exists $v \in \operatorname{Dom} L(t,x,u,\cdot\,)$

(a) $|v - v_0| \leq C(1 + |v_0| + L(t_0,x_0,u_0,v_0))(|t-t_0| + |x-x_0| + |u-u_0|)$,

(b) $L(t,x,u,v) \leq L(t_0,x_0,u_0,v_0) + C(1 + |v_0| + L(t_0,x_0,u_0,v_0))$
$\cdot(|t-t_0| + |x-x_0| + |u-u_0|)$,

where B denotes the n-dimensionnal unit ball.

The assumptions are written as properties of L. Some of them can be reformulated equivalently using the Hamiltonian H only.

Proposition 2.1. *Suppose that H, L are related by* (7), (8). *Then Lagrangian $L(t,x,u,f)$ satisfies* (A1)–(A5) *if and only if the following conditions hold true*

(a) *$H\colon [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous,*

(b) *$H(t,x,u,0) \geq 0$ for every t, x, u,*

(c) *$H(t,x,u,p)$ is concave with respect to p,*

(d) *$H(t,x,u,p)$ is nonincreasing with respect to u,*

(e) *For every $u \in \mathbb{R}$ there exists a concave, finite everywhere function $\psi\colon [0,\infty) \to \mathbb{R}$ and a constant $C > 0$ such that, for all t, x, p,*

$$H(t,x,u,p) \geq \min(\psi(|p|), -C(1+|x|)|p|).$$

Proof. As a direct conclusion from (7), (8) we obtain that (A2)–(A4) are equivalent to (b)–(d), respectively.

To prove (a) let us choose $t_n \to t$, $x_n \to x$, $u_n \to u$, $p_n \to p$. Let $\overline{u}$, r be upper bounds of u_n and $|x_n|$, respectively. Thus $L(t_n,x_n,u_n,f) \geq L(t_n,x_n,\overline{u},f)$ for every n and f. By (A5), we choose ϕ, C for $u = \overline{u}$. Then

$$L(t_n,x_n,u_n,f) \geq \phi(|f|) \quad \text{for } |f| > C(1+r).$$

Thus there exists a bounded sequence f_n such that

$$H(t_n,x_n,u_n,p_n) = \langle p_n, f_n\rangle + L(t_n,x_n,u_n,f_n).$$

For a convergent subsequence (denoted again f_n) $f_n \to f$ we obtain

$$\begin{aligned}\liminf_{n\to\infty} H(t_n,x_n,u_n,p_n) &= \langle p,f\rangle + \liminf_{n\to\infty} L(t_n,x_n,u_n,f_n)\\ &\geq \langle p,f\rangle + L(t,x,u,f) \geq H(t,x,u,p).\end{aligned}$$

To prove (e) we set $\psi(a) = \inf_{b\geq 0} -ab + \phi(b)$ for $a \geq 0$. Hence

$$\begin{aligned} H(t,x,u,p) &\geq \min\big(\inf_{|f|\geq C(|x|+1)} \langle p,f\rangle + \phi(|f|), \inf_{|f|<C(|x|+1)} \langle p,f\rangle\big) \\ &\geq \min\big(\inf_{|f|\geq C(|x|+1)} -|p||f| + \phi(|f|), -C|p|(|x|+1)\big) \\ &\geq \min(\psi(|p|), -C|p|(1+|x|)). \end{aligned}$$

Now, we shall prove that the properties of H imply (A1), (A5). At first we show that L is lower semicontinuous. Let $t_n \to t$, $x_n \to x$, $u_n \to u$, $f_n \to f$ and $\varepsilon > 0$. There exists p such that

$$L(t,x,u,f) < \langle p,f\rangle + H(t,x,u,-p) + \varepsilon.$$

Since H is finite valued and lower semicontinuous then for sufficiently large n we have

$$H(t_n,x_n,u_n,p) \geq H(t,x,u,p) - \varepsilon.$$

Thus

$$\begin{aligned} L(t_n,x_n,u_n,f_n) &\geq \langle p,f_n\rangle + H(t_n,x_n,u_n,-p) \\ &\geq \langle p,f\rangle + H(t,x,u,-p) - 2\varepsilon \geq L(t,x,u,f) - 3\varepsilon. \end{aligned}$$

Now, we shall prove (A5). For $|f| > C(1+|f|)$ we have

$$\begin{aligned} L(t,x,u,f) &\geq \sup_{p\in\mathbb{R}^n} \langle p,f\rangle + \min(\psi(|p|), -C(|x|+1)|p|) \\ &= \min\big(\sup_{p\in\mathbb{R}^n} \langle p,f\rangle + \psi(|p|), \sup_{p\in\mathbb{R}^n} \langle p,f\rangle - C(|x|+1)|p|\big) \\ &= \min(\varphi(|f|), \infty) \end{aligned}$$

where $\varphi(b) = \sup_{a\geq 0} ab + \psi(a)$. The function $\varphi\colon [0,\infty) \to (-\infty,\infty]$ is convex and it is of superlinear growth. To show that $\lim_{b\to\infty} \varphi(b)/b = \infty$ let us suppose the contrary that there exists $C > 0$ such that

$$\sup_{a\geq 0} ab + \psi(a) \geq Cb \quad \text{for all } b \geq 0.$$

Then we obtain that $\psi(a) = -\infty$ for $a > C$ which is a contradiction with the assumption that ψ is finite everywhere (comp. Corollary 13.3.1 in [20]). $\square$

Let us recall that the Fenchel transform of the sum of convex functions h_1, $h_2\colon \mathbb{R}^n \to \mathbb{R}$ equals to the episum (inf-convolution) of its transformations, i.e.

$$(h_1+h_2)^* = h_1^* \sharp h_2^*$$

where $h_1^*\sharp h_2^*(f) = \inf_{f_1+f_2=f} h_1^*(f_1) + h_2^*(f_2)$ Thus, for $h_3(p) = h_2(p) + (1+|p|)\varepsilon$ we obtain

$$h_3^*(f) = \inf_{|g-f|<\varepsilon} h_2^*(g) - \varepsilon$$

i.e. h_3^* is the inf-convolution of h_1^* and σ_ε, where

$$\sigma_\varepsilon(f) = \begin{cases} -\varepsilon & \text{if } |f| \le \varepsilon, \\ \infty & \text{elsewhere.} \end{cases}$$

Below we provide a sufficient condition for (A6) formulated as a property of the Hamiltonian.

Proposition 2.2. *If* $H\colon [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *is concave with respect to the last variable and, for all* $p \in \mathbb{R}^n$,

$$|H(t_1, x_1, u_1, p) - H(t_2, x_2, u_2, p)| \le C(|t_1 - t_2| + |x_1 - x_2| + |u_1 - u_2|)(1 + |p|)$$

then, for all $f_1 \in \mathrm{dom}(L(t_1, x_1, u_1, \cdot\,))$, *there exists* $f_2 \in \mathrm{dom}(L(t_2, x_2, u_2, \cdot\,))$,

$$\begin{aligned} |f_1 - f_2| &\le C(|t_1 - t_2| + |u_1 - u_2| + |x_1 - x_2|), \\ L(t_2, x_2, u_2, f_2) &\le L(t_1, x_1, u_1, f_1) + C(|t_1 - t_2| + |u_1 - u_2| + |x_1 - x_2|), \end{aligned}$$

where L *is given by* (8).

Proof. We set $h_1(p) = -H(t_1, x_1, u_1, p)$, $h_2(p) = -H(t_2, x_2, u_2, p)$, $h_3(p) = h_1(p) + C(|t_1 - t_2| + |u_1 - u_2| + |x_1 - x_2|)(1 + |p|)$. Since $h_1 \le h_3$ then $h_1^* \ge h_3^*$. Hence

$$h_1^*(f) \ge \inf_{|g - f| \le C(|t_1 - t_2| + |u_1 - u_2| + |x_1 - x_2|)} h_2^*(g) - C(|t_1 - t_2| + |u_1 - u_2| + |x_1 - x_2|).$$

Thus, for every f there exists g such that

$$\begin{aligned} |g - f| &\le C(|t_1 - t_2| + |u_1 - u_2| + |x_1 - x_2|), \\ h_2^*(g) &\le h_1^*(f) + C(|t_1 - t_2| + |u_1 - u_2| + |x_1 - x_2|). \end{aligned}$$

Since $L(t_i, x_i, u_i, f) = h_i^*(-f)$ for $i = 1, 2$, we obtain the assertion of the proposition. $\square$

Below we provide an example of discontinuous Lagrangians satisfying assumptions (A1)–(A6).

Suppose that $L\colon [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ satisfies:

(L1) $L(t, x, u, v)$ is locally Lipschitz,
(L2) $L(t, x, u, \cdot\,)$ is convex,
(L3) $L(t, x, \cdot\,, v)$ is nonincreasing,

and let a set-valued map $F\colon [0,T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be as follows

(F1) $F(t, x, u)$ is nonempty convex compact for every (t, x, u),
(F2) F is locally Lipschitz,
(F3) $F(t, x, u_1) \subset F(t, x, u_2)$ for $u_1 < u_2$.

We define a new Lagrangian $L_F\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n \to [0,\infty]$ (it takes also the value ∞) by

$$L_F(t,x,u,v) = \begin{cases} L(t,x,u,v) & \text{if } v\in F(t,x,u),\\ \infty & \text{elsewhere.}\end{cases} \tag{9}$$

The Lagrangian L_F satisfies (A1)–(A4). If additionally F is of linear growth i.e.

$$\forall u,\ \exists C,\ \forall t,\ x,\ \ |F(\,\cdot\,,\,\cdot\,,u)| \le C(1+|x|) \tag{10}$$

then also (A5), (A6) hold true and moreover, L_F is a Lipschitz regular minimizers Lagrangian (see Section 2.3).

2.2. Value function. Let us define the dynamical system we associate with (1):

Definition 2.3 (Dynamical system). Let us consider the following differential inclusion:

$$(x'(t),u'(t)) \in \widetilde{Q}(t,x(t),u(t)) \quad \text{for almost all } t\in[t_0,T] \tag{11}$$

where $\widetilde{Q}\colon [0,T]\times\mathbb{R}^n\times\mathbb{R} \rightsquigarrow \mathbb{R}^n\times\mathbb{R}$ denotes the following set-valued map

$$\widetilde{Q}(t,x,u) = \{(f,-\eta) : \eta \ge L(t,x,u,f)\}. \tag{12}$$

Let $S_{\widetilde{Q}}(t_0,x_0,u_0)$ denote the set of all absolutely continuous solutions $(x,u)\colon [t_0,T]\to\mathbb{R}^n\times\mathbb{R}$ to (11) such that $x(t_0)=x_0$, $u(t_0)=u_0$.

Let us remark that the above differential inclusion is an equivalent way to write (3).

Properties (A1)–(A5) imply that the set-valued map $\widetilde{Q}$ has the following property (cf. Section 8.5.A in [10]):

$$\widetilde{Q}(t,x,u) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\{\widetilde{Q}(t,x,u;\varepsilon)\} \tag{Q}$$

where

$$\widetilde{Q}(t,x,u;\varepsilon) := \bigcup_{|t'-t|<\varepsilon,\,|x'-x|<\varepsilon,\,|u'-u|<\varepsilon} \widetilde{Q}(t',x',u').$$

Later on property (Q) will be used for the existence of solution of the above differential inclusion.

