Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 25, 2005, 155–166

MULTIPLICITY OF POSITIVE SOLUTIONS FOR SEMILNEAR ELLIPTIC PROBLEMS WITH ANTIPODAL SYMMETRY

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Abstract. In this paper, we show the multiple existence of positive solutions of semilinear elliptic problems of the form

$$-\Delta u = |u|^{2^*-2}u + f, \quad u \in H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, 2^* is the Sobolev critical exponent and $f \in L^2(\Omega)$.

1. Introduction

Let $N \geq 3$, $2^* = 2N/(N-2)$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$, and $f \in L^2(\Omega)$ with $f \geq 0$. The existence and multiplicity of solutions of problem

$$(\mathrm{P}_f) \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u + f & \text{on } \Omega, \\ u > 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been studied by many authors. It is known that problem (P_0) has no nontrivial solution when domain Ω is star-shaped (cf. [7]). In [6], Kazdon and Warner proved the existence of a nontrivial solution of (P_0) in the case that Ω is annulus.

2000 *Mathematics Subject Classification.* 35J60.
Key words and phrases. Critical exponent, multiple existence, semilinear elliptic problem.

In [1], Bahri and Coron established the existence of a nontrivial solution of (P_0) when Ω has nontrivial topology. On the other hand, for the nonhomogeneous problem $f \neq 0$, Tarantello [10] proved the existence of two solutions of (P_f) when $\|f\|_{L^2(\Omega)}$ is small. In the case that Ω has non trivial topology, Rey [8] proved that problem (P_f) has $\operatorname{cat}(\Omega) + 1$ solutions when f is sufficiently small.

Our purpose in this paper is to consider the multiple existence of solutions of problem (P_f) for domain $\Omega \subset \mathbb{R}^N$ and $f \in L^2(\Omega)$ having antipodal symmetry.

To state our main results, we need some notations. Throughout this paper, Ω is a bounded domain with a smooth boundary $\partial\Omega$. We denote by $B_r(0) \subset \mathbb{R}^N$ the open ball centered at 0 with radius r. We put

$$\begin{aligned} \rho(\Omega) &= \sup\{r > 0 : B_r(x) \subset \Omega \text{ for some } x \in \Omega\}, \\ \theta(\Omega) &= \sup\left\{ r > 0 : \text{there exists } A \subset \mathbb{R}^N \setminus \Omega \text{ such that } \mathbb{R}^N \setminus \Omega = \bigcup_{x \in A} B_r(x) \right\} \end{aligned}$$

and

$$k(\Omega) = \frac{\rho(\Omega)}{\theta(\Omega)}.$$

We impose the following condition on Ω:

(Ω) $\Omega = -\Omega$ and there exists $r_0 > 0$ such that $B_{r_0}(0) \cap \Omega = \phi$.

For two topological spaces X, Y, we write $X \cong Y$ when X and Y are of the same homotopy type. For each topological space X, $H_*(X)$ stands for the singular homology groups with coefficients $\mathbb{Z}_2$ (cf. [3], [9]). We denote by $\widehat{\Omega}$ the set Ω identified the antipodal points, and denote by $p_\Omega : \Omega \to \widehat{\Omega}$ the covering projection defined by $p_\Omega(x) = (-x, x)$ for $x \in \Omega$. For each $p \geq 1$, we denote by $|\cdot|_p$ the norm of $L^p(\Omega)$. We put

$$L = \{v \in L^2(\Omega) : v(x) = v(-x) \text{ for } x \in \Omega\}$$

and $H = H_0^1(\Omega) \cap L$. We can now state our main results.

Theorem 1.1. *There exists $k_0 > 0$ and $\delta_0 > 0$ such that if $k(\Omega) < k_0$, then for each $f \in L$ with $f \geq 0$ and $0 < |f|_2 < \delta_0$, problem (P_f) possesses at least two solutions in H.*

Theorem 1.2. *There exists $k_1 > 0$, $\delta_1 > 0$ such that if $k(\Omega) < k_1$, then there exists a residual subset D of $\{f \in L : f \geq 0$ and $|f|_2 < \delta_1\}$ satisfying that for each $f \in D$, problem (P_f) possesses at least $\sum_{p=0}^{\infty} \operatorname{rank} H_p(\widehat{\Omega})$ solutions in H.*

Corollary 1.3. *Suppose that $\Omega \cong S^{N-1}$. Then there exists $k > 0$, $\delta > 0$ such that if $k(\Omega) < k$, then there exists a residual subset D of $\{f \in L : f \geq 0,\ |f|_2 < \delta\}$ satisfying that for each $f \in D$, problem (P_f) possesses at least N solutions in H.*

Remark 1.4. The solutions obtained in [10] as well as in [8] are solutions with critical levels smaller than the critical level c of the grand state solution of problem (P_0) with $\Omega = \mathbb{R}^N$. On the other hand, the solutions obtained in our results have critical levels close to $2c$. Then for instance under the assumption of Theorem 1.1, we have at least four solutions of problem (P_f) in $H_0^1(\Omega)$ by the result in [10] and Theorem 1.1.

