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ON THE STRUCTURE OF THE SOLUTION SET FOR A CLASS OF NONLINEAR EQUATIONS INVOLVING A DUALITY MAPPING

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Abstract. Sufficient conditions ensuring that the solution set of some operator equations involving a duality mapping is non-empty, compact and convex are given.

1. Introduction

Let X be a smooth real Banach space, $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ a gauge function and $J_\varphi\colon X \to X^*$ the duality mapping subordinated to φ (see the precise definition in Section 2 below).

Suppose that X is compactly embedded in a real Banach space Z.

Finally, let $N\colon Z \to Z^*$ be a demicontinuous operator:

$$z_n \to z \Rightarrow Nz_n \rightharpoonup Nz,$$

where we have denoted by "$\to$" (respectively, "$\rightharpoonup$") the convergence in the strong (respectively, weak) topology.

The aim of this paper is to formulate sufficient conditions ensuring that the solution set of the equation

$$J_\varphi u = Nu \tag{1.1}$$

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is non-empty, compact and convex. By solution for (1.1) we understand an element $u \in X$, which satisfies

$$J_\varphi u = (i^* N i)u \tag{1.2}$$

where $i: X \to Z$ stands for the compact injection of X into Z and i^* is its adjoint:

$$i^*: Z^* \to X^*, \quad i^* z^* = z^* \circ i, \quad \text{for all } z^* \in Z^*.$$

As usual, X^* (resp. Z^*) denotes the dual space of X (resp. Z). We shall denote by $\langle \cdot\,, \cdot \rangle_{X,X^*}$ (resp. $\langle \cdot\,, \cdot \rangle_{Z,Z^*}$) the duality pairing between X^* and X (resp. Z^* and Z). We shall often omit to indicate the spaces in duality and we shall simply write $\langle \cdot\,, \cdot \rangle$.

It should be noticed that in (1.2) the right operator $K = i^* \circ N \circ i$ is compact and also that a solution for (1.2) is equivalently defined by

$$\langle J_\varphi u, v \rangle_{X,X^*} = \langle N(iu), i(v) \rangle_{Z,Z^*}, \quad \text{for all } v \in X. \tag{1.3}$$

In what follows, the solution set of (1.1) will be denoted by $\mathcal{S}(J_\varphi, N)$. Thus:

$$\mathcal{S}(J_\varphi, N) = \{u \in X : J_\varphi u = (i^* N i)u\}. \tag{1.4}$$

The adopted strategy is as follows: first, we shall formulate sufficient conditions ensuring that J_φ is bijective with a continuous inverse J_φ^{-1}. Consequently, $\mathcal{S}(J_\varphi, N)$ rewrites as:

$$\mathcal{S}(J_\varphi, N) = \mathrm{Fix}(T) = \{u \in X : u = Tu\}. \tag{1.5}$$

where $T = J_\varphi^{-1}(i^* N i): X \to X$ is compact.

Thus, the problem reduces to that of finding out sufficient conditions ensuring that the fixed point set of the compact operator T defined by (1.5) is non-empty, compact and convex.

A special geometry of X, monotonicity proprieties of J_φ and some *techniques* arising from the Leray–Schauder topological degree theory are the main ingredients which are used in order to reach this last step of the strategy.

2. Duality mappings: definition and surjectivity properties

Let X be a real Banach space and $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ be a gauge function, i.e. φ is continuous, strictly increasing, $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.

By duality mapping corresponding to the gauge function φ we understand the multivalued mapping $J_\varphi: X \to \mathcal{P}(X^*)$ defined as follows:

$$\begin{aligned} J_\varphi 0 &= \{0\}, \\ J_\varphi x &= \varphi(\|x\|)\{u^* \in X^* : \|u^*\| = 1,\ \langle u^*, x \rangle = \|x\|\} \quad \text{if } x \neq 0. \end{aligned}$$

According to the Hahn–Banach theorem it is easy to see that the domain of J_φ is the whole space:

$$D(J_\varphi) = \{x \in X : J_\varphi x \neq \emptyset\} = X.$$

By the preceding definition, it follows that if X is real and smooth, i.e. for any $x \neq 0$ there is a unique element $u^*(x) \in X^*$ having the metric properties $\langle u^*(x), x\rangle = \|x\|$ and $\|u^*(x)\| = 1$, the duality mapping corresponding to a gauge function φ is the single valued mapping $J_\varphi\colon X \to X^*$ defined as follows:

$$J_\varphi 0 = 0; \quad J_\varphi x = \varphi(\|x\|)\|\cdot\|'(x) \quad \text{if } x \neq 0$$

where $\|\cdot\|'(x)$ denotes the Gâteaux differential of the norm at x (see Diestel [11] and Zeidler [20]).

The following surjectivity result will play an important role in what follows:

Theorem 2.1. *If X is a real reflexive and smooth Banach space then any duality mapping $J_\varphi\colon X \to X^*$ is surjective. Moreover, if X is also strictly convex then J_φ is a bijection of X onto X^*.*

For the proof of the surjectivity of J_φ see Beurling and Livingston [5], Browder [8], Lions [19], Deimling [10], Zeidler [20].

On the other hand it can be shown that if X is a strictly convex real Banach space then any duality mapping $J\varphi\colon X \to 2^{X^*}$ is strictly monotone, in the following sense: if $x, y \in X$ and $x \neq y$ then, for any $x^* \in J_\varphi x$ and $y^* \in J_\varphi y$ one has $\langle x^* - y^*, x - y\rangle > 0$. Clearly, the strict monotonicity implies the injectivity: if $x, y \in X$ and $x \neq y$ then $J_\varphi x \cap J_\varphi y \neq \emptyset$. In particular, if the strictly convex real Banach space X is also a smooth one then any duality mapping $J_\varphi\colon X \to X^*$ is strictly monotone:

$$\langle J_\varphi x - J_\varphi y, x - y\rangle > 0, \quad \text{for all } x, y \in X,\ x \neq y,$$

and, consequently, injective.

Corollary 2.2. *Let X be an infinite dimensional reflexive and smooth real Banach space. Then there are no compact duality mappings on X.*

Proof. Let $J_\varphi\colon X \to X^*$ be a duality mapping. According to Theorem 2.1, J_φ is surjective.