We define a function $V\colon [t_0,T]\times\mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ by (we set $\inf\emptyset=\infty$)

$$V(t_0,x_0) := \inf\{u_0 :\ \exists (x(\,\cdot\,),u(\,\cdot\,)) \in S_{\widetilde{Q}}(t_0,x_0,u_0),\ u(T)\ge g(x(T))\}. \tag{13}$$

When L does not depend on u, the definition of V is reduced to (5). We now prove some regularity of the value function:

Proposition 2.4. *If $L(t,x,u,f)$ satisfies* (A1)–(A5) *and $g:\mathbb{R}^n \to \mathbb{R}$ is a lower semicontinuous function bounded from below then the function V given by* (13) *is lower semicontinuous.*

The above proposition is a direct consequence of the following

Lemma 2.5. *Suppose that $L(t,x,u,f)$ satisfies* (A1)–(A5) *and a sequence of initial conditions (t_{n0},x_{n0},u_{n0}) is convergent to (t_0,x_0,u_0). If $(x_n(\,\cdot\,),u_n(\,\cdot\,)) \in S_{\widetilde{Q}}(t_{n0},x_{n0},u_{n0})$ and $u_n(T) \geq K$ for every n then there exist a subsequence (x_{n_k},u_{n_k}) ,$v_0 \leq u_0$ and a solution $(x(\,\cdot\,),v(\,\cdot\,)) \in S_{\widetilde{Q}}(t_0,x_0,v_0)$ such that*

$$\lim_{k\to\infty} x_{n_k}(t) = x(t), \quad \liminf_{k\to\infty} u_{n_k}(t) \leq v(t)$$

for every $t \in (t_0,T]$.

We shall use in the sequel the following version of the Tonelli–Nagumo theorem (comp. Theorem 10.3.i in [10]).

Proposition 2.6. *Suppose that $\phi:\mathbb{R}^+ \to \mathbb{R}^+$ is a superlinear growth function, i.e. $\lim_{r\to\infty}\phi(r)/r = \infty$ and C_1, C_2, C_3 are positive constants. Let $X := \{x_\alpha : [a_\alpha,b_\alpha] \to \mathbb{R}^n,\ \alpha \in \Lambda\}$ be a family of absolutely continuous functions satisfying*

$$\int_{A_\alpha} \phi(|x_\alpha'(t)|)\,dt \leq C_1 \tag{14}$$

where $A_\alpha = \{t \in [a_\alpha,b_\alpha] : |x_\alpha'(t)| > C_2(1+|x_\alpha(t)|)\}$,

$$\inf\{|x_\alpha(t)| : t \in [a_\alpha,b_\alpha]\} \leq C_3 \tag{15}$$

and for all $\alpha \in \Lambda$, $[a_\alpha,b_\alpha] \in [c,d]$. Then the family $\{x_\alpha'(\,\cdot\,) : \alpha \in \Lambda\}$ is equiabsolutely integrable, i.e.

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall \alpha \in \Lambda\ \forall E \subset [a_\alpha,b_\alpha]\text{-measurable } |E| < \delta \Rightarrow \int_E |x_\alpha'(t)|\,dt < \varepsilon.$$

Proof. At first we prove that the family X is equibounded. We fix $x_\alpha \in X$ and we shall skip the subscript α to simplify the notation. Let $\bar{t} \in [a,b]$ be choosen in such a way that $|x(\bar{t})| \leq C_3$. For $t \geq \bar{t}$ we define

$$f(t) = \int_{A\cap[\bar{t},t]} |x'(s)|\,ds \quad g(t) = \int_{[\bar{t},t]\setminus A} |x'(s)|\,ds.$$

The functions f, g are nonnegative, nondecreasing and continuous. There exists r_0 such that $\phi(r) > r$ for $r > r_0$. We have

$$\int_A |x'(s)|\,ds = \int_{A_{r_0}} |x'(s)|\,ds + \int_{A_{r_0}'} |x'(s)|\,ds \leq r_0(b-a) + C_1 =: C_5$$

where $A_{r_0} = \{t \in A : |x'(t)| \le r_0\}$ and $A'_{r_0} = A \setminus A_{r_0}$. Thus f is bounded by C_5. We have

$$|x(t)| \le |x(\bar t)| + \int_{\bar t}^{t} |x'(s)|\,ds = |x(\bar t)| + f(t) + g(t) \le C_3 + C_5 + g(t).$$

Hence

$$g(t) = \int_{[\bar t,t]\setminus A} |x'(s)|\,ds \le \int_{[\bar t,t]\setminus A} C_2(1 + C_3 + C_5 + g(s))\,ds.$$

By Gronwall's Lemma,

$$g(t) \le C_6 e^{C_2(t-\bar t)}, \quad \text{where } C_6 := C_2(1 + C_3 + C_5)(b-a).$$

Thus

$$|x(t)| \le C_3 + C_5 + C_6 e^{C_2(b-a)} =: C_4.$$

Let $B = [a,b] \setminus A$. If $t \in B$ then $|x'(t)| \le C_2(1 + C_4)$. Let $M = 3C_1/\varepsilon$. We choose $R > 0$ such that $\phi(r) \ge rM$ for $r > R$. We have

$$\int_{A'_R} \phi(|x'(s)|)\,ds \ge M \int_{A'_R} |x'(t)|\,dt.$$

Hence

$$\begin{aligned}\int_E |x'(t)|\,dt &= \int_{E\cap B} |x'(t)|\,dt + \int_{E\cap A_R} |x'(t)|\,dt + \int_{E\cap A'_R} |x'(s)|\,ds \\ &\le |E| C_2(1 + C_4) + |E|R + \frac{\varepsilon}{3}.\end{aligned}$$ □

Proof of Lemma 2.5. Let ϕ, C be choosen by (A5) for $u = u_0 + 1$. Setting

$$A_n = \{t \in [t_{n0}, T] : |x'_n(t)| > C(1 + |x_n(t)|)\}$$

by (A1), (A2), (A4) we obtain

$$\int_{A_n} \phi(|x'_n(t)|)\,dt \le \int_{t_{n0}}^{T} L(t, x_n(t), u_n(t), x'_n(t))\,dt \le u_n(t_{n0}) - u_n(T).$$

Since $u_n(T) \ge K$ and the sequence $(u_n(t_{n0}))$ is convergent then there exists C_1 such that

$$\int_{t_{n0}}^{T} \phi(|x'_n(t)|)\,dt < C_1.$$

By Proposition 2.5, the family $\{x'_n\}$ is equiabsolutely integrable. Thus

$$\lim_{n\to\infty} |x_n(t_n) - x_n(t_{n0})| = 0$$

for arbitrary $t_n \to t_0$, $t_n \in [t_{n0}, T]$.

Since $u_n(t_{0n}) \ge u_n(t_n) > C$ then for a subsequence (denoted again $u_n(t_n)$) we can set $v_0 = \lim_{n\to\infty} u_n(t_n)$. Without loss of generality we can assume that $t_0 \ge t_{n+1} \ge t_n$. Now, we extend x_n, u_n from $[t_n, T]$ to $[t_0, T]$ setting $x_n(t) = x_n(t_n)$, $u_n(t) = u_n(t_n)$ for $t \in [t_0, t_n)$.

The family $\{u_n\}$ is equibounded and each u_n is a nonincreasing function. By the Helly Theorem there exists a subsequence (denoted again by) u_n such that u_n converges pointwise to a nonincreasing function u.

By the Alaoglu and Dunford–Pettis Theorem there exists a subsequence (denoted again by) x_n such that x_n converges uniformely to an absolutely continuous function $x\colon [t_0,T] \to \mathbb{R}^n$ and x_n' converges weakly in L^1 to x'. By the Mazur Theorem there exist $C^s_{Nk} \geq 0$; $N = 1,2,\dots$, $k = 1,\dots,N$, $\sum_{k=1}^N C^s_{Nk} = 1$ such that $\sum_{k=1}^N C^s_{Nk} x'_{s+k}$ tends in L^1 norm to x' as $N \to \infty$. For a subsequence N_n

$$y^s_n = \sum_{k=1}^{N_n} C^s_{N_n\,k} x'_{s+k} \to x' \quad \text{a.e. in } [t_0,T].$$

We set

$$\eta_n(t) = L(t, x_n(t), u_n(t), x_n'(t)), \quad \eta^s_n = \sum_{k=1}^{N_n} C^s_{N_n k} \eta_{s+k}$$

and

$$\eta^s(t) = \liminf_{n\to\infty} \eta^s_n(t), \quad \eta(t) = \liminf_{s\to\infty} \eta^s(t).$$

We claim that for every $\tau_0 \in [t_0,T]$

$$u(t_0) - \int_{t_0}^{\tau_0} \eta(t)\,dt \geq u(\tau_0). \tag{16}$$

By the Fatou Lemma we have

$$\int_{t_0}^{\tau_0} \eta(t)\,dt \leq \liminf_{s\to\infty} \int_{t_0}^{\tau_0} \eta^s(t)\,ds \leq \liminf_{s\to\infty}\liminf_{n\to\infty} \int_{t_0}^{\tau_0} \eta^s_n(t)\,dt$$

where

$$\int_{t_0}^{\tau_0} \eta^s_n(t)\,dt = \sum_{k=1}^{N_n} C^s_{N_n k} \int_{t_0}^{\tau_0} \eta_{s+k}(t)\,dt = u(t_0) - \sum_{k=1}^{N_n} C^s_{N_n k} u_{s+k}(\tau_0).$$

Since $u(\tau_0) = \lim_{n\to\infty} u_n(\tau_0)$ hence

$$\liminf_{s\to\infty}\liminf_{n\to\infty} \Big(u(t_0) - \sum_{k=1}^{N_n} C^s_{N_n k} u_{s+k}(\tau_0)\Big) = u(t_0) - u(\tau_0).$$

Thus

$$\int_{t_0}^{\tau_0} \eta(t)\,dt \leq u(t_0) - u(\tau_0)$$

which proves (16).