2. Preliminaries

For given $R > 0$, we denote by Λ_R the set of bounded domains Ω with smooth boundary $\partial\Omega$ such that $\operatorname{diam}(\Omega) < R$. For each measurable set $A \subset \mathbb{R}^N$, we denote by $|A|$ the measure of A. For $u, v \in H_0^1(\Omega)$, we put $\langle u, v\rangle = \int_\Omega uv\,dx$. The norm $\|\cdot\|$ of $H_0^1(\Omega)$ is defined by $\|v\| = |\nabla v|_2$ for $v \in H_0^1(\Omega)$. For each $d \in \mathbb{R}$, Ω_d denotes the set defined by

$$\Omega_d = \begin{cases} \{x \in \mathbb{R}^N : d(x,\Omega) < d\} & \text{if } d > 0, \\ \{x \in \Omega : d(x,\partial\Omega) > -d\} & \text{if } d \le 0. \end{cases}$$

For each $a \in \mathbb{R}$, and a functional $F\colon H_0^1(\Omega) \to \mathbb{R}$, we denote by F^a the level set

$$F^a = \{v \in H_0^1(\Omega) : F(v) \le a\}.$$

For $f \in L^2(\Omega)$, we define a functional I_f on $H_0^1(\Omega)$ by

$$I_f(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{2^*}|u^+|^{2^*} - fu\right) dx \quad \text{for } u \in H_0^1(\Omega).$$

Here $u^+(x) = \max\{u(x), 0\}$ for $x \in \Omega$. Then the solutions of (P_f) correspond to critical points of functional I_f. Let

$$D^1(\mathbb{R}^N) = \{v \in L^{2^*}(\mathbb{R}^N) : |\nabla v|_2 \in L^{2^*}(\mathbb{R}^N)\}.$$

For each $(z,\varepsilon) \in \mathbb{R}^N \times (0,\infty)$, we put

$$u_{(z,\varepsilon)}(x) = m\left[\frac{\varepsilon^{1/2}}{\varepsilon + (x-z)^2}\right]^{(N-2)/2}, \quad x \in \mathbb{R}^N$$

where $m = (N(N-2))^{(N-2)/4}$. It is known that each $u_{(z,\varepsilon)}$ is a critical point of I_0 with the domain $H_0^1(\Omega)$ replaced by $D^1(\mathbb{R}^N)$. By the invariance of the norm of $D^1(\mathbb{R}^N)$ under translation and scaling

$$u \to u_R(x) = R^{-N/2^*}u(x/R), \quad R > 0, \tag{2.1}$$

we have that each $u_{(z,\varepsilon)}$ have the same critical value of I_0. We put $c = I_0(u_{(z,\varepsilon)})$ for $(z,\varepsilon) \in \mathbb{R}^N \times (0,\infty)$, and $c_0 = 2\cdot 2^* c/(2^*-2)$. We also set

$$\mathcal{S}_f(\Omega) = \{v \in H_0^1(\Omega) : \|v\|^2 = |v^+|_{2^*}^{2^*} + \langle f, v\rangle,\ I(v) = \sup_{t\in\mathbb{R}^+} I(tv)\},$$

for $f \in L$. It is easy to see that there exists $\overline{\varepsilon} > 0$ such that if $f \geq 0$, $|f|_2 < \overline{\varepsilon}$ and $v \in H \setminus \{0\}$ with $v^+ \not\equiv 0$, there exists a unique positive number $t_{f,v}$ such that $t_{f,v}v \in S_f(\Omega)$ (cf. [5], [10]). Throughout the rest of this paper, we assume that $f \geq 0$ and $|f|_2 < \overline{\varepsilon}$. For each $v \in H \setminus \{0\}$ with $v^+ \not\equiv 0$, we define $\mathcal{N}_f v \in S_f(\Omega)$ by $\mathcal{N}_f v = t_{f,v}v$. We have from the definition of $\mathcal{S}_f(\Omega)$ that

$$\langle \nabla I_f(v), v\rangle = 0 \quad \text{for all } v \in \mathcal{S}_f(\Omega). \tag{2.2}$$

We will seek for solutions of I_f in $\mathcal{S}_f \cap H$. For simplicity of notation, we put $\widetilde{I}_f^d = I_f^d \cap \mathcal{S}_f(\Omega) \cap H$ for each $d > 0$. Let $\varphi\colon \mathbb{R}^N \to [0,1]$ be a smooth function such that $\varphi(x) = 1$ for $x \in B_{1/2}(0)$ and $\varphi(x) = 0$ on $\mathbb{R}^N \setminus B_1(0)$. We put

$$v_{(r,z,\varepsilon)}(x) = \varphi((x-z)/r)u_{(z,\varepsilon)}(x) \quad \text{for } (r,z,\varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^+ \text{ and } x \in \mathbb{R}^N.$$

We also fix a mapping $\eta \in C^\infty([0,\infty);[0,1])$ such that $\eta(t) = 0$ for $t \in [0,1/2]$ and $\eta(t) = 1$ for $t \geq 1$. For each $x \in \mathbb{R}^N \setminus \{0\}$, we define a mapping $\tau_x\colon \mathbb{R}^N \to [0,1]$ by

$$\tau_x(z) = \eta(d(z,\{x\}^\perp)) \quad \text{for } z \in \mathbb{R}^N.$$

To prove theorems, it is sufficient to prove the assertions for each $R > 0$ and each $\Omega \in \Lambda_R$. Then, in the rest of this paper, we fix $R > 0$ and assume that $\Omega \in \Lambda_R$.

The following lemma is a simple consequence from the definition of τ_x.