Consequently, for any $r > 0$, $J_\varphi(\partial B_{X,r}) = \partial B_{X^*,\varphi(r)}$, where $\partial B_{X,r} = \{x \in X : \|x\| = r\}$, $\partial B_{X^*,\varphi(r)} = \{x^* \in X^* : \|x^*\| = \varphi(r)\}$. Indeed, if $x \in \partial B_{X,r}$, then $\|J_\varphi x\| = \varphi(\|x\|) = \varphi(r)$, i.e. $J_\varphi(\partial B_{X,r}) \subset \partial B_{X^*,\varphi(r)}$.

Reciprocally, let $x^* \in \partial B_{X^*,\varphi(r)}$. By the surjectivity of J_φ, there is $x \in X$ such that $x^* = J_\varphi x$. It follows that $\varphi(r) = \|x^*\| = \|J_\varphi x\| = \varphi(\|x\|)$, which simply implies $\|x\| = r$.

Consequently, $x^* = J_\varphi x$ with $x \in \partial B_{X,r}$, i.e. $\partial B_{X^*,\varphi(r)} \subset J_\varphi(\partial B_{X,r})$.

Because of $\dim X^* = \infty$, $\partial B_{X^*,\varphi(r)} = J_\varphi(\partial B_{X,r})$ is not compact. Consequently, J_φ is not compact. $\square$

The result given by Corollary 2.2 is not surprising. In fact, a more general result holds (see Appell, De Pascale and Vignoli [3, Theorem 3b]): if X and Y are infinite dimensional Banach spaces and $F: X \to Y$ is compact then F is not onto.

The proof is based essentially on Baire's category theorem. Notice also that an estimation of the Kuratowski measure of noncompactness of a duality mapping is given in [12].

Corollary 2.3. *If X is a reflexive and smooth real Banach space having the Kadeč–Klee property then any duality mapping $J_\varphi: X \to X^*$ is bijective and has a continuous inverse. Moreover,*

$$J_\varphi^{-1} = \chi^{-1} J^*_{\varphi^{-1}} \tag{2.1}$$

where $J^*_{\varphi^{-1}}: X^* \to X^{**}$ is the duality mapping on X^* corresponding to the gauge function φ^{-1} and $\chi: X \to X^{**}$ is the canonical isomorphism defined by $\langle \chi(x), x^* \rangle = \langle x^*, x \rangle$, for all $x \in X$ and all $x^* \in X^*$.

Proof. The existence of J_φ^{-1} follows from Theorem 2.1. Regarding formula (2.1), first we shall prove that, under the hypotheses of Corrolary 2.3, any duality mapping on X^* (in particular, that corresponding to the gauge function φ^{-1}) is single valued. This is equivalent with proving that X^* is smooth.

The smoothness of X^* will be proved by using the (partial) duality between strict convexity and smoothness given by the following theorem due to Klee (see Diestel [11, Chapter 2, §2, Theorem 2]):

if X^* is smooth (strictly convex) then X is strictly convex (smooth).

Clearly, if X is reflexive, then

X^* is smooth (strictly convex) if and only if X is strictly convex (smooth).

Now, by the hypotheses of Corollary 2.3, X is reflexive and smooth. Also, by the same hypotheses, X possesses the Kadeč–Klee property, which means: X is strictly convex and

$$\text{if } x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\| \text{ then } x_n \to x.$$

Consequently, X being reflexive, smooth and strictly convex so is X^*.

Let us prove that equality (2.1) holds or, equivalently, that

$$\chi J_\varphi^{-1} x^* = J^*_{\varphi^{-1}} x^*, \quad \text{for all } x^* \in X^*. \tag{2.2}$$

By the definition of duality mappings, $J^*_{\varphi^{-1}}x^*$ is the unique element in X^{**} having the metric properties

$$\begin{aligned} \langle J^*_{\varphi^{-1}}x^*, x^*\rangle &= \varphi^{-1}(\|x^*\|)\|x^*\|, \\ \|J^*_{\varphi^{-1}}x^*\| &= \varphi^{-1}(\|x^*\|). \end{aligned} \tag{2.3}$$

We shall show that $\chi J_\varphi^{-1}x^*$ possesses the same metric properties and then the result follows by unicity. Putting $x^* = J_\varphi x$ it follows (by definition of J_φ) that

$$\begin{gathered} \|x^*\| = \varphi(\|x\|), \\ \langle x^*, x\rangle = \varphi(\|x\|)\|x\| = \varphi^{-1}(\|x^*\|)\|x^*\|. \end{gathered}$$

and, consequently, we deduce that:

$$\begin{gathered} \langle \chi J_\varphi^{-1}x^*, x^*\rangle = \langle \chi(x), x^*\rangle = \langle x^*, x\rangle = \varphi^{-1}(\|x^*\|)\|x^*\|, \\ \|\chi J_\varphi^{-1}x^*\| = \|\chi(x)\| = \|x\| = \varphi^{-1}(\|x\|). \end{gathered} \tag{2.4}$$

Equality (2.2) follows by comparing (2.3) and (2.4) and using the uniqueness result evoked above. Formula (2.1) is fundamental in proving the continuity of J_φ^{-1}. Indeed, let $x_n^* \to x^*$ in X^*. As any duality mapping on a reflexive Banach space, $J^*_{\varphi^{-1}}$ is demicontinuous, $J^*_{\varphi^{-1}}x_n^* \rightharpoonup J^*_{\varphi^{-1}}x^*$. Consequently, we deduce that:

$$J_\varphi^{-1}x_n^* = \chi^{-1}J^*_{\varphi^{-1}}x_n^* \rightharpoonup \chi^{-1}J^*_{\varphi^{-1}}x^* = J_\varphi^{-1}x^*. \tag{2.5}$$

On the other hand,

$$\begin{aligned} \|J_\varphi^{-1}x_n^*\| &= \|\chi^{-1}J^*_{\varphi^{-1}}x_n^*\| = \|J^*_{\varphi^{-1}}x_n^*\| \\ &= \varphi^{-1}(\|x_n^*\|) \to \varphi^{-1}(\|x\|) = \|J_\varphi^{-1}x^*\|. \end{aligned} \tag{2.6}$$

From (2.5), (2.6) and the Kadeč–Klee property of X we infer that

$$J_\varphi^{-1}x_n^* \to J_\varphi^{-1}x^*. \qquad \square$$

COROLLARY 2.4. *If X is a weakly locally uniformly convex, reflexive and smooth real Banach space then any duality mapping $J_\varphi : X \to X^*$ is bijective and has a continuous inverse given by* (2.1).