We will show that for almost all $t \in [t_0,T]$

$$\eta(t) \geq L(t, x(t), u(t), x'(t)). \tag{17}$$

Fix $t \in (t_0, T)$ such that $\lim_{n\to\infty} y_n^s(t) = x'(t)$ for every s. Let $\varepsilon > 0$. There exists s_0 such that $t_{s_0} < t$ and, for $s \geq s_0$, $k \geq 1$, we have

$$|x_{s+k}(t) - x(t)| < \varepsilon, \quad |u_{s+k}(t) - u(t)| < \varepsilon.$$

For $n \geq s_0$ we have

$$(x_n'(t), -\eta_n(t)) \in \widetilde{Q}(t, x_n(t), u_n(t)).$$

If $s \geq s_0$, $k \geq 1$ then

$$(x_{s+k}'(t), -\eta_{s+k}(t)) \in \widetilde{Q}(t, x(t), u(t); \varepsilon).$$

Thus

$$(y_n^s(t), -\eta_n^s(t)) \in \operatorname{co} \widetilde{Q}(t, x(t), u(t); \varepsilon)$$

and for a subsequence where $n \to \infty$ we get

$$(x'(t), -\eta^s(t)) \in \overline{\operatorname{co}}\, \widetilde{Q}(t, x(t), u(t); \varepsilon).$$

For a subsequence where $s \to \infty$ recalling that $\varepsilon > 0$ was arbitrary we obtain

$$(x'(t), -\eta(t)) \in \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\widetilde{Q}\,(t, x(t), u(t); \varepsilon).$$

Since $\widetilde{Q}$ has property (Q) we obtain (17). By (17), (16) and (A4),

$$\eta(t) \geq L(t, x(t), u(t_0) - \int_{t_0}^{t} \eta(\tau) d\tau, x'(t)).$$

Setting $v(t) = u(t_0) - \int_{t_0}^{t} \eta(\tau)\, d\tau$ we get $v'(t) \leq -L(t, x(t), v(t), x'(t))$ for a.e. t in $[t_0, T]$. By (16), $v(t) \geq u(t)$. □

2.3. Minimizers. Let a Lagrangian L be given. We say that a pair (x, u) of absolutely continuous functions with values in $\mathbb{R}^n$ and $\mathbb{R}$ respectively is an L-solution on $[t_0, t_1]$ if for a.a. $t \in [t_0, t_1]$ we have

$$u'(t) \leq -L(t, x(t), u(t), x'(t)). \tag{18}$$

Note that (x, u) is an L-solution on $[t_0, t_1]$ if and only if $(x'(t), u'(t)) \in \widetilde{Q}(t, x(t), u(t))$ for almost all $t \in [t_0, t_1]$. To deal with the calculus of variations problem, let us introduce some useful definitions:

Definition 2.7. An L-solution $(\overline{x}, \overline{u})$ on $[t_0, t_1]$ is an *L-minimizer* on $[t_0, t_1]$ if for every L-solution (x, u) on $[t_0, t_1]$ such that $x(t_0) = \overline{x}(t_0)$, $x(t_1) = \overline{x}(t_1)$, $u(t_1) = \overline{u}(t_1)$ we have

$$u(t_0) \geq \overline{u}(t_0).$$

Lagrangian L is a *Lipschitz regular minimizer Lagrangian* if for every L-minimizer $(\overline{x}, \overline{u})$ on $[t_0, t_1]$, the function $\overline{x}$ is Lipschitz continuous on $[t_0, t_1]$.

REMARK 1. If $(\overline{x}, \overline{u})$ is an L-minimizer on $[t_0, t_1]$ and $[t'_0, t'_1] \subset [t_0, t_1]$, then $(\overline{x}, \overline{u})$ is an L-minimizer on $[t'_0, t'_1]$.

REMARK 2. If the Lagrangian L does not depend to u and $\overline{x}(\,\cdot\,)$ is a solution for a classical calculus of variations problem

$$\mathrm{Min}\bigg\{ \int_{t_0}^{t_1} L(t, x(t), x'(t))\, dt : x(t_0) = x_0,\ x(t_1) = x_1 \bigg\},$$

then $(\overline{x}, \overline{u})$ is an L minimizer, where $\overline{u}(t) = \int_t^{t_1} L(s, \overline{x}(s), \overline{x}'(s))\, ds + C$ and C is an arbitrary real constant.

PROPOSITION 2.8. *If $L(t, x, u, f)$ satisfies* (A1)–(A5)*, and $g \colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous function bounded from below then for every $(t_0, x_0) \in \mathrm{Dom}(V)$ there exists $(x(\,\cdot\,), u(\,\cdot\,)) \in S_{\widetilde{Q}}(t_0, x_0, V(t_0, x(t_0))$ such that $u(T) \geq g(x(T))$. Consequently (x, u) is an L-minimizer.*

The proof is the direct conclusion from Lemma 2.5.

Several sufficient conditions for Lipschitz regularity of minimizers for Lagrangians L not depending to u are provided by Cesari ([10]), Clarke–Vinter ([11]), Ambrosio–Ascendi–Buttazzo ([1]), see also the survey article [8].

Recently, Dal Maso and Frankowska ([12], [13]) studied value functions for Bolza problems with discontinuous Lagrangians $L(x, f)$ which are Lipschitz regular minimizers Lagrangians acording to [1].

3. Hamilton–Jacobi equations related to calculus of variations problems

The crucial fact for the further considerations is that the value function V is an extended lower semicontinuous function what has been obtained in Proposition 2.4.

3.1. Supersolutions and subsolutions of the Hamilton–Jacobi equations. Studying forward viability properties of solutions of differential inclusion (11) we shall obtain that V is a supersolution to

$$\frac{\partial U}{\partial t} + H\bigg(t, x, U, \frac{\partial U}{\partial x}\bigg) = 0, \tag{19}$$

in the sense of the following definition.

DEFINITION 3.1. Let $U \colon (0, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be an extended lower semicontinuous function. We say that U is an *epiderivative supersolution* if for every $(t, x) \in \mathrm{Dom}(U)$, $t < T$, there exists $v \in \mathrm{Dom}\, L(t, x, U(t, x), \,\cdot\,)$

$$D_{\uparrow} U(t, x)(1, v) \leq -L(t, x, U(t, x), v),$$

U is a *proximal supersolution* if for every $(t,x) \in \mathrm{Dom}(U)$, $t < T$

$$\forall (n_t, n_x, n_u) \in N_{\mathrm{Epi}(U)}(t,x,U(t,x)),\ \exists v \in \mathrm{Dom}\, L(t,x,U(t,x),\,\cdot\,): \\ n_t + \langle n_x, v\rangle - n_u L(t,x,U(t,x),v) \le 0.$$

U is a *subdifferential supersolution* if for every $(t,x) \in \mathrm{Dom}(U)$, $t < T$

$$p_t + H(t,x,U(t,x),p_x) \le 0 \quad \text{for all } (p_t,p_x) \in \partial_- U(t,x)$$

where the subdifferential of the lower semicontinuous function $w\colon \mathbb{R}^n \to \mathbb{R}$ at $x_0 \in \mathbb{R}^n$ is given by

$$\partial_- w(x_0) = \bigg\{ p \in \mathbb{R}^n : \liminf_{x \to x_0} \frac{w(x) - w(x_0) - \langle p, x - x_0\rangle}{\|x - x_0\|} \ge 0 \bigg\}.$$

The epiderivative of w in the direction u is given by

$$D_\uparrow w(x)(u) := \liminf_{h \to 0^+,\, u' \to u} \frac{w(x + hu') - w(x)}{h}$$

(see [4]). A proximal normal at a point x belonging to a closed set Z of $\mathbb{R}^d$ is a vector $p \in \mathbb{R}^d$ such that

$$\exists a > 0,\ \mathrm{dist}(x + ap, Z) = a|p|.$$

The set of such proximal normals is denoted by $N_Z(x_0)$. This notion was introduced by Bony in [7] and used in [9] to study viability property and its extensions to Differential Games.

Studying backward invariance properties of solutions of (11) we obtain that V is a subsolution (see the definition below).

Definition 3.2. Let $U\colon (0,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be an extented lower semicontinuous function. We say that U is an *epiderivative subsolution* if for every $(t,x) \in \mathrm{Dom}(U)$, $t < T$,

$$D_\uparrow U(t,x)(-1,-v) \le L(t,x,U(t,x),v) \quad \text{for all } v \in \mathrm{Dom} L(t,x,U(t,x),\,\cdot\,).$$

U is a *proximal subsolution* if for every $(t,x) \in \mathrm{Dom}(U)$, $t < T$,

$$-n_t + \langle n_x, -v\rangle + n_u L(t,x,U(t,x),v) \le 0$$

for all $(n_t,n_x,n_u) \in N_{\mathrm{Epi}(U)}(t,x,U(t,x))$ and all $v \in \mathrm{Dom}\, L(t,x,U(t,x),\,\cdot\,)$.

U is a *subdifferential supersolution* if for every $(t,x) \in \mathrm{Dom}(U)$, $t < T$,

$$p_t + H(t,x,U(t,x),p_x) \ge 0 \quad \text{for all } (p_t,p_x) \in \partial_- U(t,x).$$

Let us state some comparison statements for these different concept of solutions.

PROPOSITION 3.3. *Let us restrict our attention to extended lower semicontinuous functions.*

(a) *Every epiderivative sub/super solutions is a proximal sub/supersolutions* (*respectively*)
(b) *Every proximal sub/supersolution is a subdifferential sub/supersolution* (*respectively*).
(c) *If L satisfies* (A1)–(A6)*, then every subdifferential subsolution is a proximal subsolution.*
(d) *Assume that for any (t,x,u) the domain of $L(t,x,u,\,\cdot\,)$ is bounded. If L satisfies* (A1)–(A6)*, then every subdifferential supersolution is a proximal supersolution.*

PROOF. Statements (a) and (b) are consequences of well-known relation between subdifferentials, proximal normals to epigraphs and epiderivatives (cf. [21]). To prove the third statement we shall need the following Lemma due to Rockaffellar (cf. [14] for instance). $\square$

LEMMA 3.4. *Let $\phi\colon \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function and $y_0 \in \mathrm{Dom}(\phi)$. If $(p,0) \in N_{\mathrm{Epi}(\phi)}(y_0, \phi(y_0))$ then there exist $y_k \to y_0$, $p_k \to p$, $q_k \to 0$ such that $q_k < 0$, $\phi(y_k) \to \phi(y_0)$ and*

$$(p_k, q_k) \in [T_{\mathrm{Epi}(\phi)}(y_k, \phi(y_k)]^-.$$

PROOF. Let us consider U an extended lower semicontinuous function which is moreover a subdifferential subsolution. Let us fix $t_0 \in (0,T]$, $x_0 \in \mathbb{R}^n$ and $(n_t, n_x, n_u) \in N_{\mathrm{Epi}\,U}(t_0, x_0, U(t_0,x_0))$. If $n_u < 0$ then $(n_t/|n_u|, n_x/|n_u|) \in \partial_- U(t_0,x_0)$. By the definition of subdifferential subsolution, for every $v_0 \in \mathrm{Dom}\, L(t_0,x_0,U(t_0,x_0),\,\cdot\,)$ we have

$$-n_t - \langle n_x, v\rangle + n_u L(t_0, x_0, U(t_0,x_0), v_0) \leq 0.$$

Now consider the case $n_u = 0$. By Lemma 3.4, there exist $t_k \to t_0$, $x_k \to x_0$ and $(n_{t_k}, n_{x_k}, n_{u_k}) \to (n_t, n_x, n_u)$ such that $n_{u_k} < 0$, $U(t_k,x_k) \to U(t_0,x_0)$ and

$$(n_{t_k}, n_{x_k}, n_{u_k}) \in [T_{\mathrm{Epi}(U)}(t_k, x_k, U(t_k,x_k))]^-.$$

Denote $d_k = |t_k - t_0| + |x_k - x_0| + |U(t_k,x_k) - U(t_0,x_0)|$. By (A6), there exist v_k such that

$$\begin{aligned} |v_k - v_0| &\leq C(1 + |v_0| + L(t_0,x_0,U(t_0,x_0),v_0))d_k, \\ L(t_k,x_k,U(t_k,x_k),v_k) &\leq L(t_0,x_0,U(t_0,x_0),v_0) \\ &\qquad + C(1 + |v_0| + L(t_0,x_0,U(t_0,x_0),v_0))d_k. \end{aligned}$$

Thus $v_k \to v_0$ and the sequence $L(t_k, x_k, U(t_k, x_k), v_k)$ is bounded. Since $n_{u_k} < 0$ then

$$-n_{t_k} - \langle n_{x_k}, v_k \rangle + n_{u_k} L(t_k, x_k, U(t_k, x_k), v_k) \leq 0.$$

When $k \to \infty$ we obtain

$$-n_t - \langle n_x, v_0 \rangle \leq 0.$$

The prove of statement (c) is complete.

