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A GENERALIZATION OF NADLER'S FIXED POINT THEOREM AND ITS APPLICATION TO NONCONVEX INTEGRAL INCLUSIONS

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Abstract. In this paper, a generalization of Nadler's fixed point theorem is presented. In the sequel, we consider a nonconvex integral inclusion and prove a Filippov type existence theorem by using an appropriate norm on the space of selection of the multifunction and a H^+-type contraction for set-valued maps.

1. Introduction

Dynamical systems described by differential equations with continuous right-hand sides was the areas of vigorous steady in the later half of the 20th century in applied mathematics, in particular, in the study of viscous fluid motion in a porous medium, propagation of light in an optically non-homogeneous medium, determining the shape of a solid of revolution moving in a flow of gas with least resistance, etc. Euler's equation plays a key role in dealing with the existence of the solution of such problems. On the other hand, Filipopov [6] has developed a solution concept for differential equations with a discontinuous right-hand side. In practice, such dynamical systems do arise and require analysis. Examples of such systems are mechanical systems with Coulomb friction modeled as

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a force proportional to the sign of a velocity, systems whose control laws have discontinuities.

In a parallel development, the study of fixed points for multivalued contraction maps using the Hausdoff metric was initiated by Nadler [13]. Later, an interesting and rich fixed point theory for such maps has been developed, see, for instance, the work of Feng and Liu [5], Kaneko [7], Klim and Wardowski [8], Lim [10], Lami Dozo [11], Mizoguchi and Takahashi [12], Pathak and Shahzad [14], Reich [16], [17], Suzuki [18], and many others. For details, see [15].

2. Preliminaries and definitions

Let (X,d) be a metric space. Let $\mathrm{CB}(X)$ and $\mathrm{C}(X)$ denote the collection of all nonempty closed and bounded subsets of X and the collection of all compact subsets of X, respectively.

For $A,B \in \mathrm{CB}(X)$, let

$$H(A,B) = \max\{\rho(A,B), \rho(B,A)\}, \qquad H^+(A,B) = \frac{1}{2}\{\rho(A,B) + \rho(B,A)\},$$

where $\rho(A,B) = \sup\limits_{x\in A} d(x,B)$ and $d(x,B) = \inf\limits_{y\in B} d(x,y)$. It is well known that H is a metric on $\mathrm{CB}(X)$. Such a map H is called *Hausdorff metric* induced by d. In Proposition 2.1 below we show that H^+ is also a metric on $\mathrm{CB}(X)$. For $A \subset X$, $\overline{A}$ denotes the closure of A.

A set-valued mapping $T\colon X \to \mathrm{CB}(X)$ is said to be a

(i) *multi-valued contraction mapping* if there exists a fixed real number L, $0 < L < 1$ such that

$$H(Tx,Ty) \le L\, d(x,y) \tag{2.1}$$

(ii) *multi-valued nonexpansive mapping* if

$$H(Tx,Ty) \le d(x,y) \quad \text{for all } x,y \in X. \tag{2.2}$$

Proposition 2.1. *H^+ is a metric on* $\mathrm{CB}(X)$.

Proof. Let $A,B \in \mathrm{CB}(X)$ such that $H^+(A,B) = 0$. Then this is equivalent to $\rho(A,B) = 0$ and $\rho(B,A) = 0$; i.e. $\inf\limits_{y\in B} d(x,y) = 0$ for any $x \in A$ and $\inf\limits_{x\in A} d(y,x) = 0$ for any $y \in B$. Therefore, these are equivalent to $x \in \overline{B} = B$ for any $x \in A$, and $y \in \overline{A} = A$ for any $y \in B$. It follows that $A \subset B$ and $B \subset A$. Hence $A = B$.

The symmetry of the function H^+ follows directly from the definition.