PROOF. Since any weakly locally uniformly convex Banach space has the Kadeč–Klee property (see Diestel [11, Chapter 2, §2, Theorems 3 and 4(iii)]) the result follows from Corollary 2.3. $\square$

3. The main result

Theorem 3.1. *Let X be a reflexive and smooth real Banach space having the Kadeč–Klee property. Suppose that X is compactly embedded in the real Banach space Z.*

Denote by $i: X \to Z$ the compact injection of X into Z and by c_Z — the best constant in the inequality

$$\|ix\|_Z \leq c\|x\|_X \quad \text{for all } x \in X. \tag{3.1}$$

Let φ and ψ be gauge functions, $J_\varphi: X \to X^*$ be the duality mapping subordinated to φ and $N: Z \to Z^*$ be a demicontinous operator satisfying the growth condition:

$$\|Nz\| \leq c_1\psi(\|z\|) + c_2 \quad \text{for all } z \in i(X), \tag{3.2}$$

with constants $c_1 \geq 0, c_2 \geq 0$. Suppose that there is a constant $r > 0$ such that

$$\varphi(t) - c_1 c_Z \psi(c_Z t) - c_2 c_Z > 0 \quad \text{for all } t \geq r. \tag{3.3}$$

Then the solution set of the equation

$$J_\varphi u = Nu \tag{3.4}$$

is nonempty, compact and contained in $B(0, r)$.

The following result (Proposition 4 in [13]) will be involved in the proof.

Proposition 3.2. *Let X be a real reflexive Banach space, compactly embedded in the real Banach space Z. Denote by i the compact injection of X into Z and, for any $p \in [1, \infty)$, define*

$$\lambda_{1,p} := \inf\left\{\frac{\|x\|_X^p}{\|i(x)\|_Z^p} : x \in X \setminus 0\right\}. \tag{3.5}$$

Then $\lambda_{1,p}$ is attained and $\lambda_{1,p}^{-1/p}$ is the best constant c_Z in inequality (3.1). In particular, $\|i\| = \|i^*\| = \lambda_{1,p}^{-1/p} = c_Z$.

Proof of Theorem 3.1. According to Corollary 2.3, J_φ is bijective and has a continuous inverse J_φ^{-1} given by (2.1). Consequently, the solution set of (3.4) (defined by (1.4)) coincides with the fixed point set of the compact operator

$$T = J_\varphi^{-1}(i^* N i): X \to X. \tag{3.6}$$

Thus the problem reduces to that of proving, under the hypotheses of Theorem 3.1, that the fixed point set of the compact operator T defined by (3.6) is nonempty, compact and contained in $B(0, r)$.

We shall prove this by using the method of a priori estimate, namely we shall show that

$$\{x \in X : \exists\, t \in [0, 1],\ x = tTx\} \subset B(0, r), \tag{3.7}$$

with r given by (3.3). As for $t = 0$ the only point in X which satisfies $x = tTx$ is $x = 0$, let $t \in (0, 1]$ and $x \in X$ be such that

$$x = tTx = tJ_\varphi^{-1}(i^* N i)x.$$

It follows that

$$\varphi(\|x\|) \le \varphi\left(\frac{\|x\|}{t}\right) = \left\| J_\varphi\left(\frac{x}{t}\right)\right\| = \|(i^* N i)x\|. \tag{3.8}$$

By using the growth condition (3.2) we infer that

$$\begin{aligned}\|(i^* N i)x\| &\le \|i^*\|\|N(ix)\| \le \|i^*\|[c_1\psi(\|ix\|) + c_2]\\ &\le \|i^*\|[c_1\psi(\|i\|\|x\|) + c_2] = c_1 c_Z \psi(c_Z\|x\|) + c_2 c_Z.\end{aligned}$$

Thus, we have obtained

$$\|(i^* N i)x\| \le c_1 c_Z \psi(c_Z\|x\|) + c_2 c_Z. \tag{3.9}$$

By combining (3.8) and (3.9) one obtains

$$\varphi(\|x\|) - c_1 c_Z \psi(c_Z\|x\|) - c_2 c_Z \le 0$$

and from this, by virtue of (3.3), it follows that $\|x\| < r$.

Once the a priori estimate (3.7) obtained, the invariance under homotopy of compact transforms of the Leray–Schauder degree gives,

$$d_{LS}(I - (tT), B(0, r), 0) = d_{LS}(I, B(0, r), 0) = 1, \quad \text{for all } t \in [0, 1],$$

where I denotes the identity.

Consequently, $\mathrm{Fix}(tT) \ne \emptyset$ and $\mathrm{Fix}(tT) \subset B(0, r)$ for any $t \in [0, 1]$.

Since $\mathrm{Fix}(tT)$ is closed and bounded, by standard arguments, the compactness of (tT) implies the compactness of $\mathrm{Fix}(tT)$.

We conclude that, under the hypotheses of Theorem 3.1, $\mathrm{Fix}(tT)$ is nonempty, compact and included in $B(0, r)$, for any $t \in [0, 1]$. For $t = 1$, we derive the conclusion of Theorem 3.1. $\square$

Remark 3.3. An alternative proof of Theorem 3.1 may be given by using the Schauder's fixed point theorem.

Remark 3.4. Clearly, if the hypotheses of Theorem 3.1 are satisfied with X infinite dimensional then the solution set of equation (3.4) has empty interior. This follows by $\mathcal{S}(J_\varphi, N)$ being a compact set in an infinite dimensional Banach space.