Lemma 2.1. *Let $\{\Omega^{(n)}\}$, $\{x_n\} \subset \mathbb{R}^N \setminus \{0\}$ and $\{u_n\}$ be sequences such that $\Omega^{(n)} \in \Lambda_R$, $\rho(\Omega^{(n)}) = 1$ for each $n \geq 1$, $u_n \in H_0^1(\Omega^{(n)})$ for $n \geq 1$, and*

$$\lim_{n\to\infty} \int_{F(x_n)} |\nabla u_n|^2 = \lim_{n\to\infty} \int_{F(x_n)} |u_n|^{2^*} = 0,$$

where $F(x_n) = \{z \in \mathbb{R}^N : d(z,\{x_n\}^\perp) \leq 1\}$. Then

$$\lim_{n\to\infty} \int_{F(x_n)} |\nabla(\tau_{x_n}u_n)|^2 = \lim_{n\to\infty} \int_{F(x_n)} |\tau_{x_n}u_n|^{2^*} = 0.$$

Proof. Let $\{\Omega^{(n)}\}$, $\{x_n\}$ and $\{u_n\}$ satisfy the assumption. From the definition of τ_x, we have that there exists, $C > 0$ such that $|\nabla\tau_x|_\infty \leq C$ for all $x \in \mathbb{R}^N$. On the other hand, since $\Omega^{(n)} \in \Lambda_R$ for $n \geq 1$, we have that

$$\begin{aligned}\int_{F(x_n)} |u_n|^2 &\leq |F(x_n) \cap \Omega^{(n)}|^{(2^*-2)/2^*} \bigg(\int_{F(x_n)} |u_n|^{2^*}\bigg)^{2/2^*} \\ &\leq R^2 \bigg(\int_{F(x_n)} |u_n|^{2^*}\bigg)^{2/2^*}\end{aligned}$$

for each $n \geq 1$. Then from the assumption, we have

$$\begin{aligned}\lim_{n\to\infty}\int_{F(x_n)}|\nabla(\tau_{x_n}u_n)|^2 &= \lim_{n\to\infty}\int_{F(x_n)}|\tau_{x_n}\nabla u_n+\nabla\tau_{x_n}u_n|^2\\ &\leq 2\lim_{n\to\infty}\bigg(\int_{F(x_n)}|\nabla u_n|^2+C^2\int_{F(x_n)}|u_n|^2\bigg)=0.\end{aligned}$$

It is also easy to see that $\lim_{n\to\infty}\int_{F(x_n)}|\tau_{x_n}u_n|^{2^*}=0$ holds. □

Lemma 2.2. *There exist positive numbers $\overline{\delta}$ and k_0 such that if $k(\Omega)\leq k_0$, then there exists $r>0$ satisfying that the following conditions:*

(a) $\Omega\cong\Omega_{3r}$,
(b) *for each $u\in\widetilde{I}_0^{2c+\overline{\delta}}\cap S_0(\Omega)$, there is $x\in\Omega_r$ such that $B_{4r}(x)\cap B_{4r}(-x)=\phi$ and*

$$\int_{B_r(x)\cup B_r(-x)}|u|^{2^*}\,dx\geq\frac{4}{3}c_0.$$

Proof. We first note that if $\{u_n\}\subset\mathcal{S}_0(\mathbb{R}^N)$ satisfies $\lim_{n\to\infty}I_0(u_n)=c$, then there exists a sequence $\{(z_n,\varepsilon_n)\}\subset\mathbb{R}^N\times\mathbb{R}^+$ such that $\lim_{n\to\infty}\|u_n-u_{(z_n,\varepsilon_n)}\|=0$ and $\lim_{n\to\infty}|u_n-u_{(z_n,\varepsilon_n)}|_{2^*}=0$ (cf. [1], [10]).

Now suppose contrary that there exists a sequence $\{\Omega^{(n)}\}\subset\mathbb{R}^N$ and $\{u_n\}\subset H_0^1(\Omega)$ such that $\Omega^{(n)}\in\Lambda_R$ for each $n\geq 1$, $\lim_{n\to\infty}k(\Omega^{(n)})=0$, $u_n\in\mathcal{S}_0(\Omega^{(n)})\cap H$ with $\lim_{n\to\infty}I_0(u_n)=2c$ and

$$\int_{B_r(x)\cup B_r(x)}|u_n|^{2^*}\,dx<\frac{4}{3}c_0$$

for any $(r,x)\in\mathbb{R}^+\times\Omega_r$ with $B_{4r}(x)\cap B_{4r}(-x)=\phi$ and $\Omega^{(n)}\cong(\Omega^{(n)})_{3r}$ for all $n\geq 1$. By the invariance of the norms $\|\cdot\|$ and $|\cdot|_{2^*}$ under the scaling (2.1), we may assume that $\rho(\Omega^{(n)})=1$ for all $n\geq 1$. Since $\lim_{n\to\infty}k(\Omega^{(n)})=0$, we find that

$$\overline{r}_n=\sup\{r>0:B_r(0)\subset\mathbb{R}^N\setminus\Omega^{(n)}\}\to\infty,\quad\text{as } n\to\infty. \tag{2.3}$$

Then it is easy to see that there exists a sequence $\{x_n\}\subset\mathbb{R}^N\setminus\{0\}$ such that

$$\lim_{n\to\infty}\int_{F(x_n)}|\nabla u_n|^2=\lim_{n\to\infty}\int_{F(x_n)}|u_n|^{2^*}=0,$$