To show the triangle inequality, let $A,B,C \in \mathrm{CB}(X)$. Then for any $(x,y,z) \in A \times B \times C$, we have

$$d(x,z) \le d(x,y) + d(y,z),$$

whence

$$\inf_{z\in C} d(x,z) \le d(x,y) + \inf_{z\in C} d(y,z) \le d(x,y) + \rho(B,C).$$

Since the above inequality holds for any $y \in B$, we get

$$\inf_{z\in C} d(x,z) \le \inf_{y\in B} d(x,y) + \rho(B,C) \le \rho(A,B) + \rho(B,C).$$

Hence

$$\rho(A,C) \le \rho(A,B) + \rho(B,C). \tag{2.3}$$

Interchanging the roles of A and C, we get

$$\rho(C,A) \le \rho(C,B) + \rho(B,A). \tag{2.4}$$

Adding (2.3) and (2.4), and then dividing by 2, we get

$$H^+(A,C) \le H^+(A,B) + H^+(B,C). \tag{2.5}$$

Notice that the two metrics H and H^+ are equivalent (see [9]) since

$$\frac{1}{2}H(A,B) \le H^+(A,B) \le H(A,B). \qquad \square$$

It is routine to prove the following:

Proposition 2.2. *Let $(X, \|\cdot\|)$ be a normed linear space. For any λ (real or complex), $A, B \in \mathrm{CB}(X)$*

(i) $H^+(\lambda A, \lambda B) = |\lambda| H^+(A,B)$,
(ii) $H^+(A+a, B+a) = H^+(A,B)$.

In a classical approach one can easily prove Theorems 2.3 and 2.5 stated below (see, also Banaś and Goebel [1]).

Theorem 2.3. *If $a, b \in X$ and $A, B \in \mathrm{CB}(X)$, then the relations*:

(1) $d(a,b) = H^+(\{a\},\{b\})$,
(2) $A \subset \overline{S}(B; r_1)$, $B \subset \overline{S}(A; r_2) \Rightarrow H^+(A,B) \le r$ *where* $r = (r_1 + r_2)/2$, *and*
(3) $H^+(A,B) < r \Rightarrow \exists r_1, r_2 > 0$ *such that* $(r_1+r_2)/2 = r$ *and* $A \subset S(B; r_1)$, $B \subset S(A; r_2)$ *hold.*

Proof. The relation (1) follows immediately from the definition of the function H^+.

To prove relation (2), from the inclusions $A \subset \overline{S}(B; r_1)$, $B \subset \overline{S}(A; r_2)$, it follows that

$$\forall x \in A \ \exists y_x \in B \quad \text{such that} \quad d(x, y_x) \le r_1$$

and

$$\forall y \in B \ \exists x_y \in A \quad \text{such that} \quad d(x_y, y) \le r_2.$$

From here it follows that

$$\inf_{y\in B} d(x,y) \le r_1 \quad \text{for every } x\in A \quad \text{and} \quad \inf_{x\in A} d(x,y)\le r_2 \quad \text{for every } y\in B.$$

Hence

$$\sup_{x\in A}\Big(\inf_{y\in B} d(x,y)\Big)\le r_1 \quad \text{and} \quad \sup_{y\in B}\Big(\inf_{x\in A} d(x,y)\Big)\le r_2.$$

Therefore $H^+(A,B)\le r$ where $r=(r_1+r_2)/2$.

To prove relation (3), let $H^+(A,B)=k<r$. Then there exist $k_1,k_2>0$ such that $k=(k_1+k_2)/2$ and

$$\sup_{x\in A}\Big(\inf_{y\in B} d(x,y)\Big)=k_1, \qquad \sup_{y\in B}\Big(\inf_{x\in A} d(x,y)\Big)=k_2.$$

As $0<k<r$, it follows that there exist $r_1,r_2>0$ such that $k_1<r_1$, $k_2<r_2$ and $r=(r_1+r_2)/2$. Then from the above inequalities it follows that

$$\inf_{y\in B} d(x,y)\le k_1<r_1 \quad \text{for every } x\in A$$

and

$$\inf_{x\in A} d(x,y))\le k_2<r_2 \quad \text{for every } y\in B.$$

Then, for any $x\in A$ there exists $y_x\in B$ such that

$$d(x,y_x)<\inf_{y\in B} d(x,y)+r_1-k_1\le r_1.$$

and, for any $y\in B$, there exists $x_y\in A$ such that

$$d(x_y,y)<\inf_{x\in A} d(x,y)+r_2-k_2\le r_2.$$

Hence, for any $x\in A$ and $y\in B$ it follows that

$$x\in\bigcup_{y\in B} S(y;r_1) \quad \text{and} \quad y\in\bigcup_{x\in A} S(x;r_2),$$

that is

$$A\subset S(B;r_1) \quad \text{and} \quad B\subset S(A;r_2). \qquad \square$$

Remark 2.4. From the relations (2) and (3) it follows immediately that the relations:

(2′) $A\subset S(B;r_1), B\subset S(A;r_2)\Rightarrow H^+(A,B)\le r$ where $r=(r_1+r_2)/2$, and

(3′) $H^+(A,B)<r \Rightarrow \exists r_1,r_2>0$ such that $(r_1+r_2)/2=r$ and $A\subset \overline{S}(B;r_1)$, $B\subset \overline{S}(A;r_2)$ hold.

