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MULTIPLE NONNEGATIVE SOLUTIONS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS INVOLVING THE p-LAPLACIAN

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Abstract. In this paper we present a result concerning the existence of two nonzero nonnegative solutions for the following Dirichlet problem involving the p-Laplacian

$$\begin{cases} -\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

using variational methods. In particular, we will determine an explicit real interval Λ for which these solutions exist for every $\lambda \in \Lambda$. We also point out that our result improves and extends to higher dimension a recent multiplicity result for ordinary differential equations.

1. Introduction

Here and in the sequel, Ω is a bounded open set in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^1. In this paper we deal with the following Dirichlet problem

$$(\mathrm{P}_{\lambda,f}) \qquad \begin{cases} -\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p > 1$, $\lambda \in \mathbb{R}$ is a parameter, $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\Delta_p = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. In particular, under the

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condition $f(x,0) = 0$ for a.a. $x \in \Omega$, we look for multiple nonzero weak nonnegative solutions of problem $(\mathrm{P}_{\lambda,f})$. Our main result is Theorem 3.4 which is stated and proved in Section 3. It assures the existence of at least two nonzero nonnegative weak solutions for problem $(\mathrm{P}_{\lambda,f})$ determining, at the same time, an explicit real interval for which the solutions exist when λ belongs to it besides a stability property of the solutions with respect to λ. When $p = 2$, several papers are devoted to the study of problem $(\mathrm{P}_{\lambda,f})$. When $p > 1$ only, a less number of works are available. For this latter case, among the most recent ones we find interesting the papers [3], [9] (see also reference therein, in particular we refer to [3] for motivations in studying problem $(\mathrm{P}_{\lambda,f})$) where multiplicity of nonzero solutions for problem $(\mathrm{P}_{\lambda,f})$ is obtained using variational methods. In particular, the results of [3] give the existence of multiple nonnegative, multiple nonpositive and multiple sign changing solutions under a suitable oscillatory behavior of the nonlinearity. However, there the assumptions are essentially different from ours. The main result of [9] is Theorem 1.2 whose thesis and assumptions are directly comparable with ours. In fact, apart the additional properties of the solutions, Theorem 3.4 and Theorem 1.2 of [9] give the same multiplicity result. Moreover, conditions (F_1)–(F_3) of Theorem 1.2 of [9] are very similar to conditions (a)–(c) of Theorem 3.4. Specifically, (F_1) turns out a particular case of condition (a) while (F_2), (F_3) are slightly weaker than (b), (c) respectively. Anyway, it is easy to check that the previous two results are mutually independent. For example, Theorem 3.4 allows us to consider nonlinearities f such that $\sup_{\xi\in[0,\delta]} \int_0^\xi f(x,t)\,dt > 0$ for all $\delta > 0$, contrarily to Theorem 1.2 of [9]. Finally, note also that, unlike of Theorem 3.4, a real interval Λ for which the solutions exist when $\lambda \in \Lambda$ is non explicitly determined in Theorem 1.2 of [9].

2. Basic definitions and notations

Throughout this paper, for every Carathéodory function $g\colon \Omega\times\mathbb{R} \to \mathbb{R}$, every $\lambda \in \mathbb{R}$ and $u \in W_0^{1,p}(\Omega)$ we put

$$J_{\lambda,g}(u) = \frac{1}{p}\|u\|^p - \lambda \int_\Omega \left(\int_0^{u(x)} g(x,t)\,dt \right) dx \tag{2.1}$$

where

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p\,dx \right)^{1/p}$$

is the usual norm in $W_0^{1,p}(\Omega)$. Also, we put

$$g_+(x,t) = \begin{cases} g(x,t) & \text{for all } (x,t) \in \Omega\times[0,\infty[, \\ 0 & \text{for all } (x,t) \in \Omega\times\,]-\infty,0[. \end{cases}$$

Moreover, we say that g satisfies condition (I) (resp. (I^+)) when one of the following conditions holds

(i_1) $p < N$ and there exists $M > 0$ such that

$$\sup_{t\in\mathbb{R}} \frac{|g(x,t)|}{1+|t|^q} \le M$$

for a.a. $x \in \Omega$ and for some $q \in \,]0, (N(p-1)+p)/(N-p)[$,

(i_2) $p = N$ and there exists $M > 0$ such that

$$\sup_{t\in\mathbb{R}} \frac{|g(x,t)|}{1+|t|^q} \le M$$

for a.a. $x \in \Omega$ and for some $q > 0$,

(i_3) $p > N$ and, for all $r > 0$,

$$\sup_{|t|\le r} |g(\,\cdot\,,t)| \in L^1(\Omega),$$

(resp. when (I) is satisfied by the function g_+).