The fourth statement uses also Lemma 3.4 and arguments very similar to those developed in [14] for the control case. So we omit the proof. □

The next subsection is devoted to statements of results which proofs are direct consequences of the considerations in Sections 3.3 and 3.4.

3.2. Main results. We can state our main results in the two following Theorems:

THEOREM 3.5 (Existence). *Suppose that L is a Lipschitz regular minimizer Lagrangian and satisfies* (A1)–(A6) *and $g: \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a bounded from below lower semicontinuous function. Then the value function V given by* (13) *is together an epiderivative, proximal and subdifferential super- and subsolution, i.e. for every $(t, x) \in \mathrm{Dom}(U)$, $t < T$,*

$$p_t + H(t, x, U(t, x), p_x) = 0 \quad \textit{for all } (p_t, p_x) \in \partial_- U(t, x).$$

Moreover, for every $x \in \mathrm{Dom}(V(T, \cdot))$ there exist $x_n \to x$, $t_n \to T^-$ such that

$$\lim_{n \to \infty} V(t_n, x_n) = V(T, x) = g(x).$$

The proof of the above Theorem is postponed to Section 3.3.

We provide an example of a *non* Lipschitz regular minimizer Lagrangian L such that the corresponding value function is not an epiderivative supersolution.

For the uniqueness of solution to (1) we formulate the result in terms of proximal normal super- and subsolutions.

THEOREM 3.6 (Uniqueness). *Suppose that L satisfies assumptions* (A1)–(A6) *and $U: (0, T] \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous bounded from below and*

(20) *for every $x \in \mathrm{Dom}(U(T, \cdot))$ there exist $x_n \to x$, $t_n \to T^-$ such that*
$$\lim_{n \to \infty} U(t_n, x_n) = U(T, x).$$

Let us consider V_h the value function associated with $h(\cdot) := U(T, \cdot)$. If U is a proximal normal super- and subsolution then $U = V_h$.

The proof is postpone to Section 3.4.

Observe that in the above theorem we do not need the assumption that L is a Lipschitz regular minimizer Lagrangian. Moreover, to obtain that $U \leq V$ it suffices to assume that U is a subdifferential subsolution so we have

COROLLARY 3.7. *Under the assumptions of Theorem* 3.5*, the value function V is the greatest among subdifferential subsolutions U satisfying* (20) *and* $U(T, \cdot\,) = g(\,\cdot\,)$.

Moreover, we have the following important

COROLLARY 3.8. *Under the assumptions of Theorems* 3.5 *and* 3.6*, the value function V is the unique proximal super and subsolution satisfying* (20).

and

COROLLARY 3.9. *Assume that for any t, x, u the domain of $L(t,x,u,\cdot\,)$ is bounded. Under the assumptions of Theorems* 3.5 *and* 3.6*, the value function V is the unique subdifferential super and subsolution such that there exist $x_n \to x$, $t_n \to T^-$ and* $\lim_{n\to\infty} V(t_n,x_n) = V(T,x) = g(x)$.

When L is replaced by L_F given by (9) then all previous notions of subsolutions and supersolutions coincide. So we obtain the following

COROLLARY 3.10. *Assume that L_F is given by* (9)*, where L, F satisfy* (L1)–(L3)*,* (F1)–(F3)*,* (10) *and $g\colon \mathbb{R}^n \to \mathbb{R}$ is a lower semicontinuous function bounded from below, then the value funtion is the unique lower semicontinuous solution to the corresponding Hamilton–Jacobi equation.*

So let us point out that this corollary contains the control case (6), comp. [14].

3.3. Existence and representation formula. To obtain that the function $V\colon (0,T]\times\mathbb{R}^n \to \mathbb{R}$ given by (13) is the epiderivative supersolution of (1) we have to assume that L is Lipschitz regular minimizer Lagrangian. Below we provide a (counter-) example of Bolza problem such that the value function *is not* an epiderivative supersolution of the corresponding Hamilton–Jacobi equation.

EXAMPLE. We set

$$g(x) = \begin{cases} 0 & \text{if } x \geq 1, \\ 1 & \text{if } x < 1, \end{cases}$$

$$L(t,f) = \begin{cases} |f|^{3/2} & \text{if } f = 1/2\sqrt{t} \text{ and } t \in (0,1], \\ \infty & \text{if } f \neq 1/2\sqrt{t} \text{ and } t \in (0,1], \\ |f|^{3/2} & \text{if } t = 0, \\ 0 & \text{if } t < 0 \text{ and } f = 0, \\ \infty & \text{if } t < 0 \text{ and } f \neq 0. \end{cases}$$

The Lagrangian L satisfies (A1)–(A5). The value function $V\colon(-\infty,1]\times\mathbb{R}\to\mathbb{R}$ defined by

$$V(t_0,x_0)=\inf\int_{t_0}^{1}L(t,x'(t))\,dt+g(x(1))$$

can be directly calculated from the definition and is equal to

$$V(t_0,x_0)=\begin{cases}\sqrt{2}(1-t_0^{1/4}) & \text{if } t_0\in(0,1] \text{ and } x_0\geq\sqrt{t_0},\\ 1+\sqrt{2}(1-t_0^{1/4}) & \text{if } t_0\in(0,1] \text{ and } x_0<\sqrt{t_0},\\ \sqrt{2} & \text{if } t\leq 0 \text{ and } x_0\geq 0,\\ 1+\sqrt{2} & \text{if } t\leq 0 \text{ and } x_0<0.\end{cases}$$

For $(t_0,x_0)=(0,0)$ and for arbitrary $f\in\mathbb{R}^n$ we have

$$D_{\uparrow}V(0,0)(1,f)=\infty.$$

Thus it is *not* thrue that there exists $f\in\mathbb{R}^n$ such that

$$D_{\uparrow}V(0,0)(1,f)\leq -L(0,f)=-|f|^{3/2}.$$

Proposition 3.11. *Assume that $L\colon[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}$ is a Lipschitz regular minimizer Lagrangian and* (A1)–(A5) *hold true. If $g\colon\mathbb{R}\to\mathbb{R}\cup\infty$ is a lower semicontinuous function bounded from below then the value function V given by* (13) *satisfies the condition that for every $(t,x)\in\mathrm{Dom}(V)$, $t<T$, there exists $f\in\mathbb{R}^n$ such that*

$$D_{\uparrow}V(t,x)(1,f)\leq -L(t,x,V(t,x),f)$$

i.e. V is a supersolution (*in every meaning of Definition* 3.1) *of* (19).

Proof. By Proposition 2.4, the value function V is lower semicontinuous. Fix (t_0,x_0) such that $V(t_0,x_0)<\infty$. By Proposition 2.8, there exists an L-solution $(\widetilde{x},\widetilde{u})$ such that

$$\widetilde{u}(t_0)=V(t_0,x_0),\quad \widetilde{u}(T)\geq g(\widetilde{x}(T))\quad\text{and}\quad \widetilde{x}(t_0)=x_0.$$

By Lemma 2.5, there exists $(\overline{x},\overline{u})$ an L-minimizer on $[t_0,T]$ such that $\overline{x}=\widetilde{x}$ and $\widetilde{u}(t_0)=V(t_0,x_0)$. Since L is a Lipschitz regular minimizer Lagrangian then $x\colon[t_0,T]\to\mathbb{R}^n$ is Lipschitz continuous. Thus there exists $h_n\to 0^+$ and $f\in\mathbb{R}^n$ such that

$$\int_{t_0}^{t_0+h_n}\overline{x}(s)\,ds=\frac{\overline{x}(t_0+h_n)-\overline{x}(t_0)}{h_n}\to f.$$

We claim that

$$D_{\uparrow}V(t_0,x_0)(1,f)\leq -L(t_0,x_0,V(t_0,x_0),f).$$

Since $\overline{u}(t) \geq V(t, \overline{x}(t))$ thus

$$V(t_0 + h_n, \overline{x}(t_0 + h_n)) - V(t_0, x_0) \leq \int_{t_0}^{t_0+h_n} -L(s, \overline{x}(s), \overline{u}(s), \overline{x}'(s))\, dt.$$

Monotony properties of functions $s \mapsto \overline{u}(s)$ and $u \mapsto L(s, \overline{x}(s), u, \overline{x}'(s))$ yield

$$-L(s, \overline{x}(s), \overline{u}(s), \overline{x}'(s)) \leq -L(s, \overline{x}(s), \overline{u}(t_0), \overline{x}'(s)). \tag{21}$$

Let $r > 0$ be large enough such that $\overline{x}(t) \in rB$ and $\overline{u}(t) \in [-r, +r]$ for every $t \in [t_0, T]$. Fix $s > 0$. If we apply assumption (A6) to the point $(s, \overline{x}(s), \overline{u}(t_0), \overline{x}'(s))$ instead of (t_0, x_0, u_0, v_0) then because $(t_0, x_0, \overline{u}(t_0)) \in [0, T) \times rB \times [-r, r]$ there exists some $v_0(s) \in \operatorname{Dom} L(t_0, x_0, \overline{u}(t_0), \cdot\,)$ such that

$$\begin{aligned} |v_0(s) - \overline{x}'(s)| \leq{}& C(1 + |\overline{x}'(s)| + L(s, \overline{x}(s), \overline{u}(t_0), \overline{x}'(s)) \\ &\cdot (|t_0 - s| + |x_0 - \overline{x}(s)|), \end{aligned} \tag{22}$$

$$\begin{aligned} L(t_0, x_0, \overline{u}(t_0), v_0(s)) \leq{}& L(s, \overline{x}(s), \overline{u}(t_0), \overline{x}'(s)) \\ &+ C(1 + |\overline{x}'(s)| + L(s, \overline{x}(s), \overline{u}(t_0), \overline{x}'(s)) \\ &\cdot (|t_0 - s| + |x_0 - \overline{x}(s)|). \end{aligned} \tag{23}$$

So for any $s \in [t_0, T]$, we have

$$\begin{aligned} -L(s, \overline{x}(s), \overline{u}(s), \overline{x}'(s)) &\leq -L(s, \overline{x}(s), \overline{u}(t_0), \overline{x}'(s)) \\ &\leq -L(t_0, x_0, \overline{u}(t_0), v_0(s)) \\ &\quad + C(1 + |\overline{x}'(s)| + L(s, \overline{x}(s), \overline{u}(t_0), \overline{x}'(s)) \\ &\quad \cdot (|t_0 - s| + |x_0 - \overline{x}(s)|) \end{aligned}$$

and consequently

$$\begin{aligned} -L(s, \overline{x}(s), \overline{u}(s), \overline{x}'(s)) \leq{}& -L(t_0, x_0, \overline{u}(t_0), v_0(s)) \\ &+ C(1 + |\overline{x}'(s)| - \overline{u}'(s))(|t_0 - s| + |x_0 - \overline{x}(s)|) \end{aligned} \tag{24}$$

because $L(s, \overline{x}(s), \overline{u}(t_0), \overline{x}'(s))) \leq L(s, \overline{x}(s), \overline{u}(s), \overline{x}'(s))) \leq -\overline{u}'(s)$. From (22) we obtain

$$|v_0(s) - \overline{x}'(s)| \leq C(1 + |\overline{x}'(s)| - \overline{u}'(s))(|t_0 - s| + |x_0 - \overline{x}(s)|). \tag{25}$$

Let us remark that

$$\liminf_{n\to\infty} \frac{1}{h_n} \int_{t_0}^{t_0+h_n} (1 + |\overline{x}'(s)| - \overline{u}'(s))(|t_0 - s| + |x_0 - \overline{x}(s)|)\, ds = 0. \tag{26}$$

Indeed $(|t_0 - s| + |x_0 - \overline{x}(s)|) \leq (k+1)h_n$ (k is the Lipschitz constant of $\overline{x}$ and $s \mapsto 1 + |\overline{x}'(s)| - \overline{u}'(s)$) is integrable.

