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BOUNDED SOLUTIONS TO NONLINEAR DELAY DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Abstract. This paper gives some sufficient conditions for every solution of delay differential equation

$$\begin{aligned}\dddot{x}\ (t) &+ f(t, x(t), x(t-r), \dot{x}(t), \dot{x}(t-r), \ddot{x}(t), \ddot{x}(t-r)) \\ &+ b(t)g(x(t-r), \dot{x}(t-r)) + c(t)h(x(t)) \\ &= p(t, x(t), x(t-r), \dot{x}(t), \dot{x}(t-r), \ddot{x}(t))\end{aligned}$$

to be bounded.

1. Introduction

In 1999, Mehri and Shadman [2] considered third order nonlinear differential equation without delay:

$$\dddot{x}\ (t) + a(t)f(\ddot{x}) + b(t)g(\dot{x}) + c(t)h(x) = e(t), \tag{1.1}$$

and via an energy function they discussed boundedness of solutions of equation (1.1). Later, in 2008, Tunç [3] investigated the same problem for nonlinear delay differential equation of third order:

$$\dddot{x}\ (t) + f(t, x(t), \dot{x}(t), \ddot{x}(t), \ddot{x}(t-r)) + b(t)g(\dot{x}(t-r)) + c(t)h(x(t)) = e(t).$$

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In this paper, we consider third order nonlinear delay differential equation of the form:

$$\dddot{x}(t) + f(t, x(t), x(t-r), \dot{x}(t), \dot{x}(t-r), \ddot{x}(t), \ddot{x}(t-r)) \\ + b(t)g(x(t-r), \dot{x}(t-r)) + c(t)h(x(t)) \\ = p(t, x(t), x(t-r), \dot{x}(t), \dot{x}(t-r), \ddot{x}(t)) \tag{1.2}$$

whose equivalent system is

$$\begin{aligned} &\dot{x} = y, \quad \dot{y} = z, \\ &\dot{z} = -f(t, x, x(t-r), y, y(t-r), z, z(t-r)) - b(t)g(x, y) - c(t)h(x) \\ &\quad + b(t)\int_{t-r}^{t} g_x(x(s), y(s))y(s)ds + b(t)\int_{t-r}^{t} g_y(x(s), y(s))z(s)\,ds \\ &\quad + p(t, x, x(t-r), y, y(t-r), z), \end{aligned} \tag{1.3}$$

in which r is a constant delay, $r > 0$; the functions b, c, f, g, h and p depend only on the arguments displayed explicitly; the dots in (1.2) denote differentiation with respect to t. It is assumed as basic that $b(t)$ and $c(t)$ are continuous on $\mathbb{R}^+$, $\mathbb{R}^+ = (0, \infty)$, and $f(t, x, x(t-r), y, y(t-r), z, z(t-r))$, $g(x, y)$, $h(x)$ and $p(t, x, x(t-r), y, y(t-r), z)$ are continuous in their respective arguments on $\mathbb{R}^+ \times \mathbb{R}^6$, $\mathbb{R}^2$, $\mathbb{R}$ and $\mathbb{R}^+ \times \mathbb{R}^5$, respectively; the derivatives $b'(t)$, $(\partial/\partial x)g(x, y) \equiv g_x(x, y)$, $(\partial/\partial y)g(x, y) \equiv g_y(x, y)$ exist and are continuous for all t, x and y; throughout the paper $x(t), y(t), z(t)$ are abbreviated as x, y and z, respectively.

We establish here some sufficient conditions which guarantee to the boundedness of solutions of (1.2). Obviously, equations investigated by Mehri and Shadman [2] and Tunç [3] are special case of our equation (1.2).

2. Main results

The first main result is the following theorem.