By standard results, the condition (I) above assures that the functional $J_{\lambda,g}$ introduced in (2.1) is well defined, sequentially weakly lower semicontinuous and Gateâux differentiable in $W_0^{1,p}(\Omega)$.

As a usual, a weak solution of problem ($\mathrm{P}_{\lambda,f}$) is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx - \lambda\int_\Omega f(x,u(x))v(x)\,dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$. Hence, the weak solutions of problem ($\mathrm{P}_{\lambda,f}$) are exactly the critical points of the functional $J_{\lambda,f}$. As quoted in the introduction, we will assume the nonlinearity f in problem ($\mathrm{P}_{\lambda,f}$) satisfying the condition:

(i_0) $f(x,0) = 0$ for a.a. $x \in \Omega$.

Note that if f is a Carathéodory function satisfying (i_0), then f_+, is a Carathéodory function as well.

3. The results

Before proving the main result we need of the following lemmas concerning the regularity of the solutions. Although the proofs of these lemmas follow standard arguments, nevertheless we prefer to give some of them for sake of clearness.

Lemma 3.1. *Let $\lambda \in \mathbb{R}$ and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that*

(a) $|g(x,t)| \le a(x)(1+|t|^{p-1})$ *for a.a.* $x \in \Omega$ *and all* $t \in \mathbb{R}^n$

where $a \in L^{N/p}(\Omega)$. Then, if $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem $(\mathrm{P}_{\lambda,g})$, *one has $u \in L^m(\Omega)$ for all $m \geq 1$.*

Proof. Clearly, without loss of generality, we suppose $\lambda = 1$. Moreover, we only consider the case $p < N$ otherwise the thesis is obvious by the Sobolev embedding theorem. We can proceed as in Lemma B.3 of [11, Appendix B] where the same result is proved for $p = 2$. Let u be a weak solution of $(\mathrm{P}_{\lambda,g})$ and fix $M > 0$ and $s > 0$ such that $u \in L^{(s+1)p}(\Omega)$. Further, fix $B > 0$ such that

$$\left(\int_{|a(x)|\geq B} |a(x)|^{N/p}\,dx\right)^{p/N} < \frac{1}{c_p 2^p \max\{1, s^{p-1}/p\}}$$

where

$$c_p = \sup_{v\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{(\int_\Omega |v(x)|^{pN/(N-p)}\,dx)^{(N-p)/(pN)}}{\|v\|} < \infty$$

by Sobolev embedding theorem. We have

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx = \int_\Omega g(x,u(x))v(x)\,dx$$

for all $v \in W_0^{1,p}(\Omega)$. In particular, choosing $v = u\max\{|u|^{sp}, M^p\}$ and taking into account condition (a) of Lemma 3.1, we obtain the following inequalities

$$\begin{aligned}
&\int_\Omega |\nabla u(x)|^p \max\{|u(x)|^{sp}, M^p\}\,dx + sp\int_{|u(x)|^s\leq M} |\nabla u(x)|^p|u(x)|^{sp}\,dx \\
&\quad\leq \int_\Omega |a(x)||u(x)|^{sp+1}\,dx + \int_\Omega |u(x)|^p \max\{|u(x)|^{sp}, M^p\}\,dx \\
&\quad\leq \left(\int_\Omega |a(x)|^{N/p}\,dx\right)^{p/n}\left(\int_\Omega |u(x)|^{N(sp+1)/(N-p)}\,dx\right)^{(N-p)/N} \\
&\qquad + B\int_{|a(x)|\leq B} |u(x)|^{sp+p}\,dx \\
&\qquad + \int_{|a(x)|\geq B} |a(x)||u(x)|\max\{|u(x)|^{sp}, M^p\}\,dx \\
&\quad\leq c_1 + \frac{1}{c_p 2^p\max\{1, s^{p-1}/p\}}\left(\int_\Omega (|u|\max\{|u(x)|^s, M\})^{pN/(N-p)}\,dx\right)^{(N-p)/N} \\
&\quad\leq c_1 + \frac{1}{2^p\max\{1, s^{p-1}/p\}}\left(\int_\Omega |\nabla(|u|\max\{|u(x)|^s, M\})|^p\,dx\right)
\end{aligned}$$