By (25) and (26), we obtain

$$\liminf_{n\to\infty} \frac{1}{h_n} \int_{t_0}^{t_0+h_n} v_0(s)\, ds = f.$$

Using (24), (26) and Jensen's inequality we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{h_n}\int_{t_0}^{t_0+h_n} -L(s,\overline{x}(s),\overline{u}(s),\overline{x}'(s))\,ds \\
&\quad\le \liminf_{n\to\infty}\frac{1}{h_n}\bigg[\int_{t_0}^{t_0+h_n} -L(t_0,x_0,\overline{u}(t_0),v_0(s))\,ds \\
&\qquad + \int_{t_0}^{t_0+h_n} C(1+|\overline{x}'(s)|-\overline{u}'(s))(|t_0-s|+|x_0-\overline{x}(s)|)\,ds\bigg] \\
&\quad\le -L(t_0,x_0,V(t_0,x_0),f)+0.
\end{aligned}$$

The proof is complete. □

Before showing that the value function is a subsolution we provide some set-valued results. Let us recall the pseudo-Lipschitz property for a set valued map (see [4]) also called Aubin's property in [21].

Definition 3.12. A set-valued map $S\colon X \rightsquigarrow \mathbb{R}^m$ ($X \subset \mathbb{R}^n$) is pseudo-Lipschitz at $x_0 \in X$ for $y_0 \in S(x_0)$ if there are constant $\varepsilon > 0$ and $l > 0$ such that

$$S(x_1)\cap B(y_0,\varepsilon) \subset S(x_2) + l|x_1-x_2|B \quad \text{for all } x_1,x_2 \in X\cap B(x_0,\varepsilon). \tag{27}$$

If S is a single valued pseudo-Lipschitz map then S is locally Lipschitz.

Proposition 3.13. *If $L\colon[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}\cup\{\infty\}$ satisfies* (A6) *then the set-valued map $\widetilde{Q}$ given by* (12) *is pseudo-Lipschitz.*

Proof. Fix t_0, x_0, u_0, $(v_0,-\eta_0) \in \widetilde{Q}(t_0,x_0,u_0)$ and choose $\varepsilon > 0$ such that $L(t,x,u,v) \le L(t_0,x_0,u_0,v_0)+1$ for $|t-t_0|<\varepsilon$, $|x-x_0|<\varepsilon$, $|u-u_0|<\varepsilon$, $|v-v_0|<\varepsilon$. We set $l = C(1+|v_0|+L(t_0,x_0,u_0,v_0)+1)$, where C is choosen by (A6) for $r=\sqrt{(|x_0|+|u_0|+\varepsilon)}$. Let us take $t_1,t_2 \in (t_0-\varepsilon,t_0+\varepsilon)$, $x_1,x_2 \in B(x_0,\varepsilon)$, $u_1,u_2 \in (u_0-\varepsilon,u_0+\varepsilon)$ and $(v_1,-\eta_1)$ such that $|v_1-v_0|<\varepsilon$, $|\eta_1-\eta_0|<\varepsilon$ and $\eta_1 \ge L(t_1,x_1,u_1,v_1)$. By (A6), there exists v_2 such that

$$\begin{aligned}
|v_2-v_1| &\le l(|t_2-t_1|+|x_2-x_1|+|u_2-u_1|),\\
L(t_2,x_2,u_2,v_2) &\le L(t_1,x_1,u_1,v_1)+l(|t_2-t_1|+|x_2-x_1|+|u_2-u_1|).
\end{aligned}$$

We set $\eta_2 = \eta_1 + l(|t_2-t_1|+|x_2-x_1|+|u_2-u_1|)$. We have

$$\eta_2 \ge L(t_1,x_1,u_1,v_1)+l(|t_2-t_1|+|x_2-x_1|+|u_2-u_1|) \ge L(t_2,x_2,u_2,v_2).$$

Thus $(v_2,-\eta_2) \in \widetilde{Q}(t_2,x_2,u_2)$. □

PROPOSITION 3.14. *Let $F: \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a pseudo-Lipschitz map with nonempty closed values. Then, for every $x_0 \in \mathbb{R}^n$, $f_0 \in F(x_0)$, there exists $T > 0$ and a C^1-solution $x: [0,T] \to \mathbb{R}^n$ to the problem*

$$\begin{cases} x'(t) \in F(x(t)), \\ x(0) = x_0, \\ x'(0) = f_0. \end{cases} \tag{28}$$

PRROF. The idea of the proof is based on ideas developed in the proof of Theorem 2.3.2. in [3].

We choose $\varepsilon > 0$ and $l > 0$ such that for all $x_1, x_2 \in B(x_0, \varepsilon)$ and for every $f_1 \in F(x_1) \cap B(f_0, \varepsilon)$ there exists $f_2 \in F(x_2)$ such that $|f_2 - f_1| \le l|x_2 - x_1|$. We choose $T > 0$ such that $e^{lT} T \max\{1, |f_0|l\} < \varepsilon$.

Fix $n \in N$. We construct a couple of sequences x_i^n, f_i^n such that

$$\begin{gathered} x_0^n = x_0, \quad f_0^n = f_0, \quad x_{i+1}^n = x_i^n + \frac{T}{n} f_i^n, \\ f_{i+1}^n \in F(x_i^n), \quad |f_{i+1}^n - f_i^n| \le l\frac{T}{n}|f_i^n|. \end{gathered} \tag{29}$$

The construction is inductive. To choose f_{i+1}^n satisfying (29) we have to know that x_i^n, $x_{i+1}^n \in B(x_0, \varepsilon)$ and $|f_i^n - f_0| < \varepsilon$. To show this let us observe that $|f_k^n| \le |f_0| e^{lk/n}$ for $k = 0, \dots, i$. Thus $|x_k^n - x_0| \le k(T/n)|f_0|e^{lT} < \varepsilon$ for $k = 1, \dots, i+1$. Moreover, $|f_i^n - f_0| \le il(T/n)|f_0|e^{l(i/n)T} < \varepsilon$. We define the functions x_n and v_n on $[0,T]$ by interpolating linearly the sequences $x_n(iT/n) = x_i^n$, $v_n(iT/n) = f_i^n$ at the node points iT/n. Every v_n is Lipschitz continuous with the constant $l|f_0|e^{lT}$.

For a subsequence we obtain that $v_n \to \overline{v}$ uniformely on $[0,T]$. Define $\overline{y}(\cdot)$ to be

$$\overline{y}(t) = x_0 + \int_0^t \overline{v}(s)\, ds.$$

By standard method (cf. [3, p. 117]) we can check that $x_n(\cdot)$ uniformely converges to $\overline{y}(\cdot)$.

It remain to show that $\overline{v}(t) \in F(\overline{y}(t)$. For a given n we choose i such that $t \in [iT/n, (i+1)T/n)$.

$$\operatorname{dist}(\overline{v}(t), F(\overline{y}(t)) \le \left|\overline{v}(t) - \overline{v}(i\frac{T}{n})\right| + \left|\overline{v}(i\frac{T}{n}) - v_n(i\frac{T}{n})\right| + \operatorname{dist}(f_i^n, F(\overline{y}(t)).$$

Since $|f_i^n - f_0| < \varepsilon$ and $|\overline{y}(t) - x_0| < \varepsilon$ then

$$\operatorname{dist}(f_i^n, F(\overline{y}(t)) \le l|x_i^n - \overline{y}(t)|.$$

Since F has closed values then $\overline{v}(t) \in F(\overline{y}(t))$. $\square$

Proposition 3.15. *Assume that $L(t,x,u,f)$ satisfies assumptions* (A1)–(A6) *and $g:\mathbb{R}^n \to \mathbb{R}$ is a lower semicontinuous function bounded from below. Then the value function given by* (13) *satisfies for every $(t,x) \in Dom(V)$, $t > 0$, $v \in \operatorname{Dom} L((t,x,V(t,x),\,\cdot\,)$*

$$D_\uparrow V(t,x)(-1,-v) \le L(t,x,V(t,x),v)$$

i.e. V is a subsolution to (19) *in every meaning of Definition* 3.2. *Moreover, for every $x_0 \in \operatorname{Dom}(g)$ there exist $x_n \to x_0$ and $t_n \to T^-$ such that*

$$V(T,x) = \lim_{n\to\infty} V(t_n,x_n).$$

Proof. We define $F:\mathbb{R}^{1+n+1} \rightsquigarrow \mathbb{R}^{1+n+1}$ setting first, for $t \in [0,T]$,

$$F(t,x,u) = \{(-1,-v,\eta) : \eta \ge L(t,x,u,v)\}$$

and extending next to $F(t,x,u) = F(T,x,u)$ for $t > T$ and $F(t,x,u) = F(0,x,u)$ for $t < 0$. By Proposition 3.13, F is pseudo-Lipschitz.

Fix $(t_0,x_0) \in \operatorname{Dom}(V)$, $v_0 \in \operatorname{Dom} L(t_0,x_0,V(t_0,x_0))$. By Proposition 3.14, there exist C^1 functions $u:(t_0-\varepsilon,t_0] \to \mathbb{R}$, $x:(t_0-\varepsilon,t_0] \to \mathbb{R}^n$ such that $y(s) = (t_0-s,x(t_0-s),u(t_0-s))$ ($s \in [0,\varepsilon)$) is a solution of differential inclusion $y'(s) \in F(y(s))$ with initial conditions $y(0) = (0,x_0,V(t_0,x_0))$, $y'(0) = (-1,-v_0,L(t_0,x_0,V(t_0,x_0),v_0))$. By Proposition 2.8, there exists an L-solution (x_1,u_1) on $[t_0,T]$ such that $x_1(t_0) = x_0$, $u_1(t_0) = V(t_0,x_0)$ and $u_1(T) \ge g(x_1(T))$. It follows that $u(t) \ge V(t,x(t))$ for $t \in (t_0-\varepsilon,t_0]$. The left-hand side derivatives of $x(\,\cdot\,)$ and $u(\,\cdot\,)$ at t_0 exist and are equal to v_0 and $-L(t_0,x_0,V(t_0,x_0),v_0)$, respectively. Thus,

$$\begin{aligned}
D_\uparrow V(t_0,x_0)(-1,-v_0) &= \liminf_{v\to v_0,\ h\to 0^+} \frac{V(t_0-h,x_0-hv_0)-V(t_0,x_0)}{h} \\
&\le \liminf_{h\to 0^+} \frac{V(t_0-h,x(t_0-h))-V(t_0,x_0)}{h} \\
&\le \lim_{h\to 0^+} \frac{u(t_0)-u(t_0)}{h} = L(t_0,x_0,V(t_0,x_0),v_0).
\end{aligned}$$

