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A DECOMPOSITION FORMULA FOR EQUIVARIANT STABLE HOMOTOPY CLASSES

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Abstract. For any compact Lie group G, we give a decomposition of the group $\{X,Y\}_G^k$ of (unpointed) stable G-homotopy classes as a direct sum of subgroups of fixed orbit types. This is done by interpreting the G-homotopy classes in terms of the generalized fixed-point transfer and making use of conormal maps.

1. Introduction

A description of the homotopy classes, or of the stable homotopy classes of maps between two topological spaces has been a classical question in topology. A variant of the question arises when we assume that a compact Lie group G acts on all spaces involved and that all the maps considered commute with the group action, that is, that the maps are G-equivariant — G-maps for short. Then the corresponding question is to provide a description of the stable G-homotopy classes between G-spaces.

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In this paper we give a decomposition of the group of equivariant stable homotopy classes of maps between two G-spaces X and Y, provided that X has trivial G-action (Theorem 2.6). A similar result was proven by Lewis, Jr., May, and McClure in [8, V.10.1] under other assumptions (they consider more general symmetry and their space X is a finite CW-complex) and using rather different methods. Using classical methods in algebraic topology, tom Dieck gives a decomposition of the equivariant homotopy groups in his book [3, II(7.7)]. An advantage of our approach is that it gives a short proof showing the geometric interpretation of the maps that form a term of this decomposition, even in the unstable range as in [9]. In particular, we do not need the Adams and Wirthmüller isomorphisms to define the splitting homomorphism. To carry out the decomposition, we use the equivariant fixed-point transfer given by the second author in [11], which is the equivariant generalization of the classical Dold fixed-point transfer [4], and the fixed-point theoretical arguments used in [9].

One of the purposes of this paper is to make our result clear to nonlinear analysts. In the framework of nonlinear analysis, the equivariant stems, or some of their subgroups, are the ranges for various equivariant degrees (cf. [2] and [6]). With respect to the problem about the existence of periodic solutions, the most interesting seems to be the case $G = \mathbb{S}^1$. Another interesting application of this trend of ideas is the case of a homotopy class given by a gradient of a smooth function (cf. [14] for a survey article).

A special case of our main Theorem 2.6 yields a decomposition of the G-equivariant 1-stem, that was given using different methods by Kosniowski [7], Hauschild [5], and Balanov-Krawcewicz [1]. This decomposition was also used by us [10] to give a full description of the first G-stem as follows:

$$\pi_1^{G\,\mathrm{st}} = \bigoplus_{\substack{(H)\in \mathrm{Or}_G \\ \dim W(H)\leq 1}} \Pi_1(H),$$

where, if $\dim W(H) = 0$,

$$\Pi_1(H) \cong \mathbb{Z}_2 \oplus W(H)_{\mathrm{ab}},$$

and $W(H)_{\mathrm{ab}}$ is the abelianization of $W(H)$, and, if $\dim W(H) = 1$,

$$\Pi_1(H) \cong \begin{cases} \mathbb{Z} & W(H) \text{ is biorientable,} \\ \mathbb{Z}_2 & \text{if } W(H) \text{ is not biorientable.} \end{cases}$$

2. The general decomposition formula

In this section, we use the generalized fixed-point transfer to give a direct sum decomposition of $\{X, Y\}_G^k$. All along the paper, G will denote a compact Lie group. We shall assume that X and Y are metric spaces with a G-action.

DEFINITION 2.1. Let V, W, M, and N denote finite dimensional real G-modules, namely, orthogonal representations of G, and let ρ be the element $[M]-[N] \in \mathrm{RO}(G)$. Then the elements of $\{X,Y\}_G^\rho$ are *stable homotopy classes* represented by equivariant maps of pairs

$$\alpha\colon (N\times V, N\times V-0)\times X \to (M\times V, M\times V-0)\times Y.$$

Such a map will be *stably homotopic* to another

$$\alpha'\colon (N\times V', N\times V'-0)\times X \to (M\times V', M\times V'-0)\times Y,$$

if after taking the product of each map with the identity maps of some pairs $(W, W-0)$ and $(W', W'-0)$, respectively, they become G-homotopic, where $V\times W \cong_G V'\times W'$. Denote the class of α by $\{\alpha\}$.