Corollary 3.5. *Suppose that the Banach spaces X and Z and the functions φ, ψ and J_φ are like in Theorem* 3.1. *Suppose that $N\colon Z \to Z^*$ satisfies the following generalized Lipschitz condition:*

$$\|Nu - Nv\| \leq c_1\psi(\|u - v\|) \quad \text{for all } u \text{ and } v \in i(X) \tag{3.10}$$

with constant $c_1 \geq 0$. If

$$\varphi(t) - c_1 c_Z \psi(c_Z t) \to \infty \quad \text{as } t \to \infty \tag{3.11}$$

then the solution set of the equation

$$J_\varphi u = Nu$$

is nonempty, compact and contained in $B(0,r)$, where r is a positive constant such that

$$\varphi(t) - c_1 c_Z \psi(c_Z t) > c_Z\|N0\| \quad \text{for all } t \geq r.$$

Proof. Clearly, condition (3.10) implies

$$\|Nu\| \leq c_1\psi(\|u\|) + \|N0\|, \quad \text{for all } u \in i(X),$$

i.e. N satisfies the growth condition (3.2) with $c_2 = \|N0\|$.

By virtue of (3.11), there is a constant $r > 0$ such that

$$\varphi(t) - c_1 c_Z \psi(c_Z t) - c_Z\|N0\| > 0 \quad \text{for all } t \geq r,$$

saying that condition (3.3) of Theorem 3.1 is also satisfied. The result follows from Theorem 3.1. □

Remark 3.6. Suppose that the hypotheses of Corollary 3.5 are satisfied with $\psi = \varphi$, the gauge function φ being, in addition, a-positively homogeneous: $\varphi(\alpha t) = \alpha^a \varphi(t)$ for some constant $a > 0$ and all $\alpha, t \geq 0$. Then condition (3.11) is satisfied if and only if $c_1 < \lambda_{1,a+1}$. That's because, under the above assumptions, we have

$$\varphi(t) - c_1 c_Z \varphi(c_Z t) = (1 - c_1 c_Z^{a+1})\varphi(t) \quad \text{and} \quad c_Z = \lambda_{1,a+1}^{-1/(a+1)}.$$

Notice that this remark will be evoked later in proving the next corollary of Theorem 3.1 (see Corollary 3.11 below).

Indeed, in what follows we shall derive some consequences of Theorem 3.1 assuming that the gauge function φ is a-positively homogeneous.

Some influences of this assumption on the properties of J_φ are described below.

PROPOSITION 3.7. *A duality mapping $J_\varphi: X \to \mathcal{P}(X^*)$ on a real Banach space is a-positively homogeneous (i.e. $J_\varphi(\alpha x) = \alpha^a J_\varphi x$ for all $\alpha \in \mathbb{R}_+$ and $x \in X$) if and only if φ is a-positively homogeneous.*

PROOF. Suppose that $J_\varphi(\alpha x) = \alpha^a J_\varphi x$, for all $\alpha \geq 0$ and all $x \in X$. Taking the norm we derive that $\varphi(\alpha\|x\|) = \alpha^a \varphi(\|x\|)$, i.e. φ is a-positively homogeneous.

Conversely, suppose that φ is a-positively homogeneous and let us show that $J_\varphi(\alpha x) = \alpha^a J_\varphi x$, for all $\alpha \geq 0$ and all $x \in X$. Since the equality is obvious for $\alpha = 0$ or $x = 0$, we shall prove it assuming that $\alpha > 0$ and $x \neq 0$.

In order to prove that $J_\varphi(\alpha x) \subset \alpha^a J_\varphi x$, take $x^* \in J_\varphi(\alpha x)$ and show that $\dfrac{x^*}{\alpha^a} \in J_\varphi x$. That is true because of:

$$\left\langle \frac{x^*}{\alpha^a}, x \right\rangle = \frac{1}{\alpha^{a+1}} \langle x^*, \alpha x \rangle = \frac{1}{\alpha^{a+1}} \varphi(\alpha\|x\|)\alpha\|x\| = \varphi(\|x\|)\|x\|$$

and

$$\frac{\|x^*\|}{\alpha^a} = \frac{\varphi(\alpha\|x\|)}{\alpha^a} = \varphi(\|x\|).$$

In order to prove that $\alpha^a J_\varphi x \subset J_\varphi(\alpha x)$, take $x^* \in J_\varphi x$ and show that $\alpha^a x^* \in J_\varphi(\alpha x)$. That is true because of:

$$\langle \alpha^a x^*, \alpha x \rangle = \alpha^{a+1} \langle x^*, x \rangle = \alpha^{a+1} \varphi(\|x\|)\|x\| = \varphi(\alpha\|x\|)\alpha\|x\|$$

and

$$\|\alpha^a x^*\| = \alpha^a \|x^*\| = \alpha^a \varphi(\|x\|) = \varphi(\alpha\|x\|). \qquad \square$$

A more general result is given by the next proposition:

PROPOSITION 3.8 (see [15]). *Let X be a real Banach space, φ and ψ two gauge functions and $J_\varphi: X \to \mathcal{P}(X^*)$ the duality mapping subordinated to φ. Then J_φ is ψ-homogeneous,*

$$J_\varphi(\alpha x) = \psi(\alpha) J_\varphi x, \quad \text{for all } \alpha \geq 0 \text{ and all } x \in X,$$

if and only if $\varphi(t) = \varphi(1)t^a$ with $a > 0$ and $\psi(t) = t^a$, for all $t \geq 0$. In particular, J_φ is φ-homogeneous if and only if $\varphi(t) = t^a$ with $a > 0$, for all $t \geq 0$.

PROPOSITION 3.9. *Let X be a smooth real Banach space and φ a gauge function.*

(a) *If the duality mapping J_φ satisfies the accretivity condition*

$$\langle J_\varphi u - J_\varphi v, u - v \rangle \geq \alpha \varphi(\|u - v\|)\|u - v\| \tag{3.12}$$

for some constant $\alpha > 0$ and any $u, v \in X$ then, necessarily, $\alpha \leq 1$.

(b) *If, in addition, φ is a-positively homogeneous then $\alpha \leq 1/2^{a-1}$.*

Proof. Indeed, putting $v = 0$ and any $u \neq 0$ in (3.12) it easily follows that $\alpha \leq 1$.

If, in addition, φ is a-positively homogeneous then, by putting $v = -u$ for any $u \neq 0$ in (3.12) and taking into account the oddness of J_φ and the a-positive homogeneity of φ it easily follows that $\alpha \leq 1/2^{a-1}$. □

Remark 3.10. It follows from Proposition 3.9 that if φ is a-positively homogeneous with $a > 1$ and J_φ satisfies (3.12) then, necessarily, $\alpha < 1$.

