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HOMOTOPY PERTURBATION METHOD FOR MULTI-DIMENSIONAL NONLINEAR COUPLED SYSTEM OF PARABOLIC AND HYPERBOLIC EQUATIONS

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Abstract. In this paper, the homotopy perturbation method (HPM) proposed by J. H. He is adopted for solving multi-dimensional nonlinear coupled system of parabolic and hyperbolic equations. The numerical results of the present method are compared with the exact solution of an artificial multi-dimensional nonlinear coupled system of parabolic and hyperbolic model to show the efficiency of the method. Moreover, comparison is made between the results obtained by the present method and that obtained by the Adomian decomposition method (ADM). It is found that the present method works extremely well, very efficient, simple and convenient.

1. Introduction

In the past few years, we observe a growing interest towards the applications of the homotopy technique in nonlinear problems which can be described by weakly (or strongly) nonlinear partial differential equations. The HPM (see [1], [4], [6]–[9], [12], [16]) is one of the methods which has received much concern, it has the merits of simplicity and easy execution. Unlike the traditional numerical methods [10], the HPM does not need discretization, linearization. Many authors (see [2], [3], [5], [15], [17], [18] and the references cited therein) are pointed out

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that the HPM can overcome the difficulties arising in calculation of Adomian's polynomials in ADM.

The multi-dimensional coupled systems of parabolic and hyperbolic equations often appear in the study of a circled fuel reactor, high-temperature hydrodynamics and thermo-elasticity problems, see [14] and the references cited therein. For some kinds of nonlinear thermo-elasticity coupled systems, there are several publications (see [10], [11], [19] and the references cited therein) studying the numerical computations, existence and smooth properties of their solutions. In this paper we will use the merits of simplicity of HPM to solve the following multi-dimensional coupled system of non-linear partial differential equations (see [14]).

$$u_t - \nabla.(\alpha(X,t,u,v)\nabla u) = f(X,t,u,v,u_x,v_x,u_y,v_y), \tag{1.1}$$

$$v_{tt} - \nabla.(\beta(X,t,u,v)\nabla v) = g(X,t,u,v,u_x,v_x,u_y,v_y,u_t,v_t). \tag{1.2}$$

Here $X=(x,y)$, $X\in\Omega=[0,d_1]\times[0,d_2]$, $t\in[0,T]$ with the following boundary and initial conditions:

$$\begin{gathered} u(X,t)=v(X,t)=0, \quad X\in\partial\Omega, \quad t\in[0,T], \\ u(X,0)=u_0(X), \quad v(X,0)=v_0(X), \quad v_t(X,0)=v_1(X),\ X\in\Omega, \end{gathered} \tag{1.3}$$

where α, β, f, g, u_0, v_0, v_1 are known functions. For more details on such model see [14] and the references cited therein.

2. Analysis and implementation of the HPM

To illustrate the basic idea of HPM, let us consider the following system of partial differential equations:

$$L_1u(X,t)+N_1(u(X,t),v(X,t))-f(X,t) = 0, \tag{2.1}$$

$$L_2v(X,t)+N_2(u(X,t),v(X,t))-g(X,t) = 0, \tag{2.2}$$

where L_1, L_2 are linear operators and N_1, N_2 are nonlinear operators. With suitable initial and boundary conditions, f and g are known analytic functions. By the homotopy technique, we can construct homotopies

$$\widetilde{u}(X,p)\colon\Omega\times[0,1]\to R \quad\text{and}\quad \widetilde{v}(X,p)\colon\Omega\times[0,1]\to R$$

such that:

$$H_1(\widetilde{u},p) = L_1(\widetilde{u})-L_1(u_0)+p\,L_1(u_0)+p\,[N_1(\widetilde{u},\widetilde{v})-f(X,t)]=0, \tag{2.3}$$

$$H_2(\widetilde{v},p) = L_2(\widetilde{v})-L_2(v_0)+p\,L_2(v_0)+p\,[N_2(\widetilde{u},\widetilde{v})-g(X,t)]=0, \tag{2.4}$$