where c_1 is a constant independent of M. Consequently, we have

$$\begin{aligned}\int_\Omega &|\nabla(u(x)\max\{|u(x)|^s, M\})|^p\,dx\\ \leq &2^{p-1}\max\left\{1,\frac{s^{p-1}}{p}\right\}\int_\Omega|\nabla u(x)|^p\max\{|u(x)|^{sp},M^p\}\,dx\\ &+sp\int_{|u(x)|^s\leq M}|\nabla u(x)|^p|u(x)|^{sp}\,dx\\ \leq &c_1 2^{p-1}\max\left\{1,\frac{s^{p-1}}{p}\right\}+\frac{1}{2}\int_\Omega|\nabla(u(x)\max\{|u(x)|^s,M\})|^p\,dx.\end{aligned}$$

Then

$$\int_{|u(x)|^s\leq M}|\nabla|u(x)|^{s+1}|^p\,dx\leq c_1 2^p\max\left\{1,\frac{s^{p-1}}{p}\right\}$$

uniformly with respect to M. This imply $|u|^{s+1}\in W_0^{1,p}(\Omega)$ and thus $u\in L^{(s+1)pN/(N-p)}$. So, we have proved that if our solution u belongs to $L^{(s+1)p}(\Omega)$ then $u\in L^{(s+1)pN/(N-p)}(\Omega)$. Consequently, if $m\geq 1$, iterating we obtain $u\in L^m(\Omega)$ after a finite number of steps. $\square$

LEMMA 3.2. *Let $f\in L^r(\Omega)$ with $r>N/p$ and let $u\in W_0^{1,p}(\Omega)$ be a weak solution of problem*

$$\begin{cases}-\Delta_p u=f(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Then, $u\in L^\infty(\Omega)$.

PROOF. The lemma can be easily proved by applying the Moser's iterative scheme (see [5], [8]). $\square$

LEMMA 3.3. *Let $\lambda\in\mathbb{R}$ and $g:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying condition* (I). *Then, every solution of problem* $(\mathrm{P}_{\lambda,g})$ *belongs to $C^{1+\gamma}(\Omega)$, where $\gamma\in\,]0,1[$.*

PROOF. Let u be a solution of $(\mathrm{P}_{\lambda,g})$. If $p\geq N$ we have $u\in L^\infty(\Omega)$ by Sobolev embedding theorem. Suppose $p<N$. We observe that u solves the problem

$$\begin{cases}-\Delta_p v=\lambda\dfrac{g(x,u(x))}{1+|u(x)|^{p-1}}(1+|v|^{p-1}) & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega,\end{cases}$$

where $g(\,\cdot\,,u(\,\cdot\,))/(1+|u(\,\cdot\,)|^{p-1})\in L^{N/p}(\Omega)$ as it is easy to check. Thus, by Lemma 3.1 we have $u\in L^m(\Omega)$ for all $m\geq 1$. Consequently, by condition (I) we infer that $g(\,\cdot\,,u(\,\cdot\,))\in L^r(\Omega)$ with $r>N/p$. Since u solves also the problem

$$\begin{cases}-\Delta_p v=\lambda g(x,u(x)) & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega,\end{cases}$$

then, by Lemma 3.2, we have $u \in L^\infty(\Omega)$. So, in any case, we infer $g(\,\cdot\,, u(\,\cdot\,)) \in L^\infty(\Omega)$. Hence $u \in C^{1+\gamma}(\Omega)$, for some $\gamma \in \,]0,1[$, by standard regularity results (see, for instance, [4]). $\square$

At this point we are able to proof the main result.

THEOREM 3.4. *Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying* (i_0) *and* (I^+). *Assume that there exist $\xi_0, \xi_1 \in [0, \infty[$ with $\xi_0 < \xi_1$ and $u_0 \in W_0^{1,p}(\Omega)$ with $u_0(x) \geq 0$ for a.a. $x \in \Omega$ such that*

(a) $\displaystyle\int_0^{\xi_0} f(x,t)\,dt = \sup_{\xi \in [\xi_0, \xi_1]} \int_0^{\xi} f(x,t)\,dt \quad$ *for a.a. $x \in \Omega$;*