REMARK 2.2. Taking the product of X with a pair $(L, L-0)$ for some orthogonal representation L of G amounts to the same as smashing $X^+ = X\sqcup\{*\}$ with the sphere $\mathbb{S}^L$ that is obtained as the one-point compactification of L (which is G-homeomorphic to the unit sphere $S(L\oplus\mathbb{R})$ in the representation $L\oplus\mathbb{R}$, with trivial action on the last coordinate). Thus

$$\begin{aligned}\{X,Y\}_G^\rho &\cong \mathrm{colim}_V[\mathbb{S}^N\wedge\mathbb{S}^V\wedge X^+, \mathbb{S}^M\wedge\mathbb{S}^V\wedge Y^+]_G\\ &\cong \mathrm{colim}_V[\mathbb{S}^{N\oplus V}\wedge X^+, \mathbb{S}^{M\oplus V}\wedge Y^+]_G\\ &\cong \mathrm{colim}_V[X^+, \Omega_{N\oplus V}\mathbb{S}^{M\oplus V}\wedge Y^+]_G,\end{aligned}$$

where the colimit of pointed G-homotopy classes is taken over a cofinal system of G-representations V. Observe that this does not coincide with the usual definition, when X is infinite dimensional. For homotopy theoretical purposes, the definition is given by

$$G\text{-}\mathfrak{S}tab^\rho(X,Y) = [X^+, \mathrm{colim}_V\Omega_{N\oplus V}\mathbb{S}^{M\oplus V}\wedge Y^+]_G$$

with the colimit taken "inside". However, for the purposes of nonlinear analysis, our definition seems to be more adequate.

In [11] (see also [12]) one proves that any $\{\alpha\}\in\{X,Y\}_G^\rho$ can be written as a composite

$$\{\alpha\} = \varphi\circ\tau(f), \tag{2.1}$$

where $\tau(f)$ is the equivariant fixed-point transfer of an equivariant fixed-point situation

$$\begin{tikzcd} N\times E\supset\mathcal{U} \arrow[rr, "f"] \arrow[dr, "p\cdot\mathrm{proj}_E"'] & & M\times E \arrow[dl, "p\cdot\mathrm{proj}_E"] \\ & X & \end{tikzcd} \tag{2.2}$$

where $E \to X$ is a G-ENR$_X$ and the *fixed point set* $\mathrm{Fix}(f) = \{(s,e) \in \mathcal{U} \mid f(s,e) = (0,e) \in M \times E\}$ lies properly over X, $\rho = [M] - [N] \in \mathrm{RO}(G)$. The transfer is a stable map

$$\tau(f): (N \times V, N \times V - 0) \times X \to (M \times V, M \times V - 0) \times \mathcal{U},$$

for some orthogonal representation V, and $\varphi: \mathcal{U} \to Y$ is a nonstable equivariant map (by the localization property of the fixed-point transfer, $\mathcal{U}$ can always be assumed to be a very small open G-neighbourhood of the fixed point set $\mathrm{Fix}(f)$; see [12, 4.4]), (the composite is made after suspending φ by taking its product with the identity of $(M \times V, M \times V - 0)$).

We denote by Or_G the set of orbit types of G, that is the set of conjugacy classes (H) of subgroups $H \subset G$. For any G-ENR$_X$ E, where X has trivial G-action, the set of orbit types in E, denoted by $\mathrm{Or}_G(E)$, is always finite.

In what follows, we shall only be concerned with the special case $N = \mathbb{R}^n$, $M = \mathbb{R}^{n+k}$, $k \in \mathbb{Z}$, and we shall assume that X is a space with trivial G-action.

For the statement of the main result of this section we need the following definitions. The first of them was originally given in [9, 5.4].

Definition 2.3. Consider the fixed-point situation (2.2) above. We say that the map $f: \mathcal{U} \to \mathbb{R}^{n+k} \times E$ is *conormal* if for every orbit type $(H) \in \mathrm{Or}_G(\mathbb{R}^n \times E) = \mathrm{Or}_G(E)$, there exist an open invariant neighbourhood $\mathcal{V}$ of $\mathcal{U}^{(\underline{H})}$ in $\mathcal{U}^{(H)}$ and an equivariant retraction $r: \overline{\mathcal{V}} \to \mathcal{U}^{(\underline{H})}$ such that for the restricted map $f^{(H)} = f|_{\mathcal{U}^{(H)}}$ we have

$$f^{(H)}|_{\overline{\mathcal{V}}} = f \circ r: \overline{\mathcal{V}} \to \mathbb{R}^{n+k} \times E.$$

Here $\mathcal{U}^{(H)}$ consists of the points in $\mathcal{U}$ with isotropy larger than (H) and $\mathcal{U}^{(\underline{H})}$ to those with isotropy *strictly* larger than (H).