Theorem 2.1. *In addition to the basic assumptions imposed on functions b, c, f, g, h and p, it is assumed that the following conditions hold:*

(a) $B \geq b(t) \geq b_0 > 0$, $b'(t) \geq k_1 > 0$ *and* $C \geq c(t) > 0$ *for all* $t \in \mathbb{R}^+$, *where* B, b_0, C *and* k_1 *are some positive constants;*

(b) $f(t, x, x(t-r), y, y(t-r), z, z(t-r))/z \geq a_1$ *for all* $t \in \mathbb{R}^+$ *and* x, $x(t-r)$, y, $y(t-r)$, $z(\neq 0)$, $z(t-r) \in \mathbb{R}$, *where* a_1 *is a positive constant;*

(c) $0 < g(x, y)/y \leq b_1$, $(y \neq 0)$, $0 < g_y(x, y) \leq b_1$, $-M \leq g_x(x, y) \leq -L$ *for all* x, $y \in \mathbb{R}$, *where* M, L *and* b_1 *are some positive constants;*

(d) $0 < h(x)/x \leq c_1$ *for all* $x \in \mathbb{R}$ $(x \neq 0)$, *where* c_1 *is a positive constant;*

(e) $|p(t, x, x(t-r), y, y(t-r), z)| \leq |e(t)|$ *for all* $t \in \mathbb{R}^+$, x, $x(t-r)$, y, $y(t-r)$, $z \in \mathbb{R}$, *where* $e(t)$ *is a continuous function of* t*;*

(f) *there are arbitrary continuous functions α_0, α_1 and β on $\mathbb{R}^+ = (0,\infty)$ such that α_0 and α_1 are positive and decreasing functions and β is a positive and increasing function for all $t \in \mathbb{R}^+$, and*

$$\frac{e(t)}{\sqrt{b(t)}}, \quad \left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2}, \quad \left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{1/2}, \quad |c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{1/2} \in L^1(0,\infty),$$

where $L^1(0,\infty)$ is space of integrable Lebesgue functions. Then, for every solution of equation (1.2), $x/\sqrt{\beta/\alpha_0}$, $\dot{x}/\sqrt{\beta/\alpha_1}$ and $\ddot{x}/\sqrt{b}$, are bounded for all $t \in \mathbb{R}^+$ provided that

$$r < \min\left\{\frac{2b_0L}{M}, \frac{b_0(2a_1B + k_1)}{(2b_1 + M)B^2}\right\}.$$

Now, to prove the theorem, we introduce a differentiable energy functional $E = E(t, x_t, y_t, z_t)$ defined by:

$$\begin{aligned} E := {}& \frac{\alpha_0(t)}{\beta(t)}x^2 + \frac{\alpha_1(t)}{\beta(t)}y^2 + \frac{1}{b(t)}z^2 \\ &+ 2\int_0^y g(x,\eta)\,d\eta + \lambda\int_{-r}^0\int_{t+s}^t y^2(u)\,du\,ds + \mu\int_{-r}^0\int_{t+s}^t z^2(u)\,du\,ds, \end{aligned}$$

where λ and μ are some positive constants, which will be determined according to the purpose here; α_0, α_1, β and b are positive functions, and both α_0 and α_1 and β and b, respectively, are decreasing and increasing functions for all $t \in \mathbb{R}^+$. It is also clear that the expressions $\int_{-r}^0\int_{t+s}^t y^2(u)\,du\,ds$ and $\int_{-r}^0\int_{t+s}^t z^2(u)\,du\,ds$ are non-negative.

PROOF. Let $(x,y,z) = (x(t),y(t),z(t))$ be an arbitrary solution of system (1.3). Differentiating the functional $E = E(t,x_t,y_t,z_t)$ along system (1.3) and using the assumptions of Theorem 2.1, it can be easily verified that