(b) $\displaystyle\overline{\eta} \overset{\text{def}}{=} \int_\Omega \left(\int_0^{u_0(x)} f(x,t)\,dt - \sup_{0 \leq \xi \leq \xi_0} \int_0^{\xi} f(x,t)\,dt \right) dx > 0.$

Furthermore, suppose that there exist $C > 0$ and $s \in [0, p[$ such that

(c) $\displaystyle\sup_{\xi \geq 0} \frac{\int_0^{\xi} f(x,t)\,dt}{1 + |\xi|^s} \leq C \quad$ *for a.a. $x \in \Omega$.*

Then, for each $\lambda > \|u_0\|^p/(p\overline{\eta})$, there exist two nonzero nonnegative weak solutions $u_\lambda, v_\lambda \in W_0^{1,p}(\Omega) \cap C^{1+\gamma}(\Omega)$ (with $\gamma \in \,]0,1[$) of problem ($\mathrm{P}_{\lambda,f}$). *Moreover, one has $\sup_{\lambda \in K} \max\{\|u_\lambda\|, \|v_\lambda\|\} < \infty$ for every bounded set $K \subset \,]\|u_0\|^p/(p\overline{\eta}), \infty[$.*

PROOF. Let $\lambda > \|u_0\|^p/(p\overline{\eta})$ where $u_0 \in W_0^{1,p}(\Omega)$ is as in the hypotheses. At first, we observe that, thanks to condition (c), the functional J_{λ, f_+} is coercive. So that it attains the minimum on each nonempty convex closed subset of $W_0^{1,p}(\Omega)$. Now, let ξ_0, ξ_1 as in the hypotheses. Put

$$E = \{u \in W_0^{1,p}(\Omega) : 0 \leq u(x) \leq \xi_1\}.$$

Clearly, E is a nonempty convex closed subset of $W_0^{1,p}(\Omega)$. Then, from what above said, there exists $u_{0,\lambda} \in E$ such that

$$J_{\lambda, f_+}(u_{0,\lambda}) = \inf_E J_{\lambda, f_+}. \tag{3.2}$$

We want to prove that $u_{0,\lambda}$ turns out a local minimum for the functional J_{λ, f_+}. To this aim we follow a similar argument used in [1]. Let us consider the following real function

$$h(t) = \begin{cases} \xi_0 & \text{if } t > \xi_0, \\ t & \text{if } 0 \leq t \leq \xi_0, \\ 0 & \text{if } t < 0. \end{cases}$$

Define the operator $T\colon W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ putting $T(u)(x) = h(u(x))$ for every $u \in W_0^{1,p}(\Omega)$ and $x \in \Omega$. By the results of [7] we have that the operator T is continuous. Also, it turns out

$$T(W_0^{1,p}(\Omega)) \subseteq E. \tag{3.3}$$

At this point, we put $X_0 = \{x \in \Omega : u_{0,\lambda}(x) \geq \xi_0\}$ and we observe that, taking into account of condition (a) and being $u_{0,\lambda} \in E$, one has

$$\int_0^{u_{0,\lambda}(x)} f(x,t)\,dt \leq \int_0^{\xi_0} f(x,t)\,dt = \int_0^{T(u_{0,\lambda})(x)} f(x,t)\,dt$$

and $\nabla T(u_{0,\lambda})(x) = \nabla \xi_0 = 0$ for a.a. $x \in X_0$. Consequently, one has

$$\begin{aligned} &J_{\lambda,f_+}(T(u_{0,\lambda})) - J_{\lambda,f_+}(u_{0,\lambda}) \\ &\quad = -\frac{1}{p}\int_{X_0} |\nabla u_{0,\lambda}|^p\,dx - \lambda \int_{X_0} \bigg(\int_{u_{0,\lambda}(x)}^{T(u_{0,\lambda})(x)} f(x,t)\,dt \bigg)\,dx \\ &\quad \leq -\frac{1}{p}\int_{X_0} |\nabla u_{0,\lambda}|^p\,dx. \end{aligned}$$

By the previous inequality, (3.2) and (3.3) we deduce $\int_{X_0} |\nabla u_{0,\lambda}|^p\,dx = 0$. Hence, $\|T(u_{0,\lambda}) - u_{0,\lambda}\|^p = \int_{X_0} |\nabla u_{0,\lambda}|^p\,dx = 0$ from which $u_{0,\lambda}(x) = T(u_{0,\lambda})(x) \in [0,\xi_0]$ for a.a. $x \in \Omega$.