If φ is a-positively homogeneous with $a = 1$ (in this case, we shall simply say that φ is positively homogeneous) there are duality mappings J_φ satisfying (3.12) with $\alpha = 1$.

For example, the duality mapping on $\overset{\circ}{H}{}^1(\Omega)$ subordinated to the identity gauge function $\varphi(t) = t$ is the minus Laplacian, $-\Delta\colon \overset{\circ}{H}{}^1(\Omega) \to H^{-1}\Omega$, defined by

$$\langle -\Delta u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{for all } u, v \in \overset{\circ}{H}{}^1(\Omega).$$

Clearly, for any $u, v \in \overset{\circ}{H}{}^1(\Omega)$ one has

$$\langle -\Delta u - (-\Delta v), u - v\rangle = \|u - v\|^2_{\overset{\circ}{H}{}^1(\Omega)} = \varphi(\|u - v\|_{\overset{\circ}{H}{}^1(\Omega)})\|u - v\|_{\overset{\circ}{H}{}^1(\Omega)},$$

i.e. condition (3.12) is satisfied with equality and $\alpha = 1$.

Corollary 3.11. *Suppose that the Banach spaces X and Z as well and the functions φ, ψ and J_φ are like in Theorem 3.1. Suppose, in addition, that φ is $(p-1)$-positively homogeneous with $p \in [2, \infty)$ and J_φ satisfies the accretivity condition* (3.12). *Suppose that $N\colon Z \to Z^*$ satisfies the following generalized Lipschitz condition:*

$$\|Nu - Nv\| \leq c\varphi(\|u - v\|), \quad \text{for all } u, v \in i(X). \tag{3.13}$$

Then, either

(a) $p \in [2, \infty)$ *and* $0 \leq c_1 < \lambda_{1,p}\alpha$,

or

(b) $p \in (2, \infty)$ *and* $c_1 = \lambda_{1,p}\alpha$,

implies that the solution set of the equation

$$J_\varphi u = Nu \tag{3.14}$$

is nonempty, compact, convex and contained in $B(0, r)$ with

$$r > \lambda_{1,p}^{1/p}\varphi^{-1}(1)\bigg(\frac{\|N0\|}{\lambda_{1,p} - c_1}\bigg)^{1/(p-1)}. \tag{3.15}$$

Moreover, if condition (a) *is satisfied, then the solution set of equation* (3.14) *reduces to one point.*

Proof. First, let us remark that the hypotheses of Corollary 3.11 entail the fulfillment of those of Corollary 3.5.

Indeed, we deduce from (3.13) that condition (3.10) is satisfied with $\psi = \varphi$. Since $\psi = \varphi$ and φ is $(p-1)$-positively homogeneous one has that condition (3.11) is satisfied if and only if $c_1 < \lambda_{1,p}$ (see Remark 3.6). Under the hypotheses of any of the alternative statements of Corollary 3.11, this condition is satisfied. Indeed, if $p \in [2, \infty)$ and $c_1 < \lambda_{1,p}\alpha$ condition $c_1 < \lambda_{1,p}$ is satisfied since $\alpha \le 1$ (see Proposition 3.9). If $p \in (2, \infty)$ and $c_1 = \lambda_{1,p}\alpha$, condition $c_1 < \lambda_{1,p}$ is also satisfied since, by Proposition 3.9 again, $\alpha \le 1/2^{p-2} < 1$.

According to Corollary 3.5 we infer that, in both situation, $\mathcal{S}(J_\varphi, N)$ is nonempty compact and contained in $B(0, r)$, the positive constant r being such that

$$\varphi(t) - c_1 c_Z \varphi(c_Z t) > c_Z \|N0\| \quad \text{for all } t \ge r.$$

Since φ is $(p-1)$-positively homogeneous and $c_Z = \lambda_{1,p}^{-1/p}$, the previous inequality rewrites as $\varphi(t) > (\lambda_{1,p}^{1/p-1}/(\lambda_{1,p} - c_1))\|N0\|$ for all $t \ge r$, which is equivalent with (3.15).

Now we shall prove the convexity of $\mathcal{S}(J_\varphi, N)$. As it was already shown, $\mathcal{S}(J_\varphi, N) = \mathrm{Fix}(T)$ with $T = J_\varphi^{-1}K\colon X \to X$ compact. Recalling that $K = (i^* N i)\colon X \to X^*$, using (3.13), the fact that $\|i\| = \|i^*\| = \lambda_{1,p}^{-\frac{1}{p}}$ and the $(p-1)$-positive homogeneity of φ one obtains that K satisfies

$$\|Ku - Kv\| \le \frac{c_1}{\lambda_{1,p}} \varphi(\|u - v\|) \quad \text{for all } u, v \in X. \tag{3.16}$$

On the other hand, it follows from (3.12) that

$$\|J_\varphi u - J_\varphi v\| \ge \alpha \varphi(\|u - v\|) \quad \text{for all } u, v \in X,$$

which is equivalent with

$$\|J_\varphi^{-1} u^* - J_\varphi^{-1} v^*\| \le \varphi^{-1}\left[\frac{1}{\alpha}\|u^* - v^*\|\right] = \frac{1}{\alpha^{1/(p-1)}} \varphi^{-1}(\|u^* - v^*\|) \tag{3.17}$$

for all $u^*, v^* \in X^*$.

Finally, by putting $u^* = Ku$ and $v^* = Kv$ in (3.17) we deduce that the compact operator $T = J_\varphi^{-1}K$ satisfies

$$\|Tu - Tv\| \le \left(\frac{c_1}{\lambda_{1,p}\alpha}\right)^{1/(p-1)} \|u - v\| \quad \text{for all } u, v \in X. \tag{3.18}$$

Assume that $\mathrm{Fix}(T)$ would contain at least two different points, say u and v. Then, from $\|u - v\| = \|Tu - Tv\|$ and (3.18) it follows that $\lambda_{1,p}\alpha \le c_1$. We deduce that if $c_1 < \lambda_{1,p}\alpha$ then $\mathrm{Fix}(T)$ contains at most one point. Corollary 3.11 just says that if $p \in [2,\infty)$ and $c_1 < \lambda_{1,p}\alpha$ then $\mathrm{Fix}(T)$ contains precisely one point, namely the unique fixed point of the strict contraction T defined by (3.18).