$$\begin{aligned} \frac{d}{dt}&E(t,x_t,y_t,z_t) \\ &= \left(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)x^2 + \left(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)y^2 - \frac{b'(t)}{b^2(t)}z^2 \\ &\quad - \frac{2}{b(t)}\left(\frac{f(t,x,x(t-r),y,y(t-r),z,z(t-r))}{z}\right)z^2 + \frac{2\alpha_0(t)}{\beta(t)}xy \\ &\quad + \frac{2\alpha_1(t)}{\beta(t)}yz - \frac{2c(t)}{b(t)}zh(x) + 2y\int_0^y g_x(x,\eta)\,d\eta \\ &\quad + \frac{2}{b(t)}zp(t,x,x(t-r),y,y(t-r),z) \\ &\quad + \frac{2z}{b(t)}\int_{t-r}^t g_x(x(s),y(s))y(s)\,ds + \frac{2z}{b(t)}\int_{t-r}^t g_y(x(s),y(s))z(s)\,ds \\ &\quad + \lambda ry^2 + \mu rz^2 - \lambda\int_{t-r}^t y^2(s)\,ds - \mu\int_{t-r}^t z^2(s)\,ds \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
&\leq \left(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)x^2 + \left(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)y^2 \\
&\quad - \frac{k_1}{B^2}z^2 - \frac{2a_1}{B}z^2 + \frac{2\alpha_0(t)}{\beta(t)}|x||y| + \frac{2\alpha_1(t)}{\beta(t)}|y||z| \\
&\quad + \frac{2c(t)}{b(t)}\frac{h(x)}{x}|x||z| + 2y\int_0^y g_x(x,\eta)\,d\eta \\
&\quad + \frac{2}{b(t)}|z||p(t,x,x(t-r),y,y(t-r),z)| \\
&\quad + \frac{2z}{b(t)}\int_{t-r}^t g_x(x(s),y(s))y(s)\,ds \\
&\quad + \frac{2z}{b(t)}\int_{t-r}^t g_y(x(s),y(s))z(s)\,ds \\
&\quad + \lambda r y^2 + \mu r z^2 - \lambda\int_{t-r}^t y^2(s)\,ds - \mu\int_{t-r}^t z^2(s)\,ds.
\end{aligned}
$$

Now, by the assumptions of Theorem 2.1 and inequality $2|cd| \leq c^2 + d^2$, we have the following:

$$
\left(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)x^2 \leq 0, \qquad \left(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)y^2 \leq 0,
$$

$$
2y\int_0^y g_x(x,\eta)d\eta \leq -2Ly^2,
$$

$$
\begin{aligned}
\frac{2}{b(t)}|z||p(t,x,x(t-r),y,y(t-r),z)| &\leq \frac{2|e(t)|}{b(t)}|z|, \qquad (2.2)\\
\frac{2z}{b(t)}\int_{t-r}^t g_x(x(s),y(s))y(s)\,ds &\leq \frac{Mr}{b_0}z^2 + \frac{M}{b_0}\int_{t-r}^t y^2(s)\,ds, \\
\frac{2z}{b(t)}\int_{t-r}^t g_y(x(s),y(s))z(s)\,ds &\leq \frac{b_1 r}{b_0}z^2 + \frac{b_1}{b_0}\int_{t-r}^t z^2(s)\,ds.
\end{aligned}
$$

Further, the functional $E = E(t, x_t, y_t, z_t)$ implies

$$
\begin{gathered}
|x| \leq \left(\frac{\beta(t)}{\alpha_0(t)}\right)^{1/2} E^{1/2}, \quad |y| \leq \left(\frac{\beta(t)}{\alpha_1(t)}\right)^{1/2} E^{1/2}, \\
|z| \leq \sqrt{b(t)}E^{1/2} \leq \sqrt{b(t)}\left(\frac{1}{2} + \frac{E}{2}\right),
\end{gathered}
$$

respectively. Hence

$$
\frac{2\alpha_0(t)}{\beta(t)}|x||y| \leq 2\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2} E, \tag{2.3}
$$

$$
\frac{2\alpha_1(t)}{\beta(t)}|y||z| \leq 2\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{1/2} E, \tag{2.4}
$$

$$2\frac{|e(t)|}{b(t)}|z| \le \frac{|e(t)|}{\sqrt{b(t)}} + \frac{|e(t)|}{\sqrt{b(t)}}E, \tag{2.5}$$

$$\frac{2c_1|c(t)|}{b(t)}|z||x| \le 2c_1|c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{1/2}E. \tag{2.6}$$