Put $u_n'=\tau_{x_n}u_n$ for $n\geq 1$. Then we have by Lemma 2.1 that

$$\lim_{n\to\infty}\int_{F(x_n)}|\nabla u_n'|^2=\lim_{n\to\infty}\int_{F(x_n)}|u_n'|^{2^*}=0, \tag{2.4}$$

holds. Therefore we have that

$$\lim_{n\to\infty}|\nabla u_n'|_2^2=\lim_{n\to\infty}\bigg(\int_{\Omega^{(n)}\setminus F(x_n)}|\nabla u_n|^2+\int_{F(x_n)}|\nabla u_n'|^2\bigg)=\lim_{n\to\infty}|\nabla u_n|_2^2. \tag{2.5}$$

Similarly, we have

$$\lim_{n\to\infty} |u'_n|_{2_*}^{2^*} = \lim_{n\to\infty} |u_n|_{2_*}^{2^*}. \tag{2.6}$$

From the definition of u'_n, we have that

$$u'_n = v_n^1 + v_n^2, \quad \text{where } v_n^1, v_n^2 \in H_0^1(\Omega^{(n)}),$$
$$\operatorname{supp} v_n^1 \cap \operatorname{supp} v_n^2 = \phi,$$
$$v_n^1(x) = v_n^2(-x)$$

for each $n \geq 1$. It then follows from (2.5) and (2.6) that

$$\lim_{n\to\infty} \|u_n - (v_n^1 + v_n^2)\| = \lim_{n\to\infty} |u_n - (v_n^1 + v_n^2)|_{2^*} = 0. \tag{2.7}$$

It then follows that there exists $\{(z_n, \varepsilon_n)\} \subset \mathbb{R}^N \times \mathbb{R}^+$ such that

$$\lim_{n\to\infty} \|v_n^1 - u_{(z_n,\varepsilon_n)}\| = \lim_{n\to\infty} |v_n^1 - u_{(z_n,\varepsilon_n)}|_{2^*} = 0. \tag{2.8}$$

One can see that $\sup_n \varepsilon_n < \infty$. In fact, noting that $\lim_{n\to\infty} \theta(\Omega_n) = \infty$, we have

$$\lim_{n\to\infty} |\Omega^{(n)}|/|B_{r_n}(z_n)| = 0, \tag{2.9}$$

where $r_n = \inf\{r > 0 : \Omega_n \subset B_r(z_n)\}$ for each $n \geq 1$. Then if $\sup_n \varepsilon_n = \infty$, we have from (2.9) that

$$c_0 = \lim_{n\to\infty} |v_n^1|_{2^*}^{2^*} = \lim_{n\to\infty} \int_{\Omega^{(n)}} |v_n^1|^{2^*} = \lim_{n\to\infty} \inf \int_{\Omega^{(n)}} |u_{(z_n,\varepsilon_n)}|^{2^*} = 0.$$

This is a contradiction. Thus we have $\varepsilon = \sup_n \varepsilon_n < \infty$. Now we fix $r_1 > 0$ such that

$$\int_{B_{r_1}(0)} |u_{(0,\varepsilon)}|^{2^*} = \frac{3}{4} c_0. \tag{3.10}$$

Since $\lim_{n\to\infty} \theta(\Omega^{(n)}) = \infty$, we have that there exists $n_0 \geq 1$ such that $\Omega^{(n)} \cong (\Omega^{(n)})_{3r_1}$. We can choose $n_1 \geq n_0$ such that $\bar{r}_n \geq 5r_1$ for all $n \geq n_1$. Now suppose that $\liminf_{n\to\infty} |z_n| \leq 4r_1$. Then noting that $B_{r_1}(z_n) \subset \mathbb{R}^N \setminus \Omega^{(n)}$ in case that $|z_n| \leq 4r_1$, we have

$$0 = \lim_{n\to\infty} \inf \int_{B_{r_1}(z_n)} |v_n^1|^{2^*} = \lim_{n\to\infty} \inf |u_{(z_n,\varepsilon_n)}|^{2^*} \geq \frac{3}{4} c_0.$$

This is a contradiction. Thus we find that $\liminf_{n\to\infty} |z_n| > 4r_1$. This implies that $B_{4r_1}(z_n) \cap B_{4r_1}(-z_n) = \phi$. We also have that $z_n \in \Omega_{r_1}^{(n)}$ for $n \geq 1$. In fact if $z_n \notin \Omega_{r_1}^{(n)}$, then $\int_{B_{r_1}(z_n)} |v_n^1|^{2^*} = 0$. Then again we reaches to a contradiction. Now we have by (2.7), (2.8) and (2.10) that

$$\int_{B_{r_1}(z_n) \cup B_{r_1}(-z_n)} |u_n|^{2^*} \, dx \geq \frac{4}{3} c_0$$

for n sufficiently large. This contradicts to the assumption. Then the assertion follows. □