If $p \in (2,\infty)$ and $c_1 = \lambda_{1,p}\alpha$ then it follows from (3.18) that T is nonexpansive. Consequently, since X is strictly convex, $\mathcal{S}(J_\varphi, N) = \mathrm{Fix}(T)$ is convex (see Brezis [6, Theorem 1.2]). □

Remark 3.12. Under the hypotheses of Corollary 3.11 and, in addition, if $N0 = 0$, then the only solution of equation (3.14) is the trivial one. Indeed, since $J_\varphi 0 = 0$ it follows that, if $N0 = 0$, $0 \in \mathcal{S}(J_\varphi, N)$. On the other hand, $\mathcal{S}(J_\varphi, N) \subset B(0,r)$ (see (3.15)) for any $r > 0$, thus $\mathcal{S}(J_\varphi, N) = \{0\}$.

A direct proof of this result may be given as follows. Suppose that, even one has $N0 = 0$, it would exist $u \in \mathcal{S}(J_\varphi, N)$, $u \neq 0$. Then, from $J_\varphi u = Ku$ and (3.16) we derive that

$$\varphi(\|u\|) = \|J_\varphi u\| = \|Ku\| = \|Ku - K0\| \le \frac{c_1}{\lambda_{1,p}}\varphi(\|u\|),$$

which implies $\lambda_{1,p} \le c_1$, contradiction.

4. Applications

In the sequel, Ω will be a bounded domain in $\mathbb{R}^N$, $N \ge 2$, with regular boundary. For $p \in (1,\infty)$ we shall use the standard notations

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega),\ i = 1,\dots N \right\}$$

equipped with the norm

$$\|u\|^p_{W^{1,p}(\Omega)} = \|u\|^p_{L^p} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L^p}.$$

It is well known that $(W^{1,p}(\Omega), \|\cdot\|_{W^{1,p}(\Omega)})$ is separable, reflexive and uniformly convex (see Adams [1], Brezis [7]). We need the space

$$\begin{aligned} W_0^{1,p}(\Omega) &= \text{the closure of } \mathcal{C}_0^\infty(\Omega) \text{ in the space } W^{1,p}(\Omega) \\ &= \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

the value of u on $\partial\Omega$ being understood in the sense of the trace.

The dual space $(W_0^{1,p}(\Omega))^*$ will be denoted by $W^{-1,p'}$, when $1/p + 1/p' = 1$.

For each $u \in W^{1,p}(\Omega)$ we put

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right), \qquad |\nabla u| = \left(\sum_{i=1}^N \left(\frac{\partial u}{\partial x_i} \right)^2 \right)^{1/2}$$

and let us remark that

$$|\nabla u| \in L^p(\Omega), \quad |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \in L^{p'}(\Omega) \quad \text{for } i = 1,\dots,N.$$

Therefore, by the theorem concerning the form of the elements of $W^{-1,p'}(\Omega)$ (see Brezis [7] or Lions [19]) it follows that the operator $-\Delta_p$ (usually called the minus p-Laplacian) defined by

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

may be seen acting from $W_0^{1,p}$ into $W^{-1,p'}(\Omega)$ by

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega). \tag{4.1}$$

By virtue of the Poincaré inequality

$$\|u\|_{L^p} \le \text{Const.}(\Omega, n)\||\nabla u|\|_{L^p} \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

the functional $W_0^{1,p}(\Omega) \ni u \to \|u\|_{1,p} := \||\nabla u|\|_{L^p}$ is a norm on $W_0^{1,p}(\Omega)$, equivalent with $\|\cdot\|_{W^{1,p}(\Omega)}$-norm. Moreover, $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ is reflexive smooth and uniformly convex (see e.g. [14, Theorems 6 and 7]).

Being uniformly convex, $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ has, in particular, the Kadeč–Klee property.

Moreover, $-\Delta_p = J_\varphi$ where $J_\varphi\colon (W_0^{1,p}(\Omega), \|\cdot\|_{1,p}) \to (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})^* = W^{-1,p'}$ is the duality mapping corresponding to the gauge function, $\varphi(t) = t^{p-1}$.

The idea to present the operator $-\Delta_p$, $1 < p < \infty$, as a duality mapping $J_\varphi\colon (W_0^{1,p}, \|\cdot\|_{1,p}) \to W^{-1,p'}$, $1/p + 1/p' = 1$ corresponding to the gauge function $\varphi(t) = t^{p-1}$ originates in the well known book of Lions [19]. This presentation allows us to apply the results of the preceding sections (given for duality mappings) to the particular case of the p-Laplacian. It is what we shall do in the last part of this paper.

Below, the space $W_0^{1,p}(\Omega)$ will always be considered to be endowed with the norm $\|\cdot\|_{1,p}$. Some detours concerning the Nemytskiĭ (or superposition) operator are needed. Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function, i.e.

(a) for each $s \in \mathbb{R}$, the function $x \to f(x, s)$ is Lebesque measurable in Ω;
(b) for almost every $x \in \Omega$, the function $s \to f(x, s)$ is continuous in $\mathbb{R}$.

We make the convention that in the case of a Caratheodory function, the assertion "$x \in \Omega$" to be understood in the sense "almost every $x \in \Omega$".

Let $\mathbf{S}$ be the set of all measurable functions $u\colon \Omega \to \mathbb{R}$.

Proposition 4.1. *If $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is Caratheodory then, for each $u \in \mathbf{S}$, the function $(N_f u)\colon \Omega \to \mathbb{R}$ defined by*

$$(N_f u)(x) = f(x, u(x)) \quad \textit{for } x \in \Omega$$

belongs to $\mathbf{S}$ too.

In view of this proposition, a Caratheodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defines an operator $N_f\colon \mathbf{S} \to \mathbf{S}$, which is called Nemytskiĭ (or superpositions) operator. The following proposition states a necessary and sufficient condition that a Nemytskiĭ operator maps a L^p space into another L^q space.