Theorem 2.5. *If $A, B \in \mathrm{CB}(X)$, then the equalities:*

(4) $H^+(A,B) = \inf\{r > 0 : A \subset S(B;r_1),\ B \subset S(A;r_2),\ r = (r_1+r_2)/2\}$,

(4′) $H^+(A,B) = \inf\{r > 0 : A \subset \overline{S}(B;r_1),\ A \subset \overline{S}(B;r_2),\ r = (r_1+r_2)/2\}$

hold.

Proof. From the relation (2′) it follows that

$$H^+(A,B) \le \inf\{r > 0 : A \subset S(B;r_1),\ A \subset S(B;r_2),\ r = (r_1+r_2)/2\}.$$

To prove the opposite inequality, let $H^+(A,B) = k$ and let $t > 0$. Then $H^+(A,B) < k+t$. From (3) it follows that there exist $t_1, t_2 > 0$ with $(t_1+t_2)/2 = t$ such that $A \subset S(B;k+t_1)$ and $B \subset S(A;k+t_2)$. Hence

$$\{r > 0 : A \subset S(B;r_1),\ B \subset S(A;r_2)\} \supset \{k+t : t > 0,\ A \subset S(B;k+t_1),\ B \subset S(A;k+t_2)\}.$$

From this inclusion relation it follows that

$$\inf\{r > 0 : A \subset S(B;r_1), B \subset S(A;r_2)\} \le \inf\{k+t : t > 0\} = k = H^+(A,B).$$

In conclusion we have

$$H^+(A,B) = \inf\{r > 0 : A \subset S(B;r_1), B \subset S(A;r_2), r = (r_1+r_2)/2\}. \qquad \square$$

Theorem 2.6. *If the metric space (X,d) is complete, then so is* $(\mathrm{CB}(X), H^+)$ *and also* $\mathrm{C}(X)$ *is a closed subspace of* $(\mathrm{CB}(X), H^+)$.

Proof. Let (X,d) be a complete metric space and let $\{A_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{CB}(X)$. We claim that the sequence $\{A_n\}_{n\in\mathbb{N}}$ is convergent to the set $B = Ls\,A_n = \{x \in X : \forall\varepsilon > 0,\ \forall m \in \mathbb{N}\ \exists n \in \mathbb{N},\ n \ge m$ such that $S(x;\varepsilon) \cap A_n \ne \emptyset\}$.

Since the sequence $\{A_n\}_{n\in\mathbb{N}}$ is Cauchy, for any $\varepsilon > 0$ there exists $m(\varepsilon) \in \mathbb{N}$ such that

$$H^+(A_n, A_{m(\varepsilon)}) < \varepsilon \quad \text{for any } n \in \mathbb{N},\ n \ge m(\varepsilon).$$

Hence, by relation (4), it follows that there exist $\varepsilon_1, \varepsilon_2 > 0$ with $(\varepsilon_1+\varepsilon_2)/2 = \varepsilon$ and $m(\varepsilon_1), m(\varepsilon_2) \in \mathbb{N}$ such that $\min\{m(\varepsilon_1), m(\varepsilon_2)\} \ge m(\varepsilon)$, $A_n \subset S(A_{m(\varepsilon_1)};\varepsilon_1)$ for any $n \in \mathbb{N}$, $n \ge m(\varepsilon_1)$ and $A_{m(\varepsilon_2)} \subset S(A_n;\varepsilon_2)$ for any $n \in \mathbb{N}$, $n \ge m(\varepsilon_2)$.