Lemma 2.3. *Let $f \in L$ such that $f \geq 0$ and $0 < |f|_2 < \overline{\varepsilon}$. Let $r' > 0$ such that $\Omega_{-r'} \cong \Omega$ and*

$$\int_{\Omega_{-r'}} |f|^2\,dx > |f|_2^2/2.$$

Then there exists $\varepsilon_0 > 0$ and a positive function $w_{(z,\varepsilon)} \in H$ for each $(z,\varepsilon) \in \Omega_{-r'} \times (0,\varepsilon_0)$ such that

$$\sup\{I_f(\mathcal{N}_f(v_{(r',z,\varepsilon)}+v_{(r',-z,\varepsilon)}+w_{(z,\varepsilon)})) : z \in \Omega_{-r'}\} < 2c \quad \text{for } \varepsilon \in (0,\varepsilon_0). \tag{2.11}$$

Proof. The argument is standard. For completeness, we give a proof. Let $f \in L$ and $r' > 0$ satisfy the assumption. We choose $d_0 > 0$ so small that

$$\int_{\Omega_{-r'}\setminus(B_{d_0}(z)\cup B_{d_0}(-z))} |f|^2\,dx > |f|_2^2/3 \quad \text{for all } z \in \Omega. \tag{2.12}$$

Let $\psi\colon\overline{\Omega} \to [0,1]$ be a mapping such that $\psi \in C^2(\overline{\Omega})$, $\psi(x) = \psi(-x)$ on Ω, $\psi(x) = 1$ on $\Omega_{-r'}$ and $\psi(x) = 0$ on $\partial\Omega$. We fix $d \in (0,\min\{d_0/2, r'\})$ and put

$$w_{(z,\varepsilon)}(x) = \varepsilon^{1/4}[\psi(x) - \varphi((x-z)/2d) - \varphi((x+z)/2d)] \quad \text{for } x, z \in \Omega \text{ and } \varepsilon > 0.$$

By (Ω), we have that $|x| \geq r'$ for each $x \in \Omega_{-r'}$. That is $B_{r'}(x) \cap B_{r'}(-x) = \phi$. Fix $z \in \Omega$. Then, for $\varepsilon > 0$ sufficiently small, we have

$$|\nabla v_{(d,z,\varepsilon)}|_2^2 = c_0 + O(\varepsilon^{(N-2)/2}), \tag{2.13}$$

$$|v_{(d,z,\varepsilon)}|_{2^*}^{2^*} = c_0 + O(\varepsilon^{N/2}), \tag{2.14}$$

(cf. [2]). On the other hand, we have by the definition of $w_{(z,\varepsilon)}$ and (2.12) that

$$|\nabla w_{(z,\varepsilon)}|_2^2 = O(\varepsilon^{1/2}), \quad |w_{(z,\varepsilon)}|_{2^*}^{2^*} = O(\varepsilon^{N/2(N-2)}), \quad \langle f, w_{(z,\varepsilon)}\rangle = O(\varepsilon^{1/4}) \tag{2.15}$$

for ε sufficiently small. We put $y_{(z,\varepsilon)}(x) = v_{(d,z,\varepsilon)} + v_{(d,-z,\varepsilon)} + w_{(z,\varepsilon)}$. Let $t = t_{f,y_{(z,\varepsilon)}}$. Then t satisfies

$$t^2|\nabla y_{(z,\varepsilon)}|_2^2 = t^{2^*}|y_{(z,\varepsilon)}|_{2^*}^{2^*} + t\langle f, y_{(z,\varepsilon)}\rangle.$$

Then noting that

$$|\nabla y_{(z,\varepsilon)}|_2^2 = |\nabla v_{(r',z,\varepsilon)}|_2^2 + |\nabla v_{(r',-z,\varepsilon)}|_2^2 + |\nabla w_{(z,\varepsilon)}|_2^2$$

and

$$|y_{(z,\varepsilon)}|_{2^*}^{2^*} = |v_{(r',z,\varepsilon)}|_{2^*}^{2^*} + |v_{(r',-z,\varepsilon)}|_{2^*}^{2^*} + |w_{(z,\varepsilon)}|_{2^*}^{2^*},$$

we find from (2.13)–(2.15) that $t = 1 - O(\varepsilon^{1/4})$. Then we have

$$I(\mathcal{N}_f y_{(z,\varepsilon)}) = \frac{(2^*-2)t^{2^*}}{2\cdot 2^*}|y_{(z,\varepsilon)}|_{2^*}^{2^*} - \frac{t}{2}\langle f, y_{(z,\varepsilon)}\rangle \leq 2(1 - O(\varepsilon^{1/4}))c.$$

Thus we find that the assertion holds by taking ε_0 sufficiently small. □

Throughout the rest of this paper, we assume that $k(\Omega) \le k_0$ holds. We fix $r > 0$ and $\overline{\delta} > 0$ satisfying the assertion of Lemma 2.2. From the definition of $\mathcal{S}_f(\Omega)$, we have that $\mathcal{N}_f(u) \to \mathcal{N}_0(u)$ and $I_f(\mathcal{N}_f u) \to I_0(\mathcal{N}_0 u)$, as $f \to 0$, uniformly on $I_f^d \cap \mathcal{S}_f(\Omega)$ for each $d > 0$. That is we have

Lemma 2.4. *Let $d > 0$ and $\delta > 0$. Then there exists $\varepsilon \in (0, \overline{\varepsilon})$ such that for each $f \in H$ with $|f|_2 < \varepsilon$,*

$$I_0(\mathcal{N}_0 u) \le I_f(u) + \delta \quad \text{for all } u \in I_f^d \cap \mathcal{S}_f(\Omega).$$