Proposition 4.2. *Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be Caratheodory, and $1 \leq p < \infty$, $1 \leq q < \infty$. Then N_f maps the space $L^p(\Omega)$ into the space $L^q(\Omega)$ if and only if f satisfies the growth condition*

$$|f(x,s)| \leq c_1 |s|^{p/q} + b(x), \quad x \in \Omega,\ s \in \mathbb{R}$$

for some $b \in L^q$ and $c_1 = \text{constant} \geq 0$. Moreover, in this case N_f is always continuous and bounded and satisfies the growth condition

$$\|N_f u\|_{L^q} \leq c_1 \|u\|_{L^p}^{\frac{p}{q}} + \|b\|_{L^q} \quad \textit{for all } u \in L^p.$$

For the proof of Propositions 3.2 and 3.7 as well as that for other properties of the superposition operator acting in some function spaces see, for example, Appell [2], Appell and Zabreyko [4] and the references therein.

Theorem 4.3. *Let $p \in [2,\infty)$ and*

$$\lambda_{1,p} = \inf\colon \bigg\{ \frac{\|u\|_{1,p}^p}{\|i(u)\|_{L^p}^p} : u \in W_0^{1,p}(\Omega) \setminus \{0\} \bigg\}$$

where i stands for the compact embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$. Suppose that $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that:

(a) *N_f acts from $L^p(\Omega)$ into $L^{p'}(\Omega)$, $1/p + 1/p' = 1$;*
(b) *$\|N_f u - N_f v\|_{L^{p'}} \leq c_1 \|u - v\|_{L^p}^{p-1}$, for all $u, v \in L^p(\Omega)$.*

Let $\alpha(p)$ be the best constant for which the inequality

$$\langle -\Delta_p u - (-\Delta_p v), u - v \rangle \geq \alpha(p) \|u - v\|_{1,p}^p \tag{4.2}$$

is satisfied for all u and v in $W_0^{1,p}(\Omega)$. Under these hypotheses, consider the Dirichlet problem

$$(\mathcal{P}) \qquad \begin{aligned} -\Delta_p u &= f(x,u) && \textit{in } \Omega, \\ u &= 0 && \textit{on } \partial\Omega. \end{aligned}$$

Then, if either

(i) *$p \in [2,\infty)$ and $c_1 < \lambda_{1,p}\alpha(p)$*

or

(ii) *$p \in (2,\infty)$ and $c_1 = \lambda_{1,p}\alpha(p)$,*

the weak solution set of problem $(\mathcal{P})$ is nonempty, compact, convex and contained in $B(0,r)$ with

$$r > \lambda_{1,p}^{1/p} \bigg(\frac{\|N_f 0\|_{L^{p'}}}{\lambda_{1,p} - c_1} \bigg)^{1/(p-1)}.$$

Moreover, if (i) *is satisfied, the weak solution set of problem* ($\mathcal{P}$) *reduces to one point.*

PROOF. It is well known that the weak solution set of problem ($\mathcal{P}$) is the set of all elements $u \in W_0^{1,p}(\Omega)$, which satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \int_\Omega f(x,u)v \, dx \tag{4.3}$$

for any $v \in W_0^{1,p}(\Omega)$. It may be easily shown that the weak solution set of problem ($\mathcal{P}$) coincides with the solution set of the equation

$$-\Delta_p u = N_f u \tag{4.4}$$

where $-\Delta_p \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is defined by (4.1) and $N_f \colon L^p(\Omega) \to L^{p'}(\Omega)$ is the Nemytskiĭ operator defined by the Carathéodory function f, which satisfies (a) and (b).

Since $-\Delta_p$ defined by (4.1) is the duality mapping on $W_0^{1,p}(\Omega)$ corresponding to the gauge function $\varphi(t) = t^{p-1}$, $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$ and $N_f \colon L^p(\Omega) \to L^{p'}(\Omega)$ is continuous then the solution set of (4.4) is the set of all solutions in $W_0^{1,p}(\Omega)$ of the operator equation

$$-\Delta_p u = (i^* N_f i)u \tag{4.5}$$

where i stands for the compact injection of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ and i^* is its adjoint (see (1.4)). Finally, $u \in W_0^{1,p}(\Omega)$ satisfies (4.5) if and only if it satisfies (4.3).

Before giving the proper proof of Theorem 4.3, we discuss about the existence of strictly positive constants α satisfying (4.2). If $p \in [2,\infty)$ is given, the existence of a strictly positive constant α satisfying (4.2) was first proved by Glowinski and Marocco [16]. In fact, such a constant is depending on p.

First, since $-\Delta_p$ is the duality mapping on $W_0^{1,p}(\Omega)$ corresponding to the gauge function $\varphi(t) = t^{p-1}$ (which is $(p-1)$-positively homogeneous) one has $\alpha \le 1/2^{p-2}$ (Proposition 3.9).

On the other hand, for a given $p \in [2,\infty)$, $\alpha = \min(1/2^{p+1}, 1/5^{p-2})$ is a constant that satisfies (4.2) (see [17]). This constant is not the best one (one can see that by considering the case $p = 2$ for which the above estimation gives $\alpha = 1/8$ while the best constant satisfying (4.2) in this case is $\alpha = 1$).

Thus, for a given $p \in [2,\infty)$, the best constant $\alpha(p)$ in (4.2) satisfies

$$\min\left(\frac{1}{2^{p+1}}, \frac{1}{5^{p-2}}\right) \le \alpha(p) \le \frac{1}{2^{p-2}}.$$

Handling these preliminaries, the proof of theorem is as follows.

The hypotheses of Theorem 4.3 entail the satisfaction of the hypotheses of Corollary 3.11 under the following choices:

- $(X, \|\cdot\|_X) = (W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$; $(Z, \|\cdot\|_Z) = (L^p(\Omega), \|\cdot\|_{L^p})$, $p \in [2,\infty)$;
- $\varphi(t) = t^{p-1}$, $t \geq 0$; $J_\varphi = -\Delta_p\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$;
- $N = N_f\colon L^p \to L^{p'}$, $(1/p + 1/p' = 1)$;
- $\lambda_{1,p} = \inf\{\|u\|_{1,p}^p / \|i(u)\|_{L^p}^p : u \in W_0^{1,p}(\Omega) \setminus \{0\}\}$, i being the compact injection of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$

Indeed, since $p \in [2,\infty)$, the reflexive smooth and uniformly convex Banach space $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ is compactly embedded in $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ (in fact, all these properties are valid for any $p \in (1,\infty)$). Moreover, being uniformly convex, $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ has, in particular, the Kadeč–Klee property.