Now, let $u \in W_0^{1,p}(\Omega)$ and put $X_u = \{x \in \Omega : u(x) \notin [0,\xi_0]\}$. Clearly,

$$\int_{T(u)(x)}^{u(x)} f_+(x,t)\,dt = 0$$

for all $x \in \{y \in \Omega : u(y) \leq \xi_0\}$. Moreover, taking into account of condition (a), we have

$$\int_{T(u)(x)}^{u(x)} f_+(x,t)\,dt = \int_{T(u)(x)}^{u(x)} f(x,t)\,dt \leq 0$$

if $\xi_0 \leq u(x) \leq \xi_1$. Notice that, in the case $p > N$, owing to the compact embedding of $W_0^{1,p}(\Omega)$ in $C^0(\overline{\Omega})$ and being $u_{0,\lambda} \in [0,\xi_0]$, there exists a neighbourhood U of $u_{0,\lambda}$ in $W_0^{1,p}(\Omega)$ such that $v(x) \leq \xi_1$ for all $v \in U$. Finally, if $u(x) > \xi_1$ and $p \leq N$, thanks to (I^+) we have

$$\begin{aligned} \int_{T(u)(x)}^{u(x)} f_+(x,t)\,dt &= \int_{\xi_0}^{u(x)} f(x,t)\,dt \leq M \int_{\xi_0}^{u(x)} (1+t^q)\,dt \\ &= M\bigg((u(x)-\xi_0) + \frac{1}{q+1}(u(x)^{q+1} - \xi_0^{q+1}) \bigg) \\ &\leq C|u(x)-\xi_0|^{q+1} \end{aligned}$$

where

$$C = M\bigg(\sup_{\xi \geq \xi_1} \frac{\xi - \xi_0}{(\xi-\xi_0)^{q+1}} + \frac{\xi^{q+1} - \xi_0^{q+1}}{(q+1)(\xi-\xi_0)^{q+1}} \bigg) < \infty.$$

Of course, without loss of generality, we can suppose $q > p-1$.

Consequently, we have the following two cases:

$$\int_\Omega \bigg(\int_{T(u)(x)}^{u(x)} f_+(x,t)\,dt \bigg)\,dx \leq 0$$

if $p > N$ and $u \in U$;

$$\int_\Omega \left(\int_{T(u)(x)}^{u(x)} f_+(x,t)\,dt \right) dx \le C_1 \|u - T(u)\|^{q+1},$$

where

$$C_1 = C \cdot \left(\sup_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |v|^{q+1}\,dx}{\|v\|^{q+1}} \right),$$

if $p \le N$ and $u \in W_0^{1,p}(\Omega)$ (Notice that $C_1 < \infty$ thanks to the Sobolev embedding theorem). Thus, in any case we have

$$\begin{aligned} J_{\lambda,f_+}(u) \;-\; & J_{\lambda,f_+}(T(u)) \\ &= \frac{1}{p}\int_\Omega (|\nabla u|^p - |\nabla T(u)|^p)\,dx - \int_\Omega \left(\int_{T(u)(x)}^{u(x)} f_+(x,t)\,dt \right) dx \\ &\ge \frac{1}{p}\int_X |\nabla u|^p\,dx - C_1\|u - T(u)\|^{q+1} \\ &= \int_\Omega |\nabla u - \nabla T(u)|^p\,dx - C_1\|u - T(u)\|^{q+1} \\ &= \|u - T(u)\|^p \left(\frac{1}{p} - C_1 \|u - T(u)\|^{q+1-p} \right) \end{aligned}$$

for all $u \in U$. From the previous inequality, taking into account that $T(u_{0,\lambda}) = u_{0,\lambda}$, T is continuous and $q > p-1$, we easily find a beighbourhood V of $u_{0,\lambda}$ in $W_0^{1,p}(\Omega)$ such that $J_{\lambda,f_+}(u) \ge J_{\lambda,f_+}(T(u)) \ge J_{\lambda,f_+}(u_{0,\lambda})$ for all $u \in V$. Then, $u_{0,\lambda}$ turns out a local minimum for J_{λ,f_+}, as desired. Furthermore, since $0 \le u_{0,\lambda} \le \xi_0$ for a.a. $x \in \Omega$, we have

$$\begin{aligned} 0 = J_{\lambda,f_+}(0) \ge\; &= \inf_E J_{\lambda,f_+} = J_{\lambda,f_+}(u_{0,\lambda}) \\ &\ge -\lambda \int_\Omega \left(\sup_{0\le\xi\le\xi_0} \int_0^\xi f(x,t)\,dt \right) dx. \end{aligned} \tag{3.4}$$