The assertion of Lemma 2.4 is a direct consequence of the definition of $\mathcal{N}_f$. Then we omit the proof. We now put $\delta = \overline{\delta}$ and $d = c$ in Lemma 2.4. Then by Lemma 2.4, we can choose $\widetilde{\varepsilon} \in (0, \overline{\varepsilon})$ such that for $f \in H$ with $|f|_2 < \widetilde{\varepsilon}$

$$I_0(\mathcal{N}_0 u) \le 2c + \overline{\delta} \quad \text{for } u \in \widetilde{I}_f^{2c}. \tag{2.17}$$

We may assume that $\overline{\delta} < c/4$. Then again by Lemma 2.4 and Lemma 2.2 that

$$I_f(u) \ge \frac{13}{12} c \quad \text{for all } u \in \mathcal{S}_f(\Omega) \cap H. \tag{2.17}$$

Here we note that Palais–Smale (PS) condition holds in the interval $(c, 2c)$ for I_f (cf. [10], [5]). That is if $\{u_n\} \subset H_0^1(\Omega)$ with $\lim_{n\to\infty} I_f(u_n) = d \in (c, 2c)$ and $\lim_{n\to\infty} \nabla I_f(u_n) = 0$, then there exists a convergent sequence $\{u_{n_i}\} \subset \{u_n\}$ with $u_{n_i} \to u$, $I_f(u) = d$ and $\nabla I_f(u) = 0$. Therefore from (2.17), we find that (PS) condition holds on $\widetilde{I}_f^{2c-\sigma}$. In the following, we assume that $f \in H$ satisfies $|f|_2 < \widetilde{\varepsilon}$. Then there exists $r > 0$ satisfying the assertion of Lemma 2.2. Here we fix a continuous function $\xi\colon [0,\infty) \to [0,1]$ such that $\xi(t) = 1$ for $t \ge 2/3$ and $\xi(t) = 0$ for $t \le 1/2$. For each $u \in H_0^1(\Omega) \setminus \{0\}$, we define a continuous function $\beta\colon \mathbb{R}^N \to [0,1]$ by

$$\beta_u(x) = \xi\bigg(\frac{\int_{B_r(x)} |u|^{2^*}\,dx}{|u|_{2^*}^{2^*}}\bigg) \quad \text{for } x \in \mathbb{R}^N.$$

In the following we assume that $f \in L$ with $|f|_2 < \widetilde{\varepsilon}$. Then we have

Lemma 2.5. *Let $u \in \widetilde{I}_f^{2c} \cap \mathcal{S}_f(\Omega)$. Then there exists $z \in \mathbb{R}^N$ such that $|z| > 4r$, $\Omega' = \{x \in \Omega : \beta_u(x) > 0\} \subset B_{2r}(z) \cup B_{2r}(-z)$, and*

$$\frac{\int_{B_r(z)\cap\Omega'} \beta_u(x)x}{\int_{B_r(z)\cap\Omega'} \beta_u(x)} \in \Omega_{3r}, \tag{2.18}$$

Proof. Let $u \in \widetilde{I}_f^{2c}$. Then by Lemma 2.2, there exists $z \in \Omega_r$ such that

$$\int_{B_r(z)\cup B_r(-z)} |\mathcal{N}_0 u|^{2^*}\,dx \ge \frac{4}{3} c_0.$$

From the inequality above, it is obvious that

$$\beta_u(x) = \beta_{\mathcal{N}_0 u}(x) = 0 \quad \text{for } x \in \mathbb{R}^N \setminus (B_{2r}(z) \cup B_{2r}(-z)).$$

Then

$$\Omega' = \{x \in \Omega : \beta_u(x) > 0\} \subset B_{2r}(z) \cup B_{2r}(-z).$$

Since $z \in \Omega_r$, we have that $\Omega' \subset \Omega_{3r}$. Then (2.18) holds. □

From lemma above, we can define a mapping $\widetilde{\gamma}: \widetilde{I}_f^{2c} \to \widehat{\Omega}_{3r}$ by

$$\widetilde{\gamma}(u) = \left\{ \frac{\int_{B_r(z)\cap\Omega'} \beta_u(x)x}{\int_{B_r(z)\cap\Omega'} \beta_u(x)}, \frac{\int_{B_r(-z)\cap\Omega'} \beta_u(x)x}{\int_{B_r(-z)\cap\Omega'} \beta_u(x)} \right\},$$

where $z \in \mathbb{R}^N$ is the point obtained in Lemma 2.5. One can see, from the fact $\Omega' \subset B_{2r}(z) \cup B_{2r}(-z)$, that $\widetilde{\gamma}(u)$ does not depend on the choice of z, and $\widetilde{\gamma}: \widetilde{I}_f^{2c} \to \widehat{\Omega}_{3r}$ is continuous. Then we have

Lemma 2.6. *For each* $p \geq 1$, $\operatorname{rank} H_p(\widetilde{I}_f^{2c-\sigma}) \geq \operatorname{rank} H_p(\widehat{\Omega})$ *for* $\sigma > 0$ *sufficiently small.*