Substituting (2.2) and (2.3)–(2.6) into (2.1), we get

$$\begin{aligned}\frac{d}{dt}E(t,x_t,y_t,z_t) \le &-(2L-\lambda r)y^2 - \left[\frac{2a_1}{B} + \frac{k_1}{B^2} - \left(\frac{b_1}{b_0} + \frac{M}{b_0} + \mu\right)r\right]z^2 \\ &+ 2\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2}E + 2\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{1/2}E \\ &+ \frac{|e(t)|}{\sqrt{b(t)}} + \frac{|e(t)|}{\sqrt{b(t)}}E + 2c_1|c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{1/2}E \\ &- (\lambda - b_0^{-1}M)\int_{t-r}^{t} y^2(s)\,ds - (\mu - b_0^{-1}b_1)\int_{t-r}^{t} z^2(s)\,ds.\end{aligned}$$

Let us choose $\lambda = M/b_0$ and $\mu = b_1/b_0$. Hence

$$\begin{aligned}\frac{d}{dt}E(t,x_t,y_t,z_t) \le\, &2\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2}E + 2\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{1/2}E \\ &+ \frac{|e(t)|}{\sqrt{b(t)}}E + 2c_1|c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{1/2}E + \frac{|e(t)|}{\sqrt{b(t)}}\end{aligned} \tag{2.7}$$

provided

$$r < \min\left\{\frac{2b_0L}{M}, \frac{b_0(2a_1B+k_1)}{(2b_1+M)B^2}\right\},$$

which we now assume. Now, let

$$\begin{aligned}\Phi(t) = 2\Bigg[&\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2} + \left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{1/2} \\ &+ \frac{|e(t)|}{2(b(t))^{1/2}} + c_1|c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{1/2}\Bigg].\end{aligned} \tag{2.8}$$

Then, it follows from (2.7) and (2.8) that

$$\frac{d}{dt}E(t,x_t,y_t,z_t) \le \frac{|e(t)|}{\sqrt{b(t)}} + \Phi(t)E(t,x_t,y_t,z_t). \tag{2.9}$$

Integrating (2.9) from 0 to t, we obtain

$$E(t,x_t,y_t,z_t) - E(0,x_0,y_0,z_0) = \int_0^t \frac{|e(s)|}{\sqrt{b(s)}}\,ds + \int_0^t E(s,x_s,y_s,z_s)\Phi(s)\,ds.$$

By using assumption (f) of Theorem 2.1 and the Gronwall–Reid–Bellman inequality (see also Ahmad and Rama Mohana Rao [1]), we get

$$E(t,x_t,y_t,z_t) \le A\exp\left(\int_0^t \Phi(s)\,ds\right) \tag{2.10}$$

for a positive constant

$$A = E(0,x_0,y_0,z_0) + \int_0^\infty \frac{|e(s)|}{\sqrt{b(s)}}\,ds,$$

since $|e(s)|/\sqrt{b(s)} \in L^1(0,\infty)$. Finally, since $\Phi \in L^1(0,\infty)$, ones can get from (2.10) for some positive constant K that

$$E(t,x_t,y_t,z_t) \le K. \tag{2.11}$$

On the other hand, observe

$$\begin{aligned} E(t,x_t,y_t,z_t) &\ge \frac{\alpha_0(t)}{\beta(t)}x^2 + \frac{\alpha_1(t)}{\beta(t)}y^2 + \frac{1}{b(t)}z^2 + 2\int_0^y \frac{g(x,\eta)}{\eta}\eta\,d\eta \\ &\ge \frac{\alpha_0(t)}{\beta(t)}x^2 + \frac{\alpha_1(t)}{\beta(t)}y^2 + \frac{1}{b(t)}z^2. \end{aligned} \tag{2.12}$$

Now, (2.11) and (2.12) together imply that $\alpha_0 x^2/\beta$, $\alpha_1 y^2/\beta$ and z^2/b are bounded, and hence this result guarantees the boundedness of $x/\sqrt{\beta/\alpha_0}$, $x'/\sqrt{\beta/\alpha_1}$ and $x''/\sqrt{b}$. This case completes the proof of Theorem 2.1. □

The second and last result is the following theorem.