Since, as already observed, the functional J_{λ,f_+} turns out coercive, then it admits a global minimum v_λ in $W_0^{1,p}(\Omega)$. Of Course v_λ is a critical point for J_{λ,f_+} that is v_λ is a weak solution for problem $(\mathrm{P}_{\lambda,f_+})$. Therefore, by Lemma 3.3 we have $v_\lambda \in C^{1+\gamma}(\Omega)$, with $\gamma \in]0,1[$. Let us to show that v_λ is nonnegative in Ω. Assume the contrary. Then, $Y = \{x \in \Omega : v_\lambda(x) < 0\}$ turns out a nonempty open set in Ω. Moreover, we have $v_{\lambda|A} \in W_0^{1,2}(A)$. Consequently, since one has

$$\int_\Omega |\nabla v_\lambda(x)|^{p-2} \nabla v_\lambda(x) \nabla v(x)\,dx = \lambda \int_\Omega f_+(x, v_\lambda(x)) v(x)\,dx$$

for all $v \in W_0^{1,p}(\Omega)$, we infer

$$\int_A |\nabla v_\lambda(x)|^{p-2} \nabla v_\lambda(x) \nabla v(x)\,dx = 0 \tag{3.5}$$

for all $v \in C_0^\infty(A)$. By density, (3.5) actually holds for all $v \in W_0^{1,2}(A)$. In particular, choosing $v = v_{\lambda|A}$ we get $\int_A |\nabla v_\lambda(x)|^p\,dx = 0$ which is absurd. Now, we claim that $u_{0,\lambda} \neq v_\lambda$. Indeed, taking into account of condition (b) and since $\lambda > \|u_0\|^p/(p\overline{\eta})$, it is enough to note that

$$\begin{aligned} J_{\lambda,f_+}(v_\lambda) &= \inf_{W_0^{1,p}(\Omega)} J_{\lambda,f_+} \leq J_{\lambda,f_+}(u_0) \\ &= \frac{1}{p}\|u_0\|^p - \lambda \int_\Omega \left(\int_0^{u_0(x)} f_+(x,t)\,dt - \sup_{0\leq\xi\leq\xi_0} \int_0^\xi f_+(x,t)\,dt \right) dx \\ &\quad - \lambda \int_\Omega \left(\sup_{0\leq\xi\leq\xi_0} \int_0^\xi f_+(x,t)\,dt \right) dx \\ &< \frac{1}{p}\|u_{0,\lambda}\|^p - \lambda \int_\Omega \left(\int_0^{u_{0,\lambda}(x)} f_+(x,t)\,dt \right) dx = J_{\lambda,f_+}(u_{0,\lambda}). \end{aligned}$$

Observe that, the previous inequality and (3.4) imply $v_\lambda \neq 0$. Furthermore, in view of Lemma 3.2 of [6], the functional J_{λ,f_+} satisfies the Palais–Smale condition. Hence, by Theorem 1 of [10], we easily get the existence of a third critical point $u_{1,\lambda}$ of J_{λ,f_+} distinct by $u_{0,\lambda}, v_\lambda$ such that

$$J_{\lambda,f_+}(u_{1,\lambda}) = \inf_{\psi\in\Gamma_\lambda} \sup_{t\in[0,1]} J_{\lambda,f_+}(\psi(t)) \tag{3.6}$$

where $\Gamma_\lambda = \{\psi \in C^0([0,1], W_0^{1,p}(\Omega)) : \psi(0) = u_{0,\lambda} \text{ and } \psi(1) = v_\lambda\}$.

By the same argument used to show that v_λ is nonnegative in Ω we infer that $u_{1,\lambda}$ in nonnegative in Ω as well.