Proof. By Lemma 2.3, there exists positive numbers r_1, ε_0, such that $\Omega \cong \Omega_{-r_1}$ and that for each $(z, \varepsilon) \in \Omega_{-r_1} \times (0, \varepsilon_0)$,

$$\sup\{I_f(\mathcal{N}_f(v_{(r_1,z,\varepsilon)} + v_{(r_1,-z,\varepsilon)} + w_{(z,\varepsilon)}) : z \in \Omega_{-r_1}\} < 2c, \tag{2.19}$$

where $w_{(z,\varepsilon)} \in H$ the function defined in the proof of Lemma 2.3. Then we have that $\widehat{\Omega}_{3r} \cong \widehat{\Omega} \cong \widehat{\Omega}_{-r_1}$, and $H_p(\widehat{\Omega}_{3r}) \cong H_p(\widehat{\Omega}) \cong H_p(\widehat{\Omega}_{-r_1})$ for each $p \geq 0$. We denote by θ the retraction from Ω_{3r} to Ω_{-r_1}. We put

$$W_1 = \{\mathcal{N}_f(v_{(r_1,z,\varepsilon)} + v_{(r_1,-z,\varepsilon)} + w_{(z,\varepsilon)}) : z \in \Omega_{-r_1}\}.$$

Let $j: \widehat{\Omega}_{-\delta_1} \to W_1$ be the mapping defined by

$$j[(z, -z)] = \mathcal{N}_f(v_{(r_1,z,\varepsilon)} + v_{(r_1,-z,\varepsilon)} + w_{(z,\varepsilon)}) \quad \text{for each } x \in \Omega_{-r_1}.$$

From the definition of $w_{(z,\varepsilon)}$, we have that $w_{(z,\varepsilon)} \to 0$ as $\varepsilon \to 0$. Then

$$\gamma(\mathcal{N}_f(v_{(r_1,z,\varepsilon)} + v_{(r_1,-z,\varepsilon)} + w_{(z,\varepsilon)})) \to \gamma(\mathcal{N}_f(v_{(r_1,z,\varepsilon)} + v_{(r_1,-z,\varepsilon)}) = (z, -z),$$

as $\varepsilon \to 0$. That is $\theta \circ \gamma \circ j \to i$, as $\varepsilon \to 0$, where $i: \Omega_{-r_1} \to \Omega_{-r_1}$ is the identity mapping. Therefore we have by choosing $\varepsilon_1 \in (0, \varepsilon_0)$ sufficiently small that $\theta \circ \gamma \circ j(\Omega_{-r_1}) \cong \Omega_{-r_1}$. By Lemma 2.3, we have that there exists $\sigma > 0$ such that

$$\sup\{I_f(\mathcal{N}_f(v_{(r_1,z,\varepsilon_1)} + v_{(r_1,-z,\varepsilon_1)} + w_{(z,\varepsilon_1)}) : z \in \Omega_{-r_1}\} < 2c - \sigma. \tag{2.20}$$

We now consider the following sequence:

$$\widehat{\Omega}_{-r_1} \xrightarrow{j} \widetilde{I}_f^{2c-\sigma} \xrightarrow{\gamma} \widehat{\Omega}_{3r} \xrightarrow{\theta} \widehat{\Omega}_{-r_1}.$$

Then noting that $\theta_* \circ \gamma_* \circ j_*$ is the identity mapping on $H_p(\widehat{\Omega}_{-r_1})$, we have from the sequence

$$H_p(\widehat{\Omega}_{-r_1}) \xrightarrow{j^*} H_p(\widetilde{I}_f^{2c-\sigma}) \xrightarrow{\widetilde{\gamma}^*} H_p(\widehat{\Omega}_{3r}) \xrightarrow{\theta^*} H_p(\widehat{\Omega}_{-r_1}),$$

that

$$\operatorname{rank} H_p(\widetilde{I}_f^{2c-\sigma}) \geq \operatorname{rank} H_p(\widehat{\Omega}_{-r_1}) = \operatorname{rank} H_p(\widehat{\Omega}) \quad \text{for each } p \geq 1. \qquad \square$$

Proof of Theorem 1.1. From the assumption (Ω), we have that $H_0(\Omega) \neq \{0\}$ and $H_p(\Omega) \neq \{0\}$ for some $p \geq 1$. By the Thom–Gysin exact sequence

$$\cdots \to H_p(\Omega) \xrightarrow{p_*} H_p(\widehat{\Omega}) \xrightarrow{\xi\cap} H_{p-1}(\widehat{\Omega}) \longrightarrow H_{q-1}(\Omega) \to \cdots$$

where $\xi \in H^1(\widehat{\Omega})$ (cf. [9, Chapter 5.3, Theorem 11], we find that $\sum_{p=0}^{\infty} H_p(\widehat{\Omega}) \geq 2$ holds. We choose $\sigma > 0$ sufficiently small that the assertion of Lemma 2.6 holds. We may assume that $2c-\sigma$ is a regular value of I_f. Since (PS) condition holds on the interval $[13c/12, 2c-\sigma]$ for I_f on H, we have that $m = \inf\{I_f(v) : v \in \widetilde{I}_f^{2c-\sigma}\}$ is attained by an element in $\mathcal{S}_f(\Omega)$. That is there exists a subset $K \subset H$ of critical points of I_f such that

$$I_f(u) = \min\{I_f(v) : v \in \widetilde{I}_f^{2c-\sigma}\} \quad \text{for each } u \in K.$$