From the properties of upper topological limit Ls it follows that $B \subset \overline{\bigcup_{k\ge n} A_k}$ for any $n \in \mathbb{N}$. Therefore $B \subset \overline{S}(A_{m(\varepsilon_1)};\varepsilon_1)$, whence the relation:

$$B \subset \overline{S}(A_{m(\varepsilon_1)};4\varepsilon_1) \tag{2.6}$$

holds. On the other hand, taking $\overline{\varepsilon}_k = \varepsilon_1/2^k$, $k \in \mathbb{N}$, it follows that there exists $n_k = m(\overline{\varepsilon}_k) \in \mathbb{N}$ such that

$$H^+(A_n, A_{n_k}) < \overline{\varepsilon}_k, \quad \text{for all } n \ge n_k.$$

Next, we choose n_k such that the sequence $\{n_k\}_{k\in\mathbb{N}}$ to be strictly increasing. Let $p \in A_{n_0} = A_{m(\varepsilon_1)}$ arbitrarily, and let there be the sequence $\{p_{n_k}\}_{k\in\mathbb{N}}$ such that $p_{n_0} = p$ and $p_{n_k} \in A_{n_k}$ with the property that $d(p_{n_k}, p_{n_{k-1}}) < \varepsilon_1/2^{k-2}$. It follows that the sequence $\{p_{n_k}\}_{k\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (X,d). Hence it is convergent to a point $l \in X$.

Since $d(p_{n_k}, p_{n_0}) < 4\,\varepsilon_1$, it follows that $d(l,p) \le 4\,\varepsilon_1$. Therefore $\inf_{y\in B} d(p,y) \le 4\,\varepsilon_1$; that is, $p \in \overline{S}(B; 4\varepsilon_1)$, which implies that:

$$A_{n_0} \subset \overline{S}(B; 4\,\varepsilon_1). \tag{2.7}$$

Keeping in view the relations (2.6) and (2.7), (3) yields $H^+(A_{n_0}, B) \le 4\,\varepsilon_1$. Taking into account the fact that H^+ is a metric on $\mathrm{CB}(X)$, we get

$$H^+(A_n, B) \le H^+(A_n, A_{n_0}) + H^+(A_{n_0}, B) < 5\,\varepsilon_1,$$

for any $n \ge m(\varepsilon_1) = n_0$. Thus, the sequence $\{A_n\}_{n\in\mathbb{N}}$ converges to $B = Ls\,A_n$; that is, $(\mathrm{CB}(X), H^+)$ is a complete metric space. This proves the first assertion of our theorem.

To prove the second assertion, we just require to show that $\mathcal{C}(X)$ is a complete subspace of $(\mathrm{CB}(X), H^+)$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}(X)$. Then, $\{A_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}B(X)$. Let $A \in CB(X)$ be such that $A = \lim_{n\to\infty} A_n$. Then for any $\varepsilon > 0$ there exists $m(\varepsilon) \in \mathbb{N}$ such that

$$H^+(A_n, A) < \frac{\varepsilon}{2} \quad \text{for all } n \ge m(\varepsilon),\ n \in \mathbb{N}.$$

Hence, by relation (4), it follows that there exist $\varepsilon_1, \varepsilon_2 > 0$ with $(\varepsilon_1 + \varepsilon_2)/2 = \varepsilon$ and $m(\varepsilon_1), m(\varepsilon_2) \in \mathbb{N}$ such that $\min\{m(\varepsilon_1), m(\varepsilon_2)\} \ge m(\varepsilon)$, $A_n \subset S(A; \varepsilon_1/2)$ for any $n \in \mathbb{N}$, $n \ge m(\varepsilon_1)$ and $A \subset S(A_n; \varepsilon_2/2)$ for any $n \in \mathbb{N}$, $n \ge m(\varepsilon_2)$.

Suppose $n_0 \ge m(\varepsilon_2)$ is a fixed natural number. Then $A \subset S(A_{n_0}; \varepsilon_2/2)$. Since A_{n_0} is compact in X, it follows that it is totally bounded. Hence there exist $x_i^{\varepsilon_2}$, $i \in \overline{1,p}$ such that $A_{n_0} \subset \bigcup_{i=1}^{p} S(x_i^{\varepsilon_2}; \varepsilon_2/2)$, whence $A \subset \bigcup_{i=1}^{p} S(x_i^{\varepsilon_2}; \varepsilon_2)$. Therefore $A \in \mathrm{C}(X)$. □

In [13], S.B. Nadler proved the following result, which he announced earlier.