where p is an embedding parameter, u_0 and v_0 are initial approximations of the solutions. It is obvious that when $p=0$, equations (2.3) and (2.4) are linear equations and when $p=1$, they become the original non-linear equations. The embedding parameter monotonically increases from zero to the unit as the trivial problems. $L_1(\widetilde{u}) - L_1(u_0) = 0$, $L_2(\widetilde{v}) - L_2(v_0) = 0$, are continuously deformed to problems (2.3) and (2.4), respectively. This is a basic idea of homotopy method which is continuously deforming a simple problem easy to be solved into the difficult problem under study. In view of HPM, we use the homotopy parameter to expand the solutions

$$\widetilde{u}(X,t) = \widetilde{u}_0 + p\,\widetilde{u}_1 + p^2\,\widetilde{u}_2 + p^3\,\widetilde{u}_3 + \dots, \tag{2.5}$$

$$\widetilde{v}(X,t) = \widetilde{v}_0 + p\,\widetilde{v}_1 + p^2\,\widetilde{v}_2 + p^3\,\widetilde{v}_3 + \dots \tag{2.6}$$

The approximate solutions can be obtained by setting $p = 1$ in (2.5) and (2.6):

$$u(X,t) = \lim_{p\to 1} \widetilde{u} = \widetilde{u}_0 + \widetilde{u}_1 + \widetilde{u}_2 + \dots,$$
$$v(X,t) = \lim_{p\to 1} \widetilde{v} = \widetilde{v}_0 + \widetilde{v}_1 + \widetilde{v}_2 + \dots$$

Now, in this section, we apply the HPM to an artificial model like in (1.1)–(1.3) in order to demonstrate the high order accuracy and to compare the HPM solution with the exact solution. Let us consider the two dimensional nonlinear coupled system (1.1)–(1.3) with the following coefficients and functions:

$$\alpha(x,y,t,u,v) = u - 2v, \quad \beta(x,y,t,u,v) = v - 2u,$$
$$f(x,y,t) = 2t - 12x^2 - 12y^2 + 4(t^2 - x^2 - y^2 - 2(t^2 + x^2 + y^2)),$$
$$g(x,y,t) = 2 - 12x^2 - 12y^2 - 4(t^2 + x^2 + y^2 - 2(t^2 - x^2 - y^2)),$$

and the following initial conditions:

$$u(x,y,0) = -(x^2+y^2), \quad v(x,y,0) = x^2+y^2, \quad v_t(x,y,0) = 0, \quad \Omega = [0,1]\times[0,1].$$

In this case, the coupled system of parabolic and hyperbolic equations has the following exact solution:

$$u(x,y,t) = t^2 - (x^2 + y^2), \quad v(x,y,t) = t^2 + (x^2 + y^2).$$

Now, the model problem (1.1)–(1.2) can be written in the following operator form:

$$L_t u - N_1(u,v) - f(X,t) = 0, \tag{2.7}$$

$$L_{tt} v - N_2(u,v) - g(X,t) = 0, \tag{2.8}$$

where the notations $L_t = \partial/\partial t$ and $L_{tt} = \partial^2/\partial t^2$ symbolize the linear differential operators and the nonlinear operators $N_1(u,v)$ and $N_2(u,v)$ are defined by:

$$N_1(u,v) = (u_x - 2v_x)u_x + (u_y - 2v_y)u_y + (u - 2v)(u_{xx} + u_{yy}), \tag{2.9}$$

$$N_2(u,v) = (v_x - 2u_x)v_x + (v_y - 2u_y)v_y + (v - 2u)(v_{xx} + v_{yy}). \tag{2.10}$$