If K contains more than two points, the assertion holds. Then we assume that K consists of single point u_1. Then we have that there exists $\delta > 0$ such that $m + \delta < 2c - \sigma$, $H_0(I_f^{m+\delta}) = Z_2$ and $H_p(I_f^{m+\delta}) = \{0\}$ for $p \geq 1$. Then since $\sum_{p=0}^{\infty} H_p(I_f^{2c-\sigma}) \geq 2$, we find that there exists a critical point $u_2 \in \mathcal{S}_f(\Omega)$ with $u_1 \neq u_2$. $\square$

Proof of Theorem 1.2. As in the proof of Theorem 1.1, we choose $\sigma > 0$ so small that the assertion of Lemma 2.6. Since $\{g \in C^\infty(\Omega) : g > 0 \text{ on } \Omega\}$ is dense $\{g \in L^2(\Omega) : g \geq 0\}$, we may assume that $f \in C^\infty(\Omega)$ and $f > 0$ on Ω. We suppose that $n \geq 0$ and there exist critical points $u_1, \ldots, u_n \in H$ of I_f such that each of them is nondegenerate. If $\sum_{p\geq 0} \operatorname{rank} H_p(\widehat{\Omega}) \leq n$, the assertion holds. Suppose that $\sum_{p\geq 0} \operatorname{rank} H_p(\widehat{\Omega}) > n$. Then since $\sum_{p\geq 0} \operatorname{rank} H_p(\widetilde{I}_f^{2c-\sigma}) > n$, we have by the Morse inequality that there exists a critical point $u_{n+1} \in \widetilde{I}_f^{2c-\sigma}$ of I_f such that $u_{n+1} \neq u_i$ for $1 \leq i \leq n$. We define a mapping $\mathcal{F}\colon H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ by

$$\mathcal{F}(u) = -(\Delta u + |u|^{2^*-2}u) \quad \text{for } u \in H^2(\Omega) \cap H_0^1(\Omega).$$

We denote by $\mathcal{B}_r^{(2)}$, $\mathcal{B}_r^{(h)}$ and $\mathcal{B}_r^{(\infty)}$ the balls centered at 0 with radius r in $L^2(\Omega)$, $H_0^1(\Omega) \cap H^2(\Omega)$ and $C_0^\infty(\Omega)$, respectively. Since each critical point u_i is nondegenerate for $1 \leq i \leq n$, we can choose $r_i > 0$ such $\operatorname{Ker} I_f''(u) = \{0\}$ for each $u \in u_i + \mathcal{B}_{r_i}^{(h)}$ and the mapping $\mathcal{F}$:$u_i + \mathcal{B}_{r_i}^{(h)} \to \mathcal{F}(u_i + \mathcal{B}_{r_i}^{(h)})$ is an isomorphism, for each $1 \leq i \leq n$, where I_f'' denotes the Hessian of I_f. Recall that $v \in$Ker $I''(u_{n+1})$ if and only if

$$-\Delta v - (2^* - 1)|u_{n+1}|^{2^*-2}v = 0$$

and that there exists $m > 0$ such that for each

$$|\langle -\Delta v - (2^* - 1)|u_{n+1}|^{2^*-2}v, v\rangle| \geq m|v|^2, \quad \text{for } v \in (\text{Ker } I''_f(u_{n+1}))^\perp.$$

Then we can choose $r' \in (0, r)$ such that

$$\mathcal{F}(u_{n+1} + \mathcal{B}^{(h)}_{r'}) \subset \bigcap_{i=1}^{n} \mathcal{F}(u_i + \mathcal{B}^{(h)}_{r_i}),$$

and that for each $u \in u_{n+1} + \mathcal{B}^{(h)}_{r'}$,

$$|\langle -\Delta v - (2^* - 1)|u|^{2^*-2}v, v\rangle| \geq (m/2)|v|^2 \quad \text{for } v \in (\text{Ker}\, I''(u_{n+1}))^\perp. \tag{2.21}$$

We can also choose $\widehat{r} > 0$ such that $\mathcal{B}^{(\infty)}_{\widehat{r}} \subset \mathcal{B}^{(h)}_{r'}$ and for each $u \in u_{n+1} + \mathcal{B}^{(h)}_{\widehat{r}}$.

$$\mathcal{F}(u) = -\Delta u - |u|^{2^*-2}u > 0 \quad \text{on } \Omega.$$

Then since Ker $I''(u_{n+1})$ is a finite dimensional space, one can see that there exists $u' \in u_{n+1} + \mathcal{B}^{(\infty)}_{\widehat{r}}$ such that

$$-\Delta v - (2^* - 1)|u'|^{2^*-2}v \neq 0 \quad \text{for } v \in \text{Ker}\, I''(u_{n+1}) \setminus \{0\}$$

and that

$$f' = -\Delta u' - |u'|^{2^*-2}u' > 0 \quad \text{on } \Omega.$$

Then u' is nondegenerate critical point of problem $(\mathrm{P}_{f'})$. Since $f' = \mathcal{F}(u') \in \bigcap_{i=1}^{n} \mathcal{F}(u_i + \mathcal{B}^{(*)}_{r_i})$, there exist critical points $u'_1, \ldots, u'_n$ of $I_{f'}$ such that $u'_i \in u_i + \mathcal{B}^{(h)}_{r_i}$. From the definition of r_i, each u'_i is a nondegenerate critical point of $(\mathrm{P}_{f'})$. Thus we find that problem $(\mathrm{P}_{f'})$ has $n+1$ nondegenerate critical points. Repeating this procedure, we reaches to the conclusion. □

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Manuscript received January 30, 2004

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