To obtain the last statement we take an arbitrary L-solution (x,u) on the interval $[T-\varepsilon,T]$ satisfying $x(T) = x_0$, $u(T) = g(x_0)$. We have

$$V(t,x(t)) \le u(t).$$

Thus

$$\limsup_{t\to T^-} V(t,x(t)) \le \lim_{t\to T^-} u(t) = g(x_0).$$

Since V is lower semicontinuous we obtain

$$\lim_{t\to T^-} V(t,x(t)) = g(x_0). \qquad \square$$

3.4. Comparison results and uniqueness.

PROPOSITION 3.16. *Suppose that a Lagrangian* $L\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}$ *satisfies* (A1), (A2), (A6) *and the Hamiltonian* H *is given by* (7). *If* $U\colon (0,T]\times\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}$ *is a lower semicontinuous function such that*

$$p_t + H(t,x,U(t,x),p_x)\geq 0$$

for all $(t,x)\in\mathrm{Dom}(U)$, $t<T$, *and for all* $(p_t,p_x)\in\partial_-U(t,x)$, *then*

$$U(t,x)\leq V_g(t,x)\quad \textit{for every } (t,x)\in(0,T]\times\mathbb{R}^n,$$

where V_g *is the value function given by* (13) *and the function* $g\colon\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}$ *is given by*

$$g(\overline{x}) = \liminf_{t\to T^-,\ x\to\overline{x}} U(t,x).$$

PROOF. Fix an L-solution (x,u) on $[t_0,T]$ such that $u(T)\geq g(x(T))$. Our aim is to prove that

$$u(t)\geq U(t,x(t))\quad\text{for every } t\in[t_0,T].$$

We set

$$l_0(t) = 1+|x'(t)|+L(t,x(t),u(t),x'(t)).$$

Let $r>0$ be choosen such that $|x(t)|<r/2$, $|u(t)|<r/2$ for $t\in[t_0,T]$. We choose $C>0$ such that (A6) holds true for r and $C/2$. Let $M>1$. There exists $\varepsilon>0$ such that

$$\varepsilon\exp\left(C\int_0^T Ml_0(t)\,dt\right)<\frac{r}{2}. \qquad\square$$

LEMMA 3.17. *If* $\tau\in(t_0,T)$, $|x_\tau - x(\tau)|<\varepsilon/2$, $|u_\tau-u(\tau)|<\varepsilon/2$ *then there exists an* L*-solution* $(\overline{x},\overline{u})$ *on* $[t_0,\tau]$ *such that* $\overline{x}(\tau)=x_\tau$, $\overline{u}(\tau)=u_\tau$ *and*

$$\begin{aligned}(31)\qquad &|\overline{x}(t)-x(t)|+|\overline{u}(t)-u(t)|\\ &\leq(|x_\tau-x(\tau)|+|u_\tau-u(\tau)|)\exp\left(CM\int_t^\tau l_0(s)\,ds\right).\end{aligned}$$

PROOF. We succesively construct functions x_n, u_n, v_n, w_n for $n=0,1,\ldots$ For $n=0$ we set $x_0(t)=x(t)$, $u_0(t)=u(t)$, $v_0(t)=x_0'(t)$, $w_0(t)=u_0'(t)$. Having $v_{n-1}(\cdot)$, $w_{n-1}(\cdot)$ we define

$$x_n(t)=x_\tau-\int_t^\tau v_{n-1}(s)\,ds,\qquad u_n(t)=u_\tau-\int_t^\tau w_{n-1}(s)\,ds.$$

By (A6), we can select $v_n(t)$ such that

$$|v_n(t)-v_{n-1}(t)|\leq\frac{C}{2}(1+|v_{n-1}(t)|+L_{n-1}(t))(|x_n(t)-x_{n-1}(t)|+|u_n(t)-u_{n-1}(t)|)$$

and

$$L_n(t) \leq L_{n-1}(t) + \frac{C}{2}(1+|v_{n-1}(t)|+L_{n-1}(t))(|x_n(t)-x_{n-1}(t)|+|u_n(t)-u_{n-1}(t)|)$$

where $L_i(t) := L(t, x_i(t), u_i(t), v_i(t))$. We define

$$w_n(t) := w_{n-1}(t) - \frac{C}{2}(1+|v_{n-1}(t)|+L_{n-1}(t))(|x_n(t)-x_{n-1}(t)|+|u_n(t)-u_{n-1}(t)|.$$

We set

$$\begin{aligned} d_n(t) &= |x_{n+1}(\tau - t) - x_n(\tau - t)| + |u_{n+1}(\tau - t) - u_n(\tau - t)|, \\ l_n(t) &= 1 + |v_n(\tau - t)| - w_n(\tau - t). \end{aligned}$$

Since $w_0(t) \leq -L_0(t)$ and $L_n(t) \leq L_{n-1}(t) + w_{n-1}(t) - w_n(t)$ then $w_n(t) \leq -L_n(t)$ for $n \geq 1$. Hence

$$\begin{aligned} d_{n+1}(t) &\leq \int_0^t |w_{n+1}(\tau - s) - w_n(\tau - s)|\, ds + \int_0^t |v_{n+1}(\tau - s) - v_n(\tau - s)|\, ds \\ &\leq \int_0^t [C(1 + |v_n(\tau - s)| + L_n(\tau - s)) \\ &\qquad \cdot (|x_{n+1}(\tau - s) - x_n(\tau - s)| + |u_{n+1}(\tau - s) - u_n(\tau - s)|)]\, ds \\ &\leq \int_0^t C l_n(\tau - s) d_n(\tau - s)\, ds \end{aligned}$$

and

$$\begin{aligned} l_{n+1}(t) &\leq l_n(t) + (l_{n+1}(t) - l_n(t)) \\ &\leq l_n(t) + |v_{n+1}(\tau - t) - v_n(\tau - t)| + w_n(\tau - t) - w_{n+1}(\tau - t) \\ &\leq l_n(t) + C l_n(t) d_n(t). \end{aligned}$$

By Lemma 3.18, we obtain

$$|u_{n+1}(\tau - t) - u_n(\tau - t)| + |x_{n+1}(\tau - t) - x_n(\tau - t)| \leq \frac{(CM \int_t^\tau l_0(s)\, ds)^n}{n!} \varepsilon$$

and

$$1 + |v_n(\tau - t)| - w(\tau - t) \leq M l_0(\tau - t).$$

Thus

$$|x_n(\tau - t) - x_0(\tau - t)| + |u_n(\tau - t) - u_0(\tau - t)| < \varepsilon \exp\left(CM \int_t^\tau l_0(s)\, ds\right) < \frac{r}{2}. \tag{32}$$

Moreover, $u_n(\,\cdot\,)$, $x_n(\,\cdot\,)$ are uniformely convergent on the interval $[t, \tau]$ to absolutely continuous functions $\overline{u}(\,\cdot\,)$, $\overline{x}(\,\cdot\,)$ and the derivatives $u_n'(t)$, $x_n'(t)$ converge to $\overline{u}'(t)$, $\overline{x}'(t)$ for almost all t. Since L is lower semicontinuous

$$L(t, \overline{x}(t), \overline{u}(t), \overline{x}'(t)) \leq \liminf_{n \to \infty} L(t, x_n(t), u_n(t), v_n(t)) \leq \lim_{n \to \infty} -w_n(t) = -\overline{u}'(t).$$

Hence $(\overline{x}, \overline{u})$ is an L-solution on $[t, \tau]$. By (32), we obtain (31). $\square$

LEMMA 3.18. *Let $l_0\colon [0,T] \to [0,\infty)$ be an integrable function and $C > 0$. Suppose that $\varepsilon > 0$, $M > 1$ satisfy*

$$\exp\left(C\varepsilon \exp\left(M \int_0^T l_0(s)\,ds\right)\right) \le M. \tag{33}$$

If a sequences $d_n\colon [0,T] \to [0,\infty)$ of continuous functions and $l_n\colon [0,T] \to [0,\infty)$ of measurable functions satisfy

$$d_0 \equiv \varepsilon \quad l_0 \textit{ is given above,}$$
$$d_{n+1}(t) \le \int_0^t C l_n(s) d_n(s)\,ds, \quad l_{n+1} \le (1 + C d_n(t)) l_n(t),$$

then

$$l_n(t) \le M l_0(t), \tag{34}$$
$$d_n(t) \le \frac{(Cm(t))^n}{n!}\varepsilon, \tag{35}$$

where $m(t) = \int_0^t M l_0(s)\,ds$.

PROOF. Suppose that (34) holds true for $n = 0, \dots, k$. Hence

$$\begin{aligned}
d_1(t) &\le \int_0^t C l_0(s) d_0(s)\,ds = \frac{Cm(t)}{1!}\varepsilon,\\
d_2(t) &\le \int_0^t CM l_0(s) d_1(s)\,ds \le \frac{C^2 m(t)^2}{2!}\varepsilon,\\
\\
d_{k+1}(t) &\le \int_0^t CM l_0(s) d_k(s)\,ds \le \frac{C^{k+1} m(t)^{k+1}}{(k+1)!}\varepsilon.
\end{aligned}$$

Thus

$$\begin{aligned}
l_{k+1}(t) &\le (1 + C d_k(t))(1 + C d_{k-1}(t)) \dots (1 + C d_0(t)) l_0(t)\\
&\le \left(1 + C\frac{C^k m(t)^k}{k!}\varepsilon\right)\\
&\quad \cdot \left(1 + C\frac{C^{k-1} m(t)^{k-1}}{(k-1)!}\varepsilon\right) \dots \left(1 + C\frac{C^0 m(t)^0}{0!}\varepsilon\right) l_0(t)\\
&\le \exp\left(C\varepsilon \sum_{n=0}^k \frac{C^n m(t)^n}{n!}\right) l_0(t) \le M l_0(t).
\end{aligned}$$

So, (34) holds true for $n = k + 1$. Since

$$d_{n+1}(t) \le \int_0^t CM l_0(s) d_n(s)\,ds$$

therefore we have inductively proved (35). □

Let us observe that for a fixed $M > 1$ the inequality (33) holds true for sufficiently small ε.

By the definition of g, there exist $\tau_n \to T^-$ and $x_n \to x(T)$ such that

$$\lim_{n\to\infty} U(\tau_n, x_n) = g(x(T))$$

We choose $u_n \geq U(\tau_n, x_n)$ such that $u_n \to u(T)$. By Lemma 3.17, for sufficiently large n we can find an L-solution $(\overline{x}_n, \overline{u}_n)$ on $[t_0, \tau_n]$ such that

$$\overline{u}_n(\tau_n) = u_n, \quad \lim_{n\to\infty} \overline{u}_n(t) = u(t), \quad \lim_{n\to\infty} \overline{x}_n(t) = x(t)$$

for every $t \in [t_0, T)$.

We claim that $\overline{u}_n(t) \geq U(t, \overline{x}_n(t))$ for $t \in [t_0, \tau_n]$. We set $F(t,x,u) = \{(-1, -v, \theta) : \theta \geq L(t,x,u,v)\}$ and $K = \mathrm{Epi}(U)$. The function $y_n(t) = (\tau_n - t, x(\tau_n - t), u(\tau_n - t))$ is a solution to $y'(t) \in F(y(t))$ on $[0, \tau_n - t_0]$. We verify that K, F, $y_n(\,\cdot\,)$ satisfy the assumptions of Invariance Theorem 4.2 the statement and proof of which are postponed to the Appendix.