The gauge function $\varphi(t) = t^{p-1}$ is $(p-1)$ positively homogeneous and $J_\varphi = -\Delta_p\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ satisfies the accretivity condition (3.12) by virtue of (4.2) (notice again that constants α satisfying (4.2) exist and the best one satisfies $\min(1/2^{p+1}, 1/5^{p-2}) \leq \alpha(p) \leq 1/2^{p-2}$).

Finally, from (b) we deduce that inequality (3.13) in Corollary 3.11 is satisfied with $N = N_f$ and $\varphi(t) = t^{p-1}$.

The thesis of Corollary 3.11 stated in the particular framework imposed by the above indicated choices becomes the thesis of Theorem 4.3. $\square$

Example 4.4. Consider the problem

$$(\mathcal{P}_2) \qquad \begin{aligned} -\Delta u &= ah(u) + b \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0, \end{aligned}$$

where $a \in L^\infty(\Omega)$, $a(x) \geq 0$ almost everywhere in Ω, $b \in L^2(\Omega)$ and $h\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$h(s) = \begin{cases} s^{q-1} & \text{for } s \geq 1, \\ s & \text{for } -1 \leq s < 1, \\ -|s|^{q-1} & \text{for } s < -1, \end{cases}$$

q being a given number, which satisfies $1 < q < 2$.

Let i be the compact injection of $\overset{\circ}{H}{}^1(\Omega)$ into $L^2(\Omega)$. If

$$\|a\|_{L^\infty} < \lambda_{1,2} = \inf\left\{ \frac{\|u\|^2_{\overset{\circ}{H}{}^1(\Omega)}}{\|i(u)\|^2_{L^2(\Omega)}} : u \in \overset{\circ}{H}{}^1(\Omega) \setminus \{0\} \right\},$$

the problem $(\mathcal{P}_2)$ admits a unique weak solution.

The result follows from Theorem 4.3 for $p = 2$ $f(x,s) = a(x)h(s) + b(x)$ and $c_1 = \|a\|_{L^\infty}$. The only things that have to be proved are: the Nemytskiĭ operator N_f generated by f is well defined from $L^2(\Omega)$ into $L^2(\Omega)$ and satisfies

$$\|N_f u - N_f v\|_{L^2(\Omega)} \leq \|a\|_{L^\infty} \|u - v\|_{L^2(\Omega)}, \quad \text{for all } u, v \in L^2(\Omega). \tag{4.6}$$

This is elementary but not trivial. First, we shall prove that:

$$|h(s)| = \begin{cases} |s| < |s|^{q-1} & \text{if } 0 < |s| < 1, \\ |s| = |s|^{q-1} & \text{if } |s| = 0 \text{ and } |s| = 1, \\ |s|^{q-1} < |s| & \text{if } |s| > 1. \end{cases}$$

Indeed, if $0 < |s| < 1$, out of $1 < 1/|s|$ and $2-q > 0$ it follows that $1 < (1/|s|)^{2-q}$, which is equivalent with $|s| < |s|^{q-1}$.

Let $|s| > 1$. Since $0 < q-1 < 1$ we infer that $|h(s)| = |s|^{q-1} < |s|$.

Now, since $|h(s)| \le |s|$ for any $s \in \mathbb{R}$, we deduce that, for any $u \in L^2(\Omega)$,

$$|f(x, u(x))| \le a(x)|u(x)| + |b(x)|$$

with $a \in L^\infty(\Omega)$ and $b \in L^2(\Omega)$, thus $(N_f u) \in L^2(\Omega)$.

In order to prove that (4.6) is true, it is sufficient to prove that f satisfies

$$|f(x, s_1) - f(x, s_2)| \le a(x)|s_1 - s_2|, \quad x \in \Omega,\ s_1, s_2 \in \mathbb{R}. \tag{4.7}$$

Indeed, by simple computations, it may be shown that (4.7) implies (4.6). Since

$$|f(x, s_1) - f(x, s_2)| = a(x)|h(s_1) - h(s_2)|,$$

the problem reduces to proving that h satisfies

$$|h(s_1) - h(s_2)| \le |s_1 - s_2|, \quad \text{for all } s_1, s_2 \in \mathbb{R}.$$

Let $s_1, s_2 \in \mathbb{R}$. We shall examine successively the situations: $s_1, s_2 \in \mathbb{R}_+$, $s_1, s_2 \in \mathbb{R}_-$, $s_1 \in \mathbb{R}_-, s_2 \in \mathbb{R}_+$. Let $s_1, s_2 \in \mathbb{R}_+$. If $s_1, s_2 \in [0,1]$ then, by definition of h,

$$|h(s_1) - h(s_2)| = |s_1 - s_2|.$$

If $0 \le s_1 \le 1 \le s_2$, then

$$|h(s_1) - h(s_2)| = |s_1 - s_2^{q-1}|.$$

Since $s_2 \ge 1$ and $0 < q-1 < 1$ it follows that $s_2^{q-1} \ge 1 \ge s_1$. Consequently,

$$|h(s_1) - h(s_2)| = s_2^{q-1} - s_1 \le s_2 - s_1 = |s_1 - s_2|.$$

If $s_1, s_2 \ge 1, s_1 \ne s_2$ the result follows immediately by the mean value theorem. Indeed, suppose that $1 \le s_1 < s_2$. There is $\xi \in (s_1, s_2)$ such that

$$|h(s_1) - h(s_2)| = |h'(\xi)||s_1 - s_2| < |s_1 - s_2|.$$

as much as $0 < h'(\xi) = (q-1)/\xi^{2-q} < 1$.

We conclude that (4.7) is true for $s_1, s_2 \in \mathbb{R}_+$.

Due to the oddness of h, (4.7) is valid for $s_1, s_2 \in \mathbb{R}_-$.

Finally, if $s_1 \in \mathbb{R}_+$ and $s_2 \in \mathbb{R}_-$, the oddness of h, the fact that h is positive over $\mathbb{R}_+$ and inequality $|h(s)| \leq |s|$ for any $s \in \mathbb{R}$ enables us to write

$$\begin{aligned}|h(s_1) - h(s_2)| &= |h(|s_1|) + h(|s_2|)| = h(s_1) + h(|s_2|) \\ &\leq s_1 + |s_2| = |s_1 + |s_2|| = |s_1 - s_2|.\end{aligned}$$

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