Hence, to prove the first part of our thesis it is enough to observe that at least one of the functions $u_{0,\lambda}, u_{1,\lambda}$ must be nonzero, then we choose it as the nonzero solution u_λ. At this point, it remains to prove the second part of our thesis which provides the stability of the solutions with respect to λ. Fix a bounded set $K \subset \,]\|u_0\|^p/(p\overline{\eta}), \infty[$. By (3.4) and condition (I^+) we infer

$$\begin{aligned} \|u_{0,\lambda}\|^p &\leq p\lambda \int_\Omega \left(\int_0^{u_{0,\lambda}(x)} f_+(x,t)\,dt \right) dx \\ &\leq p(\sup K) \int_\Omega \left(\sup_{0\leq\xi\leq\xi_0} \int_0^\xi f(x,t)\,dt \right) dx := C_2^p < \infty \end{aligned} \tag{3.7}$$

for every $\lambda \in K$. Now, we claim that

$$\sup_{\lambda\in K} \|v_\lambda\| \leq C_3 < \infty. \tag{3.8}$$

Arguing by contradiction, assume $\sup_{\lambda\in K} \|v_\lambda\| = \infty$ and let $\{\lambda_n\}$ be a sequence in K such that $\lim_{n\to\infty} \|v_{\lambda_n}\| = \infty$. At first observe that the real function

$$\lambda \to \inf_{W_0^{1,p}(\Omega)} J_{\lambda,f_+} = \inf_{u\in W_0^{1,p}(\Omega)} \left(\frac{1}{p}\|u\|^p - \lambda \int_\Omega \left(\int_0^{u(x)} f_+(x,t)\,dt \right) dx \right)$$

is well defined and concave in $\mathbb{R}$, so in particular it is bounded on every bounded set in $\mathbb{R}$. Then, if

$$\int_\Omega \left(\int_0^{v_{\lambda_n}(x)} f_+(x,t)\,dt \right) \le 0$$

for infinite λ_n, we should have

$$\infty > \sup_{\lambda \in K} \inf_{W_0^{1,p}(\Omega)} J_{\lambda,f_+} \ge J_{\lambda_n,f_+}(v_{\lambda_n}) \ge \frac{1}{p}\|v_{\lambda_n}\|^p$$

for infinite λ_n, which is absurd. On the other hand, if

$$\int_\Omega \left(\int_0^{v_{\lambda_n}(x)} f_+(x,t)\,dt \right) \ge 0$$

for infinite λ_n, then, putting $\lambda_K = \sup K$, we should have

$$\begin{aligned} \infty > \sup_{\lambda \in K} \inf_{W_0^{1,p}(\Omega)} J_{\lambda,f_+} &\ge \frac{1}{p}\|v_{\lambda_n}\|^p - \lambda_K \int_\Omega \left(\int_0^{v_{\lambda_n}(x)} f_+(x,t)\,dt \right) dx \\ &+ (\lambda_K - \lambda_n) \int_\Omega \left(\int_0^{v_{\lambda_n}(x)} f_+(x,t)\,dt \right) dx \ge J_{\lambda_K,f_+}(v_{\lambda_n}) \end{aligned}$$

for infinite λ_n and this is absurd being J_{λ,f_+} coercive for all $\lambda \in \mathbb{R}$. Consequently, condition (3.6) holds. So, to complete the prove, it remain to show that

$$\sup_{\lambda \in K} \|u_{1,\lambda}\| < \infty. \tag{3.9}$$

To this aim, we first note that the functional

$$v \to \int_\Omega \left| \int_0^{v(x)} |f_+(x,t)|\,dt \right| dx$$

is weakly continuous (and, in particular, bounded) on every closed ball of $W_0^{1,p}(\Omega)$. Thus, in view of this and taking into account of (3.6)–(3.8), we get

$$\begin{aligned} J_{\lambda,f_+}(u_{1,\lambda}) &\le \sup_{t \in [0,1]} J_{\lambda,f_+}(tu_{0,\lambda} + (1-t)v_\lambda) \\ &\le \frac{1}{p}\|u_{0,\lambda}\|^p + \frac{1}{p}\|v_\lambda\|^p - \lambda \int_\Omega \left(\int_0^{v_\lambda(x)} f_+(x,t)\,dt \right) dx \\ &\quad + \lambda \int_\Omega \left(\int_0^{v_\lambda(x)} f_+(x,t)\,dt - \int_0^{tu_{0,\lambda}(x)+(1-t)v_\lambda(x)} f_+(x,t)\,dt \right) dx \\ &\le \frac{1}{p} C_2^p + \sup_{\lambda \in K} \inf_{W_0^{1,p}(\Omega)} J_{\lambda,f_+} \\ &\quad + \sup(K) \left[\sup_{\|v\| \le C_3} \int_\Omega \left| \int_0^{v(x)} |f_+(x,t)|\,dt \right| dx \right. \\ &\quad \left. + \sup_{\|v\| \le C_2 + C_3} \int_\Omega \left| \int_0^{v(x)} |f_+(x,t)|\,dt \right| dx \right] := C_4 < \infty \end{aligned}$$