According to the HPM, we construct the following simple homotopy

$$H_1(\widetilde{u},p) = L_t(\widetilde{u}) - L_t(\widetilde{u_0}) + p\,L_t(\widetilde{u_0}) - p\,[N_1(\widetilde{u},\widetilde{v}) + f(X,t)] = 0, \tag{2.11}$$

$$H_2(\widetilde{v},p) = L_{tt}(\widetilde{v}) - L_{tt}(\widetilde{v_0}) + p\,L_{tt}(\widetilde{v_0}) - p\,[N_2(\widetilde{u},\widetilde{v}) + g(X,t)] = 0, \tag{2.12}$$

where $p \in [0,1]$ is an embedding parameter. It is obvious that when $p = 0$, the above equations become linear equations of the form $L_t(\widetilde{u}) = L_t(\widetilde{u_0})$, $L_{tt}(\widetilde{v}) = L_{tt}(\widetilde{v_0})$, and it turns to the original equations when $p = 1$. The HPM uses the homotopy parameter as expanding parameter to obtain

$$\begin{aligned} u(X,t) &= \lim_{p\to 1} \widetilde{u} = \widetilde{u}_0 + \widetilde{u}_1 + \widetilde{u}_2 + \dots\,, \\ v(X,t) &= \lim_{p\to 1} \widetilde{v} = \widetilde{v}_0 + \widetilde{v}_1 + \widetilde{v}_2 + \dots \end{aligned}$$

Substituting (2.5) and (2.6) into (2.11) and (2.12), respectively and using equations (2.9) and (2.10), and equating the terms with the identical powers of p, we can obtain a series of linear equations. These linear equations are easy to be solved by using Mathematica software. Here we only write the first few linear equations and for simplicity, we will use the notion u_i, v_i for the approximate solution instead of $\widetilde{u}_i$, $\widetilde{v}_i$:

$$\dot{u}_0(X,t) = 0, \quad \ddot{v}_0(X,t) = 0, \tag{2.13}$$

$$\begin{aligned} \dot{u}_1(X,t) = {}& f(X,t) + (u_{0x} - 2v_{0x})u_{0x} \\ &+ (u_{0y} - 2v_{0y})u_{0y} + (u_0 - 2v_0)(u_{0xx} + u_{0yy}), \end{aligned} \tag{2.14}$$

$$\begin{aligned} \ddot{v}_1(X,t) = {}& g(X,t) + (v_{0x} - 2u_{0x})v_{0x} \\ &+ (v_{0y} - 2u_{0y})v_{0y} + (v_0 - 2u_0)(v_{0xx} + v_{0yy}), \end{aligned} \tag{2.15}$$

$$\begin{aligned} \dot{u}_2(X,t) = {}& (u_{1x} - 2v_{1x})u_{0x} + (u_{0x} - 2v_{0x})u_{1x} \\ &+ (u_{1y} - 2v_{1y})u_{0y} + (u_{0y} - 2v_{0y})u_{1y} \\ &+ (u_0 - 2v_0)(u_{1xx} + u_{1yy}) + (u_1 - 2v_1)(u_{0xx} + u_{0yy}), \end{aligned} \tag{2.16}$$

$$\begin{aligned} \ddot{v}_2(X,t) = {}& (v_{1x} - 2u_{1x})v_{0x} + (v_{0x} - 2u_{0x})v_{1x} \\ &+ (v_{1y} - 2u_{1y})v_{0y} + (v_{0y} - 2u_{0y})v_{1y} \\ &+ (v_0 - 2u_0)(v_{1xx} + v_{1yy}) + (v_1 - 2u_1)(v_{0xx} + v_{0yy}), \end{aligned} \tag{2.17}$$

the solution of equation (2.13) using the initial conditions is:

$$\begin{aligned} u_0(x,y,t) &= u(x,y,0) = -(x^2 + y^2), \\ v_0(x,y,t) &= v(x,y,0) + v_t(x,y,0)t = x^2 + y^2, \end{aligned}$$

after substituting $u_0(x,y,t)$ and $v_0(x,y,t)$ in (2.14)–(2.15), we can find the solution of (2.14)–(2.15) in the form:

$$\begin{aligned} u_1(x,y,t) &= \int_0^t [\,(u_{0x}-2v_{0x})u_{0x}+(u_{0y}-2v_{0y})u_{0y} \\ &\quad + (u_0-2v_0)(u_{0xx}+u_{0yy})+f(X,\tau)\,]\,d\tau = t^2-\frac{4}{3}t^3, \\ v_1(x,y,t) &= \int_0^t\int_0^t [\,(v_{0x}-2u_{0x})v_{0x}+(v_{0y}-2u_{0y})v_{0y} \\ &\quad + (v_0-2u_0)(v_{0xx}+v_{0yy})+g(X,\tau)]\,d\tau d\tau = t^2+\frac{1}{3}t^4, \end{aligned}$$

after substituting u_0, u_1 and v_0, v_1 in (2.16)–(2.17), we can find the solution of (2.16)–(2.17) in the form:

$$\begin{aligned} u_2 = &\int_0^t [(u_{1x}-2v_{1x})u_{0x}+(u_{0x}-2v_{0x})u_{1x}+(u_{1y}-2v_{1y})u_{0y} \\ &+(u_{0y}-2v_{0y})u_{1y}+(u_0-2v_0)(u_{1xx}+u_{1yy}) \\ &+(u_1-2v_1)(u_{0xx}+u_{0yy})]\,d\tau = \frac{4}{15}t^3(5+5t+2t^2), \\ v_2 = &\int_0^t\int_0^t [(v_{1x}-2u_{1x})v_{0x}+(v_{0x}-2u_{0x})v_{1x}+(v_{1y}-2u_{1y})v_{0y} \\ &+(v_{0y}-2u_{0y})v_{1y}+(v_0-2u_0)(v_{1xx}+v_{1yy}) \\ &+(v_1-2u_1)(v_{0xx}+v_{0yy})]\,d\tau\,d\tau = \frac{t^4}{45}(-15+24t+2t^2). \end{aligned}$$

Also, we can find the solutions $u_3(x,y,t)$, $v_3(x,y,t)$ in the form:

$$\begin{aligned} u_3(x,y,t) &= \frac{-4t^4}{315}(105+126t-28t^2-4t^3), \\ v_3(x,y,t) &= \frac{-t^5}{315}(168+126t+16t^2-t^3). \end{aligned}$$

Proceeding in the same way, we can obtain high order approximations. In order to illustrate the advantages and the accuracy of the HPM for solving the present problem, we calculate the fifteenth order perturbation, i.e. the approximate solutions are:

$$u(x,y,t)=u_0+\ldots+u_{15} \quad \text{and} \quad v(x,y,t)=v_0+\ldots+v_{15}, \tag{2.18}$$

and compare it with the exact solution, where $d_1=d_2=1, T=2$. The numerical results are shown in Table 1. We achieved a very good approximation with the actual solution of the equations. It is evident that even using few terms of the series, the overall results are getting very close to the exact solution, errors can be made smaller by adding new terms of the expanded series. From Table 1, we

t	$\|u - u_{15}\|_2$	$\|v - v_{15}\|_2$
0.25	2.77334E−32	1.92593E−34
0.50	6.87141E−23	3.30001E−25
0.75	4.28078E−18	7.90194E−20
1.00	1.78134E−16	3.51264E−17
1.25	4.85668E−12	1.18268E−13
1.50	4.30891E−12	3.11003E−11
1.75	2.64388E−07	2.70213E−09
2.00	2.13518E−05	2.19768E−06

TABLE 1. The L^2-norm error of u, v at different times

can conclude that the HPM scheme has a very high accuracy comparing with the exact solution even for long time period.