Definition 2.4. For any subgroup $H \subset G$, we define the subgroup $\{X,Y\}^k_{(H)}$ of $\{X,Y\}^k_G$ as the subgroup of those classes $\{\alpha\}$ such that $\{\alpha\} = \varphi \circ \tau(f)$, where

(a) f is a conormal map, and
(b) $\mathrm{Fix}(f) \subset \mathcal{U}_{(H)}$, where $\mathcal{U}_{(H)}$ consists of the points in $\mathcal{U}$ with isotropy group conjugate to H.

Remark 2.5. The fact that $\{X,Y\}^k_{(H)}$ is a subgroup of $\{X,Y\}^k_G$ follows easily by observing that both properties (a) and (b) are preserved by the sum of two elements $\{\alpha\} = \varphi \circ \tau(f)$, $\{\beta\} = \psi \circ \tau(g)$, that, by the additivity property of the fixed-point transfer, corresponds to the disjoint union $f + g$ of the fixed-point situations (see [13, 1.17]).

The main result in this section is the following.

THEOREM 2.6. *Let X be a space with trivial G-action. Then there is an isomorphism*

$$\{X,Y\}_G^k \cong \bigoplus_{(H)} \{X,Y\}_{(H)}^k.$$

For the proof we need some preliminary results. Consider a fixed-point situation as (2.2). First note that it is always possible to provide $\mathrm{Or}_G(E)$ with an order (H_j), $j = 1,\dots,l$ such that $(H_i) \subset (H_j)$ implies $j \le i$. Define $E_i \subset E$ as $\bigcup_{i \le j} E^{(H_j)}$. These G-subspaces determine a filtration of E such that $E_i - E_{i-1} = E_{(H_i)}$. Let $f_i = f|_{\mathcal{U}_i}\colon \mathcal{U}_i \to \mathbb{R}^{n+k} \times E_i$, where $\mathcal{U}_i = \mathcal{U} \cap (\mathbb{R}^n \times E_i)$.

PROPOSITION 2.7. *For every $i = 1,\dots,l$ there exists an invariant neighbourhood $\mathcal{V}_i$ of E_{i-1} in E_i and an equivariant retraction $r_i\colon \overline{\mathcal{V}}_i \to E_{i-1}$ that is admissibly homotopic to the identity. Thus f_i is admissibly homotopic to $f'_{i-1} = f_{i-1} \circ (\mathrm{id}_{\mathbb{R}^n} \times r_i)$.*

The proof is similar to those of [9, 5.3 and 5.7].

PROPOSITION 2.8. *The following hold:*

(a) *f is equivariantly homotopic by an admissible homotopy f_τ to a conormal map $f' = f_1\colon V \to \mathbb{R}^m \times E$. Moreover, if $A \subset \mathcal{U}$ is a closed G-*ENR *subspace, then this homotopy can be taken relative to A.*

(b) *Furthermore, if f_0 and f_1 are equivariantly homotopic by an admissible homotopy, and each of them is equivariantly homotopic by an admissible homotopy to two conormal maps f'_0, $f'_1\colon \mathcal{U} \to \mathbb{R}^m \times E$, respectively, then these two conormal maps are equivariantly homotopic by an admissible conormal homotopy.*

The proof is the same as that of [9, 5.7] (see also [13, 2.10 and 2.11] or [15, II.6.8 and III.5.2]).

We also need a lemma.

LEMMA 2.9. *Let $f\colon \mathcal{U} \to \mathbb{R}^{n+k} \times E$ be a fixed-point situation over X such that f is a conormal map and take $(H) \in \mathrm{Or}_G(E)$. Then there is a neighbourhood $\mathcal{V}$ of $\mathrm{Fix}(f|_{\mathcal{U}_{(H)}})$ such that $g = f|_{\mathcal{V}} : \mathcal{V} \to \mathbb{R}^{n+k} \times E$ is a conormal map with $\mathrm{Fix}(g) = \mathrm{Fix}(f|_{\mathcal{U}_{(H)}})$. Denote g by $f_{(H)}$. Consequently,*

$$\tau(f) = \sum_{(H) \in \mathrm{Or}_G(E)} \tau(f_{(H)}). \tag{2.3}$$

PROOF. Since f is conormal, the set $F = \mathrm{Fix}(f|_{\mathcal{U}_{(H)}})$ is separated from all other fixed points. Then there is a neighbourhood $\mathcal{V}$ of F in $\mathcal{U}$ such that $\mathrm{Fix}(f) \cap \mathcal{V} = F$. Hence $g = f|_{\mathcal{V}}\colon \mathcal{V} \to \mathbb{R}^{n+k} \times E$ is a conormal map with the desired properties. By the additivity property of the transfer we obtain the decomposition (2.3). $\square$

We now pass to the proof of Theorem 2.6.