for all $\lambda \in K$. At this point, to prove (3.9) we can follow the same argument used to prove (3.8) starting from the fact that $\sup_{\lambda\in K} J_{\lambda,f_+}(u_{1,\lambda}) < \infty$. So, since the nonnegative solutions of problems $(\mathrm{P}_{\lambda,f})$ and $(\mathrm{P}_{\lambda,f_+})$ are the same, the proof is now complete. $\square$

Remark 3.5. Let us suppose $\xi_0 = 0$ in the assumptions of Theorem 3.4. Then, conditions (a) and (b) become

(a) $\displaystyle\sup_{\xi\in[0,\xi_1]} \int_0^\xi f(x,t)\,dt \le 0$ for a.a. $x \in \Omega$,

(b) there exists $u_0 \in W_0^{1,p}(\Omega)$ with $u_0(x) \ge 0$ for a.a. $x \in \Omega$ such that

$$\int_\Omega \bigg(\int_0^{u_0(x)} f(x,t)\,dt \bigg)\,dx > 0.$$

Remark 3.6. Suppose f independent from the first variable and choose $R > 0$ and $x_0 \in \Omega$ such that $\{x \in \mathbb{R}^N : |x - x_0| \le R\} \subseteq \Omega$. After, define

$$u_0(x) = \begin{cases} c(1 - |x - x_0|/R & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \Omega \setminus \{x \in \mathbb{R}^N : |x - x_0| \le R\} \end{cases}$$

where $c > 0$. Then, we easily get

$$\int_\Omega \bigg(\int_0^{u_0(x)} f(t)\,dt \bigg)\,dx = \omega_N \bigg(\frac{R}{c}\bigg)^N \int_0^c (c-\xi)^{N-1} \int_0^\xi f(t)\,dt\,d\xi$$

and

$$\|u_0\|^p = \omega_N \bigg(\frac{c}{R}\bigg)^p$$

where $\omega_N = \int_{|x|\le 1} dx$ is the volume of the unit ball in $\mathbb{R}^N$. In particular, if in condition (b)of Remark 3.5 we choose u_0 as above, then we can restate it as follows:

(b') there exists $c > 0$ such that

$$\int_0^c (c-\xi)^{N-1} \int_0^\xi f(t)\,dt\,d\xi > 0.$$

Taking into account of Remarks 3.5 and 3.6, we obtain the following consequence of Theorem 3.4:

Theorem 3.7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = 0$ satisfying* (I^+), (a), (b') *and*

$$\sup_{\xi\ge 0} \bigg(\int_0^\xi f(t)\,dt \bigg) \Big/ (1 + |\xi|^s) < \infty \quad \textit{for some } s \in\,]0,p[.$$

Then, for all

$$\lambda > \lambda_0 \overset{\text{def}}{=} \bigg(\frac{c}{R}\bigg)^{N+p} \bigg(p \int_0^c (c-\xi)^{N-1} \int_0^\xi f(t)\,dt\,d\xi \bigg)$$

there exist two nonzero nonnegative weak solutions $u_\lambda, v_\lambda \in W_0^{1,p}(\Omega) \cap C^{1+\gamma}(\Omega)$ (*with* $\gamma \in]0,1[$) *of problem* $(\mathrm{P}_{\lambda,f})$. *Moreover, one has* $\sup_{\lambda\in K} \max\{\|u_\lambda\|, \|v_\lambda\|\} < \infty$ *for every bounded set* $K \subset]\lambda_0, \infty[$.

Observe that, when $p = 2$ and $\Omega =]0,1[$, Theorem 3.7 gives back Theorem 3.9 of [2] where, nevertheless, the stability of the solutions is not specified. Also, in the proof of latter result, the embedding of $W_0^{1,2}(]0,1[)$ in $C^0([0,1])$ turns out essential. So, in the case $p < N$, we cannot extend Theorem 3.9 of [2] to problem $(\mathrm{P}_{\lambda,f})$ using the same methods employed there. In view of this, Theorem 3.7 turns out a remarkable extension to higher dimension and to the p-Laplacian operator of Theorem 3.9 of [2].

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