THEOREM 2.2. *Let us replace conditions* (a), (b) *and* (f) *of Theorem* 2.1 *by the conditions:*

- (a′) $b(t) > 0$ *for all* $t \in \mathbb{R}^+$*;*
- (b′) *there exist a positive constant* M_1 *such that*

$$\frac{f(t,x,x(t-r),y,y(t-r),z,z(t-r))}{z} \ge M_1$$

for all $t \in \mathbb{R}^+$, x, $x(t-r)$, y, $y(t-r)$, $z (z \ne 0)$, $z(t-r) \in \mathbb{R}$, *where* M_1 *is a positive constant and* $b'(t) + M_1 b(t) > 0$ *for all* $t \in \mathbb{R}^+$*;*
- (f′) *there are arbitrary continuous functions* α_0, α_1 *and* β *on* $\mathbb{R}^+$ *such that* α_0 *and* α_1 *are positive and decreasing and* β *is positive and increasing for all* $t \in \mathbb{R}^+$, *and*

$$\frac{e^2(t)}{b'(t)+M_1 b(t)},\ \frac{e(t)}{\sqrt{b(t)}},\ \left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2},\ \left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{1/2},\ |c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{1/2}$$

in $L^1(0,\infty)$.

Then the conclusion of Theorem 2.1 *holds provided that*

$$r \le \min\left\{ \inf_t \left(\frac{b_0 M_1}{b(t)(2b_1+M)}\right), \frac{2b_0 L}{M}\right\}.$$

PROOF. Now, under the assumptions of Theorem 2.2, we easily obtain

$$\begin{aligned}
\frac{d}{dt}E(t,x_t,y_t,z_t) \le{}& \left(\frac{\alpha_0'(t)}{\beta(t)} - \frac{\alpha_0(t)\beta'(t)}{\beta^2(t)}\right)x^2 + \left(\frac{\alpha_1'(t)}{\beta(t)} - \frac{\alpha_1(t)\beta'(t)}{\beta^2(t)}\right)y^2 \\
&- \frac{b'(t)}{b^2(t)}z^2 - \frac{2M_1}{b(t)}z^2 + \frac{2\alpha_0(t)}{\beta(t)}|x||y| \\
&+ \frac{2\alpha_1(t)}{\beta(t)}|y||z| + 2c_1\frac{|c(t)|}{b(t)}|z||x| \\
&+ 2\frac{|e(t)|}{b(t)}|z| - 2Ly^2 + \frac{Mr}{b_0}z^2 + \frac{b_1 r}{b_0}z^2 + \lambda r y^2 + \mu r z^2 \\
&+ \frac{M}{b_0}\int_{t-r}^t y^2(s)\,ds + \frac{b_1}{b_0}\int_{t-r}^t z^2(s)\,ds \\
&- \lambda\int_{t-r}^t y^2(s)\,ds - \mu\int_{t-r}^t z^2(s)\,ds \\
\le{}& -(2L-\lambda r)y^2 - \left[\frac{M_1}{b(t)} - \left(\frac{M}{b_0} + \frac{b_1}{b_0} + \mu\right)r\right]z^2 \\
&- \frac{M_1}{b(t)}z^2 - \frac{b'(t)}{b^2(t)}z^2 + \frac{2\alpha_0(t)}{\beta(t)}|x||y| \\
&+ \frac{2\alpha_1(t)}{\beta(t)}|y||z| + 2c_1\frac{|c(t)|}{b(t)}|z||x| + 2\frac{|e(t)|}{b(t)}|z| \\
&- (\lambda - b_0^{-1}M)\int_{t-r}^t y^2(s)\,ds - (\mu - b_0^{-1}b_1)\int_{t-r}^t z^2(s)\,ds.
\end{aligned}$$