3. Analysis and implementation of the ADM

To explain and implement the ADM to the same model, we will consider the system (2.7)–(2.8) with respect to (2.9) and (2.10). By using the inverse operators, we can write the system (2.7)–(2.8) in the following form:

$$u(x,y,t) = u(x,y,0) + L_t^{-1} f(x,y,t) + L_t^{-1} N_1(u,v), \tag{3.1}$$
$$v(x,y,t) = v(x,y,0) + v_t(x,y,0)t + L_{tt}^{-1} g(x,y,t) + L_{tt}^{-1} N_2(u,v), \tag{3.2}$$

where the inverse operators are defined by

$$L_t^{-1} = \int_0^t (\,\cdot\,)\,dt, \qquad L_{tt}^{-1} = \int_0^t \int_0^t (\,\cdot\,)\,dt\,dt.$$

The ADM suggests that the solution $u(x,y,t)$ and $v(x,y,t)$ can be decomposed into an infinite series of components:

$$u(x,y,t) = \sum_{i=0}^{\infty} U_i(x,y,t), \qquad v(x,y,t) = \sum_{i=0}^{\infty} V_i(x,y,t),$$

and the nonlinear terms defined in (2.9) and (2.10) decomposed into the infinite series:

$$N_k(u,v) = \sum_{i=0}^{\infty} A_{ki}, \quad k = 1,2,$$

where $U_i(x,y,t)$ and $V_i(x,y,t)$, $i \geq 0$, are the components of $u(x,y,t)$, $v(x,y,t)$ that will be smartly determined and are called Adomian's polynomials and defined by

$$A_{kn} = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N_k\left(\sum_{j=0}^{n} \lambda^j u_j, \sum_{j=0}^{n} \lambda^j v_j\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{3.3}$$

From the above considerations, the decomposition method defines the components U_i and V_i for, $i \geq 0$ by the following recursive relationships

$$
\begin{aligned}
U_0(X,t) &= u(X,0),\\
U_1(X,t) &= L_t^{-1}[f(X,t)+A_{10}],\\
U_{n+1}(X,t) &= L_t^{-1}[A_{1n}], \quad n \geq 1,
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
V_0(X,t) &= v(X,0)+v_t(X,0)t,\\
V_1(X,t) &= L_{tt}^{-1}[g(X,t)+A_{20}],\\
V_{n+1}(X,t) &= L_{tt}^{-1}[A_{2n}], \quad n \geq 1.
\end{aligned}
\tag{3.5}
$$

This will enable us to determine the components U_n and V_n recurrently. However, in many cases the exact solution in a closed form may be obtained. For numerical comparisons purpose, we construct the solutions $u(x,y,t)$ and $v(x,y,t)$ such that:

$$\lim_{n\to\infty} \Psi_n = u(x,y,t), \qquad \lim_{n\to\infty} \Theta_n = v(x,y,t),$$

where

$$\Psi_n(x,y,t) = \sum_{i=0}^{n-1} U_i(x,y,t), \qquad \Theta_n(x,y,t) = \sum_{i=0}^{n-1} V_i(x,y,t), \quad n \geq 0.$$

In an algorithmic form, the ADM can be implemented to the coupled solutions as follows:

Algorithm. Let n be the iteration index, set a suitable value for the tolerance (Tol.)

Step 1. Compute the initial approximations

$$
\begin{aligned}
U_0(X,t) &= u(X,0), & U_1(X,t) &= L_t^{-1}(f(X,t)+A_{10}),\\
V_0(X,t) &= v(X,0)+v_t(X,0)t, & V_1(X,t) &= L_{tt}^{-1}(g(X,t)+A_{20})
\end{aligned}
$$

with respect to (1.3), set $n = 1$.

Step 2. Compute the Adomian polynomials A_{1n} and A_{2n} from (3.3).

Step 3. Use the calculated values of U_n and V_n to compute U_{n+1} from (3.4).

Step 4. Define $U_n := U_{n+1}$.

Step 5. Use the calculated values of U_n and V_n to compute V_{n+1} from (3.5).