F satisfies (a) of Theorem 4.2 with l given by

$$l((t,x,u),(t',x',u')) = C(1 + |x'| + L(t,x,u,x')).$$

The assumption (b) of Theorem 4.2 is a consequence of the third part of Proposition 3.3.

Now, we check that $t \to l(y_n(t), y_n'(t))$ is integrable. Since

$$\int_{t_0}^{\tau_n} L(t, x_n(t), u_n(t), x_n'(t))\,dt \leq u(t_0) - u(\tau_n)$$

and $x_n(\,\cdot\,)$ is absolutely continuous then

$$t \to C(1 + |x_n'(t)| + L(t, x_n(t), u_n(t), x_n'(t)))$$

is integrable on $[t_0, \tau_n]$. □

Theorem 3.19 (Viability Theorem). *Suppose that $U: [0,T] \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous and bounded from below and L satisfies* (A1)–(A5). *Assume that for every $t \in (0,T)$, $(t,x) \in \mathrm{Dom}(U)$ and for every $n \in N_{\mathrm{Epi}\,U}(t,x,U(t,x)$ there exists $v \in \mathbb{R}^n$ such that*

$$\langle (1, v, -L(t,x,U(t,x),v)); n\rangle \leq 0. \tag{36}$$

Then, for every $(t_0, x_0) \in \mathrm{Dom}(U)$, $t_0 \in (0,T)$, there exists an L-solution $(x, \overline{u})$ on $[t_0, T]$ such that $x(t_0) = x_0$, $\overline{u}(t_0) = U(t_0, x_0)$ and $\overline{u}(t) \geq U(t, x(t))$ for every $t \in [t_0, T]$.

For the proof see Appendix.

The fact that (x,u) is an L-solution can be equivalently formulated as

$$u(t) \le u(t_0) - \int_{t_0}^{t} L(s,x(s),u(s),x'(s))\,ds \quad \text{for every } t \in [t_0,T].$$

One can consider the above theorem as a viability result ([2]) because $(x(\,\cdot\,),u(\,\cdot\,))$ is a solution to the differential inclusion

$$(x'(t),u'(t)) \in \widetilde{Q}(t,x(t),u(t))$$

which is viable with respect to the epigraph of function U namely $(x(t),u(t)) \in \mathrm{Epi}(U)$ for all $t \in [t_0,T)$.

Corollary 3.20. *Under the assumptions of Theorem* 3.19*, if U is a proximal normal supersolution to* (19) *and $g(\,\cdot\,) := U(T,\,\cdot\,)$ then the value function V corresponding to g satisfies $U \ge V$.*

4. Appendix

4.1. Proof of Viability Theorem 3.19.

Step 1. Definition of an ε approximate solution.

We say that a family $\Sigma = \{[t_j,\tau_j) : j \in J\}$ of nonempty intervals is a subdivision of the interval $[t_0,t_\varepsilon)$ if $[t_j,\tau_j) \cap [t_i,\tau_i) = \emptyset$ for every $j \ne i$ and

$$[t_0,t_\varepsilon) = \bigcup_{j\in J} [t_j,\tau_j).$$

Let $y\colon [t_0,t_\varepsilon) \to \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ $(y(t) = (s(t),x(t),u(t)),\ y(t_0) = (t_0,x_0,u_0))$ be a continuous function. We say that a pair (y,Σ) is an ε approximate solution if

$$\begin{cases} \text{for all } t \in [t_j,\tau_j)\ y(t) = y(t_j) + (t-t_j)(f_j + (w_j - y(t_j))) \\ f_j \in \{1\}\times \widetilde{Q}(w_j),\ \langle f_j, y(t_j) - w_j\rangle \le 0,\ w_j \in \mathrm{Epi}(U), \\ |w_j - y(t_j)| = \mathrm{dist}(y(t_j),\mathrm{Epi}(U). \end{cases} \tag{37}$$

$$\begin{cases} \text{(a) If } y(t_j) \in \mathrm{Epi}(U) \text{ then } (\tau_j - t_j)|f_j| < \varepsilon. \\ \text{(b) If } y(t_j) \notin \mathrm{Epi}(U) \text{ then } |y(t) - w_j| \le |y(t_j) - w_j| \text{ for } t \in [t_j,\tau_j]. \end{cases} \tag{38}$$

$$\mathrm{dist}(y(t),\mathrm{Epi}(U)) \le \varepsilon \quad \text{for all } t \in [t_0,t_\varepsilon). \tag{39}$$

Step 2. Extension of an approximate solution.

By (A5), we choose $\phi(\,\cdot\,)$, $C > 0$ for $\widetilde{u} := u_0 + T$. Let $(y(\,\cdot\,),\sum)$ be an ε approximate solution $(\varepsilon < 1/2(2+T))$. At first we show that the limit $\lim_{t\to t_\varepsilon^-} y(t)$

exists. By (39), $u(\,\cdot\,)$ is bounded by below (say by b). We set:

$$
(40)\qquad
\begin{aligned}
(n_j^t, n_j^x, n_j^u) &= y(t_j) - w_j, \\
(t_j^w, x_j^w, u_j^w) &= w_j, (1, v_j, -\eta_j) = f_j \quad \text{where } \eta_j \geq L(w_j, v_j), \\
n^u &= \sum_{j \in J} \chi_{[t_j,\tau_j)} n_j^u, \qquad n^x = \sum_{j \in J} \chi_{[t_j,\tau_j)} n_j^x, \\
v &= \sum_{j \in J} \chi_{[t_j,\tau_j)} v_j, \qquad \eta = \sum_{j \in J} \chi_{[t_j,\tau_j)} \eta_j, \\
x_v(t) &= x_0 + \int_{t_0}^t v(s)\, ds, \qquad A := \{t : |v(t)| \geq 2C(1 + |x_v(t)|)\}.
\end{aligned}
$$

We have $u'(t) = -\eta(t) - n^u(t) \leq \varepsilon$ for a.a. $t \in [t_0, t_\varepsilon)$. If $t \in A \cap (t_j, \tau_j)$ then $|v_j| = |v(t)| \geq 2C(1 + |x_v(t)|)$. Moreover,

$$|x_j^w - x_v(t)| \leq |x_j^w - x(t)| + |x(t) - x_v(t)| \leq 2\varepsilon + \varepsilon(t - t_0) < \frac{1}{2}.$$

Thus

$$|v_j| \geq 2C\bigg(1 + |x_j^w| - \frac{1}{2}\bigg) \geq C(1 + |x_j^w|).$$

Hence

$$(41)\qquad \int_A \phi(|v(t)|)dt \leq \int_A \eta(t)\, dt \leq \int_A (-u'(t) + \varepsilon)\, dt \leq u(t_0) - b + T =: C_2.$$

By Proposition 2.6, $v(\,\cdot\,)$ is integrable on $[t_0, t_\varepsilon]$. Since $x'(t) = v(t) - n^x(t)$ and $|n^x(t)| \leq \varepsilon$ then the limit $\lim_{t \to t_\varepsilon^-} x(t)$ exists.

We have $s'(t) = 1 - u^t(t)$ and $|u^t(t)| \leq \varepsilon$. Thus the limit $\lim_{t \to t_\varepsilon^-} s(t)$ exists.

Since the function $\operatorname{dist}(\,\cdot\,, \operatorname{Epi}(U))$ is continuous hence

$$\operatorname{dist}(\lim_{t \to t_\varepsilon^-} y(t), \operatorname{Epi}(U)) \leq \varepsilon.$$

Now, suppose that $t_\varepsilon < T - \varepsilon$. We provide an extension of $y(\,\cdot\,)$. If $y(t_\varepsilon) \in \operatorname{Epi}(U)$ then we take an arbitrary $f_\varepsilon \in \{1\} \times \widetilde{Q}(y(t_\varepsilon))$ and $\tau_\varepsilon > t_\varepsilon$ such that $(\tau_\varepsilon - t_\varepsilon)|f_\varepsilon| < \varepsilon$. Setting $y(t) = y(t_\varepsilon) + (t - t_\varepsilon) f_\varepsilon$ we obtain an extension of $y(\,\cdot\,)$ satisfying (37)–(39). Suppose that $y(t_\varepsilon) \in (\operatorname{Epi}(U) + \varepsilon\overline{B}) \setminus \operatorname{Epi}(U)$. Let $w_\varepsilon = (s_\varepsilon, x_\varepsilon, u_\varepsilon) \in \operatorname{Epi}(U)$ be a proximal point to $y(t_\varepsilon)$ in $\operatorname{Epi}(U)$. We have

$$\operatorname{dist}((s_\varepsilon, x_\varepsilon, U(s_\varepsilon, x_\varepsilon)) + (y(t_\varepsilon) - w_\varepsilon), \operatorname{Epi}(U)) = |y(t_\varepsilon) - w_\varepsilon|.$$

Thus

$$y(t_\varepsilon) - w_\varepsilon \in N_{\operatorname{Epi}(U)}(s_\varepsilon, x_\varepsilon, U(s_\varepsilon, x_\varepsilon)).$$

By (36), there exists v such that

$$\langle (1, v, -L(s_\varepsilon, x_\varepsilon, U(s_\varepsilon, x_\varepsilon))), y(t_\varepsilon) - w_\varepsilon \rangle \leq 0.$$

By (A4),

$$f_\varepsilon = (1, v, -L(s_\varepsilon, x_\varepsilon, U(t_\varepsilon, x_\varepsilon)) \in \{1\} \times \widetilde{Q}(w_\varepsilon).$$

We extend $y(\,\cdot\,)$ setting for $t > t_\varepsilon$

$$y(t) = y(t_\varepsilon) + (t - t_\varepsilon)(f_\varepsilon - (y(t_\varepsilon) - w_\varepsilon)).$$

Lemma 4.1. *If* $y \neq w$ *and* $\langle f, y-w\rangle \leq 0$*, then we have*

$$|(y + h(f - (y-w)) - w| \leq |y - w| \quad \textit{for } 0 \leq h \leq \frac{2|y-w|^2 - 2\langle y-w, f\rangle}{|f-(y-w)|^2}.$$

We point out this simple geometrical fact, because it appeared to be the crucial step in our construction and moreover, according to our knowlegde, it is used for the first time in the viability theory. According to Lemma 4.1, there exists $\tau_\varepsilon > t_\varepsilon$ such that $|y(t) - w_\varepsilon| \leq |y(t_\varepsilon) - w_\varepsilon| \leq \varepsilon$ for $t \in [t_\varepsilon, \tau_\varepsilon]$. Thus $\operatorname{dist}(y(t), \operatorname{Epi}(U)) \leq \varepsilon$ for $t \in [t_\varepsilon, \tau_\varepsilon]$.

By the Kuratowski–Zorn Lemma, for every $\varepsilon > 0$ there exists an ε approximate solution on the whole interval $[t_0, T-\varepsilon]$.

Step 3. Convergence of approximate solutions.