Let $\lambda = M/b_0$ and $\mu = b_1/b_0$. Then

$$\begin{aligned}
\frac{d}{dt}E(t,x_t,y_t,z_t) \le{}& -\left(2L - \frac{M}{b_0}r\right)y^2 - \left[\frac{M_1}{b(t)} - \left(\frac{M+2b_1}{b_0}\right)r\right]z^2 \\
&- \frac{M_1}{b(t)}z^2 - \frac{b'(t)}{b^2(t)}z^2 + \frac{2\alpha_0(t)}{\beta(t)}|x||y| \\
&+ \frac{2\alpha_1(t)}{\beta(t)}|y||z| + 2c_1\frac{|c(t)|}{b(t)}|z||x| + 2\frac{|e(t)|}{b(t)}|z| \\
\le{}& -\frac{M_1}{b(t)}z^2 - \frac{b'(t)}{b^2(t)}z^2 + \frac{2\alpha_0(t)}{\beta(t)}|x||y| \\
&+ \frac{2\alpha_1(t)}{\beta(t)}|y||z| + 2c_1\frac{|c(t)|}{b(t)}|z||x| + 2\frac{|e(t)|}{b(t)}|z|
\end{aligned}$$

provided that

$$r \le \min\left\{ \inf_t \left(\frac{b_0 M_1}{b(t)(2b_1+M)}\right), \frac{2b_0 L}{M}\right\},$$

which we now assume. Hence

$$\frac{d}{dt}E(t,x_t,y_t,z_t) \le -(b'(t)+M_1b(t))\left(\frac{|z|}{b(t)}-\frac{|e(t)|}{b'(t)+M_1b(t)}\right)^2 + \frac{e^2(t)}{b'(t)+M_1b(t)} + \frac{2\alpha_0(t)}{\beta(t)}|x||y| + \frac{2\alpha_1(t)}{\beta(t)}|y||z| + 2c_1\frac{|c(t)|}{b(t)}|z||x|.$$

Therefore, it is clear that

$$\begin{aligned}
&\frac{d}{dt}E(t,x_t,y_t,z_t)\\
&\quad\le -(b'(t)+M_1b(t))\left(\frac{|z|}{b(t)}-\frac{|e(t)|}{b'(t)+M_1b(t)}\right)^2+\frac{e^2(t)}{b'(t)+M_1b(t)}\\
&\qquad+2\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2}E+2\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{1/2}E+2c_1|c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{1/2}E\\
&\quad= -(b'(t)+M_1b(t))\left(\frac{|z|}{b(t)}-\frac{|e(t)|}{b'(t)+M_1b(t)}\right)^2+\frac{e^2(t)}{b'(t)+M_1b(t)}\\
&\qquad+2\left[\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2}+\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{1/2}+c_1|c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{1/2}\right]E\\
&\quad\le 2\left[\left(\frac{\alpha_0(t)}{\alpha_1(t)}\right)^{1/2}+\left(\frac{\alpha_1(t)b(t)}{\beta(t)}\right)^{1/2}+c_1|c(t)|\left(\frac{\beta(t)}{\alpha_0(t)b(t)}\right)^{1/2}\right]E\\
&\qquad+\frac{e^2(t)}{b'(t)+M_1b(t)}.
\end{aligned}$$

This implies that

$$\frac{d}{dt}E(t,x_t,y_t,z_t) \le \frac{e^2(t)}{b'(t)+M_1b(t)} + [\Phi(t)-b^{-1/2}(t)|e(t)|]E, \tag{2.13}$$

where $\Phi(t)$ is defined as the same as in (2.8). Now, as in the proof of Theorem 2.1, integrating (2.13) from 0 to t, later using assumption (f′) of Theorem 2.2 and the Gronwall–Reid–Bellman inequality, (see also Ahmad and Rama Mohana Rao [1]), ones can easily obtain the following inequality:

$$\begin{aligned}
&E(t,x_t,y_t,z_t)-E(0,x_0,y_0,z_0)\\
&\qquad\le \int_0^t \frac{e^2(s)}{b'(s)+M_1b(s)}\,ds + \int_0^t E(s,x_s,y_s,z_s)[\Phi(s)-b^{-1/2}(s)|e(s)|]\,ds,
\end{aligned}$$

and hence a bound for the functional E. The proof of Theorem 2.2 is now complete. □

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