Step 6. If $\max_{X\in\Omega} |U_{n+1} - U_n| <$ Tol. and $\max_{X\in\Omega} |V_{n+1} - V_n| <$ Tol. stop, otherwise continue.

Step 7. Set $U_{n+1} := U_n$.

Step 8. Set $n = n + 1$ and return to Step 2.

To find the solution of the system (1.1)–(1.3) using ADM, we can give the first Adomian polynomials of the A_{ki} using equations (3.3) as follows:

$$\begin{aligned}
A_{10} &= (u_{0x} - 2v_{0x})u_{0x} + (u_{0y} - 2v_{0y})u_{0y} + (u_0 - 2v_0)(u_{0xx} + u_{0yy}),\\
A_{20} &= (v_{0x} - 2u_{0x})v_{0x} + (v_{0y} - 2u_{0y})v_{0y} + (v_0 - 2u_0)(v_{0xx} + v_{0yy}),\\
A_{11} &= (u_{1x} - 2v_{1x})u_{0x} + (u_{0x} - 2v_{0x})u_{1x} + (u_{1y} - 2v_{1y})u_{0y}\\
&\quad + (u_{0y} - 2v_{0y})u_{1y} + (u_0 - 2v_0)(u_{1xx} + u_{1yy}) + (u_1 - 2v_1)(u_{0xx} + u_{0yy}),\\
A_{21} &= (v_{1x} - 2u_{1x})v_{0x} + (v_{0x} - 2u_{0x})v_{1x} + (v_{1y} - 2u_{1y})v_{0y}\\
&\quad + (v_{0y} - 2u_{0y})v_{1y} + (v_0 - 2u_0)(v_{1xx} + v_{1yy}) + (v_1 - 2u_1)(v_{0xx} + v_{0yy}).
\end{aligned}$$

Proceeding in the same way, the above Adomian polynomials of the A_{1k} and A_{2k} can be evaluated. Now, by using the given initial conditions and the above choice of the coefficient and functions we can derive:

$$\begin{aligned}
U_1(x,y,t) &= t^2 - \frac{4}{3}t^3,\\
U_2(x,y,t) &= \frac{4}{15}t^3(5 + 5t + 2t^2),\\
U_3(x,y,t) &= \frac{-4t^4}{315}(105 + 126t - 28t^2 - 4t^3),\\
U_4 &= \frac{8t^5}{315}(378 + 126t - 261t^2 + t^3),\\
V_1(x,y,t) &= t^2 + \frac{1}{3}t^4,\\
V_2(x,y,t) &= \frac{t^4}{45}(-15 + 24t + 2t^2),\\
V_3(x,y,t) &= \frac{-t^5}{315}(168 + 126t + 16t^2 - t^3),\\
V_4 &= \frac{t^6}{14175}(5040 + 3600t - 1245t^2 + 2t^3),
\end{aligned}$$

Proceeding in the same way, we can obtain high order approximation. In order to verify numerically whether the proposed methodology leads to higher accuracy, we evaluate the numerical solutions using the n-term approximation. It is to be noted that Ψ_n and Θ_n show clearly the convergence to the correct limit. Although we have difficulties to calculate the Adomian polynomials, but we can arrive to the same order of accuracy of the solutions using $n = 15$ terms of the decomposition series derived above (3.4)–(3.5) where $d_1 = d_2 = 1$, $T = 2$.

4. Conclusions

In this paper, HPM is used to solve numerically the multi-dimensional nonlinear coupled system of parabolic and hyperbolic equations when compared with ADM the present method has some obvious merits: (1) the mathematical calculations of the approximate solution are simpler than in other methods; (2) the solution obtained by the present method has a very high accuracy comparing with the exact solution even for long time period; (3) the method does need not to calculate Adomian's polynomials. (4) the HPM is highly accurate numerical solution without spatial discretizations or linearization for nonlinear partial differential equations. Finally, we point out that the corresponding analytical and numerical solutions are obtained according to the iteration equations using Mathematica 5.

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