Proof. Any $\{\alpha\} \in \{X,Y\}^k_G$ can be written as the composite (2.1) $\varphi \circ \tau(f)$, where $\tau(f)$ is the equivariant fixed-point transfer of an equivariant fixed-point situation (2.2). By Proposition 2.8(a), f can be assumed to be a conormal map, and by Lemma 2.9, $\tau(f) = \sum_{(H)\in \mathrm{Or}_G(E)} \tau(f_{(H)})$. Defining $\{\alpha_{(H)}\}$ by $\{\alpha_{(H)}\} = \varphi|_{\mathcal{U}_{(H)}} \circ \tau(f_{(H)})$, we have immediately

$$\{\alpha\} = \sum_{(H)\in \mathrm{Or}_G(E)} \{\alpha_{(H)}\},$$

where $\{\alpha_{(H)}\} \in \{X,Y\}^k_{(H)}$. So, by Proposition 2.8(b), we may define

$$\Phi\colon \{X,Y\}^k_G \to \bigoplus_{(H)} \{X,Y\}^k_{(H)} \quad \text{by } \Phi(\{\alpha\}) = \bigoplus_{(H)} \{\alpha_{(H)}\}.$$

If $(H) \neq (K)$, then $\{X,Y\}^k_{(H)} \cap \{X,Y\}^k_{(K)} = 0$ as easily follows with the same argument used in the proof of [9, 6.2] (see also [1]). Thus we may also define

$$\Psi\colon \bigoplus_{(H)} \{X,Y\}^k_{(H)} \to \{X,Y\}^k_G \quad \text{by } \Psi\bigg(\bigoplus_{(H)} \{\alpha_{(H)}\}\bigg) = \sum_{(H)} \{\alpha_{(H)}\}.$$

Then Φ and Ψ are inverse isomorphisms. $\square$

Remark 2.10. For any fixed-point situation f (see (2.2)), it is proven in [12, 4.4] that the transfer

$$\tau(f) = \sum_{(H)\in \mathrm{O}(G)} (\tau(f^{(H)}) - \tau(f^{(\underline{H})})),$$

where

$$f^{(H)} = f|_{U^{(H)}}\colon U^{(H)} \to (M \times E)^{(H)}$$

and $U^{(H)} \subset (N \times E)^{(H)} = \mathbb{R}^n \times E^{(H)}$, resp.

$$f^{(\underline{H})} = f|_{U^{(\underline{H})}}\colon U^{(\underline{H})} \to (M \times E)^{(\underline{H})}$$

and $U^{(\underline{H})} \subset (N \times E)^{(\underline{H})} = \mathbb{R}^n \times E^{(\underline{H})}$.

As in (2.1) any $\{\alpha\} = \varphi \circ \tau(f)$ for some fixed-point situation f as above. As in the proof of [12, 4.4], we have

$$\begin{aligned} \{\alpha^{(H)}\} &= \varphi^{(H)} \circ \tau(f^{(H)})\colon X \to Y^{(H)} \subset Y, \\ \{\alpha^{(\underline{H})}\} &= \varphi^{(\underline{H})} \circ \tau(f^{(\underline{H})})\colon X \to Y^{(\underline{H})} \subset Y. \end{aligned}$$

Thus $\{\alpha\} = \sum_{(H)\in \mathrm{O}(G)} (\{\alpha^{(H)}\} - \{\alpha^{(\underline{H})}\})$. Hence it is each difference $\{\alpha^{(H)}\} - \{\alpha^{(\underline{H})}\} = \{\alpha_{(H)}\}$; that is, $\alpha_{(H)}$, as given by the conormal map, realizes the (stable) difference $\{\alpha^{(H)}\} - \{\alpha^{(\underline{H})}\}$ for each orbit type (H) (cf. also [15, III.5]).

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