Let (s_k, x_k, u_k) be an $\varepsilon = 1/k$ ($k \geq 2$) approximated solution on $[t_0, T-1/k)$. Since $u_k'(t) \leq 1/k$ hence

$$\begin{aligned}\operatorname{Var}_{[t_0,T-1/k]} u_k &= \int_{t_0}^{T-1/k} |u_k'(t)|\,dt \\ &= 2\int_{t_0}^{T-1/k} (u_k'(t))^+\,dt - \int_{t_0}^{T-1/k} u_k'(t)\,dt \\ &\leq 2\varepsilon T + (u_k(t_0) - u_k(T-1/k)\end{aligned}$$

where $(x)^+ = x$ for nonnegative x and $(x)^+ = 0$ for negative x. As $u_k(t_0) = U(t_0, x_0)$, $u_k(T-1/k) \geq C$ then the variations of u_k on $[t_0, T-1/k]$ are equibounded. By the Helly Theorem (cf. Theorem 15.1.i in [10]), there exists a subsequence (denoted again by u_k) converging pointwise (everywhere) to a bounded variation function $u\colon [t_0, T] \to \mathbb{R}$.

Let $n_k^u(\,\cdot\,)$, $n_k^x(\,\cdot\,)$, $v_k(\,\cdot\,)$, $\eta_k(\,\cdot\,)$, A_k be given by (40). We have $u_k'(t) = -\eta_k(t) - n_k^u(t)$, $|n_k^u(t)| \leq 1/k$, $x_k'(t) = v_k(t) - n_k^x(t)$, $s_k'(t) = 1 - n_k^t(t)$, $|n_k^t(t)| \leq 1/k$. Repeating the same arguments as in Step 2 we obtain

$$\begin{aligned}\int_{A_k} \phi(|v_k(t)|)\,dt &\leq \int_{A_k} \eta_k(t)\,dt \\ &\leq -\int_{t_0}^{T-1/k} u_k'(t)\,dt + \int_{t_0}^{T-1/k} n_k^u(t)\,dt \\ &\leq u_k(t_0) - u_k\left(T - \frac{1}{k}\right) + \frac{1}{k}(T - t_0) \leq C_2\end{aligned}$$

where the constant C_2 is given by (41). By Proposition 2.6, the family $\{v_k(\,\cdot\,)\}$ is equiabsolutely integrable. Since $|n_k^x(t)| \leq 1/k$ then $\{x_k'(\,\cdot\,)\}$ is equiabsolutely integrable, too. Thus, $\{x_n\}$ is equiabsolutely continuous. By the Alaoglu and

Dunford–Pettis Theorems, there exists a subsequence (denoted again by x_n) such that x_n tends uniformely to an absolutely continuous function x and x'_n tends weakly in L^1 to x'. Fix a positive integer s. Since

$$x'_{s+i} \xrightarrow{L^1} x' \quad \text{as } i \to \infty$$

there exist $C^s_{Ni} \geq 0$; $N = 1, 2, \dots$, $i = 1, \dots, N$, $\sum_{i=1}^N C^s_{Ni} = 1$ such that $\sum_{i=1}^N C^s_{Ni} x'_{s+i}$ tends in L^1 norm to x' as $N \to \infty$. For a subsequence N_n

$$z^s_n := \sum_{i=1}^{N_n} C^s_{N_n\, i} x'_{s+i} \to x' \quad \text{a.e. in } [t_0, T].$$

We set

$$\eta^s_n := \sum_{i=1}^{N_n} C^s_{N_n i} \eta_{s+i}$$

and

$$\eta^s(t) := \liminf_{n\to\infty} \eta^s_n(t), \quad \eta(t) := \liminf_{s\to\infty} \eta^s(t).$$

We claim that for $\tau_0 \in (t_0, T)$

$$u(t_0) - \int_{t_0}^{\tau_0} \eta(t) dt \geq u(\tau_0). \tag{42}$$

By the Fatou Lemma we have

$$\int_{t_0}^{\tau_0} \eta(t)\, dt \leq \liminf_{s\to\infty} \int_{t_0}^{\tau_0} \eta^s(t)\, ds \leq \liminf_{s\to\infty} \liminf_{n\to\infty} \int_{t_0}^{\tau_0} \eta^s_n(t)\, dt$$

where

$$\begin{aligned} \int_{t_0}^{\tau_0} \eta^s_n(t)\, dt &= \sum_{i=1}^{N_n} C^s_{N_n i} \int_{t_0}^{\tau_0} \eta_{s+i}(t)\, dt \\ &= u(t_0) - \sum_{i=1}^{N_n} C^s_{N_n i} u_{s+i}(\tau_0) - \sum_{i=1}^{N_n} C^s_{N_n i} \int_{t_0}^{\tau_0} n^u_{s+i}(t)\, dt. \end{aligned}$$

Since $u(\tau_0) = \lim_{n\to\infty} u_n(\tau_0)$ and $|n^u_{s+i}(t)| \leq 1/s$ hence

$$\liminf_{s\to\infty} \liminf_{n\to\infty} \bigg(u(t_0) - \sum_{i=1}^{N_n} C^s_{N_n k} u_{s+i}(\tau_0) - \sum_{i=1}^{N_n} C^s_{N_n i} \int_{t_0}^{\tau_0} n^u_{s+i}(t)\, dt \bigg) \\ = u(t_0) - u(\tau_0).$$

Thus

$$\int_{t_0}^{\tau_0} \eta(t)\, dt \leq u(t_0) - u(\tau_0)$$

which proves (42). We will show that for almost all $t \in [t_0, T]$

$$\eta(t) \geq L(t, x(t), u(t), x'(t)). \tag{43}$$

Fix $t \in (t_0, T)$ such that $\lim_{n\to\infty} z_n^s(t) = x'(t)$ for every s. We have

$$\lim_{k\to\infty}(s_k(t), x_k(t), u_k(t)) = (t, x(t), u(t)$$

and $(v_k(t), -\eta_k(t)) \in \widetilde{Q}((s_k(t), x_k(t), u_k(t)), 1/k)$.

Fix $\varepsilon > 0$. There exists k_0 such that $(v_k(t), -\eta_k(t)) \in \widetilde{Q}(t, x(t), u(t); \varepsilon)$ for $k \geq k_0$. Thus

$$(x_k'(t), -\eta_k(t)) \in \widetilde{Q}(t, x(t), u(t)); \varepsilon) + \frac{1}{k}\overline{B}.$$

We claim that

$$(x'(t), -\eta(t)) \in \overline{\text{co}}\, \widetilde{Q}(t, x(t), u(t); \varepsilon). \tag{44}$$

Let $s > k_0$. For every n

$$(z_n^s(t), -\eta_n^s(t)) \in \overline{\text{co}}\, \widetilde{Q}(t, x(t), u(t); \varepsilon) + \frac{1}{s}\overline{B}.$$

Taking the limit for a subsequence (with respect to n) we obtain

$$(x'(t), -\eta^s(t)) \in \overline{\text{co}}\, \widetilde{Q}(t, x(t), u(t); \varepsilon) + \frac{1}{s}\overline{B}.$$

Now, taking the limit of a subsequence (with respect to s) we get (44).

Since $\varepsilon > 0$ was arbitrary and $\widetilde{Q}$ has the property (Q) we have obtained that

$$(x'(t), -\eta^s(t)) \in \overline{\text{co}}\widetilde{Q}(t, x(t), u(t))$$

which is equivalent to (43). By (43), (42) and (A4), we obtain

$$\eta(t) \geq L(t, x(t), u(t_0) - \int_{t_0}^{t} \eta(\tau) d\tau, x'(t)).$$

Setting $\overline{u}(t) = u(t_0) - \int_{t_0}^{t} \eta(\tau)\, d\tau$ we get

$$\overline{u}'(t) \leq -L(t, x(t), \overline{u}(t), x'(t))$$

a.e. in $[t_0, T]$. Since

$$\text{dist}((t, x(t), u(t)), \text{Epi}(U)) = \lim_{n\to\infty} \text{dist}((s_n(t), x_n(t), u_n(t)), \text{Epi}(U)) = 0$$

and $\overline{u}(t) \geq u(t)$ hence $\overline{u}(t) \geq U(t, x(t))$. $\square$

4.2. Invariance Theorem. Define the proximal normal cone as follows [7]:

$$N_K(y) = \{n \in \mathbb{R}^d : \exists \alpha > 0\ |\alpha n| = \text{dist}(y + \alpha n; K)\}.$$

Theorem 4.2. *Let* $F: \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ *be a set valued map and* $K \subset \mathbb{R}^d$ *be locally compact. Suppose that*

(a) *for every* $r > 0$ *there exists a function* $l: \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ *such that for every* $y_1, y_2 \in \mathbb{R}^d$, $|y_1| < r$, $|y_2| < r$ *and every* $f_1 \in F(y_1)$ *there exists* $f_2 \in F(y_2)$ *such that*

$$|f_1 - f_2| \le l(y_1, f_1)|y_1 - y_2|,$$

(b) $\langle f, n \rangle \le 0$ *for all* $y \in K$, *for all* $n \in N_K(y)$ *and for all* $f \in F(y)$.

If $y: [0, T] \to \mathbb{R}^d$ *is an absolutely continuous function satisfying*

(1) $t \to l(y(t), y'(t))$ *is integrable on* $[0, T]$,
(2) $y(0) \in K$,
(3) $y'(t) \in F(y(t))$ *for a.a.* $t \in [0, T]$,

then there exists $\tau > 0$ *such that* $y(t) \in K$ *for every* $t \in [0, \tau]$

Proof. We choose $r > 0$ such that $K \cap \overline{B}(y(0), r)$ is closed. Let us take $\tau > 0$ such that $|y(t) - y(0)| < r/2$ for $t < \tau$. Define $d(t) = \operatorname{dist}(y(t), K)$. Fix $t < \tau$ such that derivatives $y'(t)$, $d'(t)$ exist, $y'(t) \in F(y(t))$ and $d(t) > 0$. There is $p \in K \cap \overline{B}(y(0), r)$ such that

$$|y(t) - p| = \operatorname{dist}(x(t), K \cap \overline{B}(y(0), r)) \le |y(t) - y(0)| < r/2.$$

If $q \notin \overline{B}(y(0), r)$ then $|y(t) - q| > r/2$. Thus $\operatorname{dist}(y(t), K) = \operatorname{dist}(y(t), K \cap \overline{B}(y(0), r))$. Then $n = y(t) - p \in N_K(p)$. We choose (by (a)) $g \in F(p)$ such that $|y'(t) - g| \le l(y(t), y'(t))|y(t) - p|$. We have

$$\begin{aligned} d(t+h) - d(t) &\le |y(t+h) - p| - |y(t) - p| \\ &\le |y(t+h) - (y(t) + hy'(t)| + h|y'(t) - g| + (|(y(t) - p) + hg| - |y(t) - p|. \end{aligned}$$

Thus

$$\begin{aligned} d'(t) &\le |y'(t) - g| + \lim_{h \to 0} \frac{|(y(t) - p) + hg| - |y(t) - p|}{h} \\ &= |y'(t) - g| + \left\langle \frac{y(t) - p}{|y(t) - p|}, g \right\rangle. \end{aligned}$$

So

$$d'(t) \le l(y(t), y'(t))\, d(t).$$

Because $t \to l(y(t), y'(t))$ is an integrable nonnegative function, Gronwall's Lemma yields $d(t) = 0$, for every $t \in [0, \tau]$. □

Remark that supposition (a) is stronger than pseudo-lipschitziannity of F as it is shown in the following example.

EXAMPLE. The set valued map $S\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ given by

$$S(x) = \begin{cases} \{-1, 1\} & \text{for } x \neq 0, \\ \{1\} & \text{for } x = 0, \end{cases}$$

is pseudo-Lipschitz but does not satisfy assumption (a) of Theorem 4.2.

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