Theorem 2.7. *Let (X,d) be a complete metric space and $T\colon X \to \mathrm{CB}(X)$ a multi-valued contraction mapping. Then T has a fixed point.*

In this paper, we intend to generalize this result by weakening the multi-valued contraction. Our main results are summarized in Section 3. Subsequently, in Section 4, first we introduce the concept of H^+-type nonexpansive mappings, then we extend some fixed point results of Lami Dozo [11] for H^+-type nonexpansive mappings. Finally, in Section 5, we consider a nonconvex integral inclusion and prove a Filippov type existence theorem by using an appropriate norm on the space of selection of the multifunction and a H^+-type contraction for set-valued maps.

3. Main results

Now we state and prove our main result. We begin our discussion with the following definition.

DEFINITION 3.1. Let (X,d) be a complete metric space. A multi-valued map $T : X \to \mathrm{CB}(X)$ is called H^+-contraction if

(1) there exists L in $(0,1)$ such that

$$H^+(Tx,Ty) \le L\, d(x,y) \quad \text{for every} x,y \in X,$$

(2) for every x in X, y in $T(x)$ and $\varepsilon > 0$, there exists z in $T(y)$ such that

$$d(y,z) \le H^+(T(y),T(x)) + \varepsilon.$$

Now we state and prove our main result.

THEOREM 3.2. *Every H^+-type multi-valued contraction mapping $T\colon X \to \mathrm{CB}(X)$ with Lipschitz constant $L < 1$ has a fixed point.*

PROOF. Let $\varepsilon > 0$ be given. Let $x_0 \in X$ be arbitrary. Fix an element x_1 in Tx_0. From (2) it follows that we can choose $x_2 \in Tx_1$ such that

$$d(x_1,x_2) \le H^+(Tx_0,Tx_1) + \varepsilon \tag{3.1}$$

In general, if x_n be chosen, then we choose $x_{n+1} \in Tx_n$ such that

$$d(x_n,x_{n+1}) \le H^+(Tx_{n-1},Tx_n) + \varepsilon. \tag{3.2}$$

Set $\varepsilon = (1/\sqrt{L} - 1)H^+(Tx_{n-1},Tx_n)$. Then from (3.2), it follows that

$$\begin{aligned} d(x_n,x_{n+1}) \le H^+(Tx_{n-1},Tx_n) + \left(\frac{1}{\sqrt{L}} - 1)\, H^+(Tx_{n-1},Tx_n\right) \\ = \frac{1}{\sqrt{L}} H^+(Tx_{n-1},Tx_n). \end{aligned}$$

Thus, we have

$$\sqrt{L}\, d(x_n,x_{n+1}) \le H^+(Tx_{n-1},Tx_n). \tag{3.3}$$

Now, from (1) we have

$$\sqrt{L}\, d(x_n,x_{n+1}) \le L\, d(x_{n-1},x_n) = (\sqrt{L})^2\, d(x_{n-1},x_n).$$

Hence, for all $n \in \mathbb{N}$ we have

$$d(x_n,x_{n+1}) \le \sqrt{L}\, d(x_{n-1},x_n).$$

Repeating the same argument n-times we get

$$d(x_n, x_{n+1}) \le L^{n/2}\, d(x_0, x_1).$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Since

$$\frac{1}{2}\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\} = H^+(Tx_n, Tu) \le L\, d(x_n, u),$$

it follows that

$$\liminf_{n\to\infty}\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\} = 0.$$

Since

$$\liminf_{n\to\infty} \rho(Tx_n, Tu) + \liminf_{n\to\infty} \rho(Tu, Tx_n) \le \liminf_{n\to\infty}\{\rho(Tx_n, Tu) + \rho(Tu, Tx_n)\},$$

we have

$$\liminf_{n\to\infty} \rho(Tx_n, Tu) + \liminf_{n\to\infty} \rho(Tu, Tx_n) = 0.$$

This implies that

$$\liminf_{n\to\infty} \rho(Tx_n, Tu) = 0.$$

Since $\lim_{n\to\infty} d(x_{n+1}, u) = 0$ exists, and

$$d(u, Tu) \le \rho(Tx_n, Tu) + d(x_{n+1}, u),$$

it follows that

$$\begin{aligned} d(u, Tu) &\le \liminf_{n\to\infty}[\rho(Tx_n, Tu) + d(x_{n+1}, u)] \\ &= \liminf_{n\to\infty} \rho(Tx_n, Tu) + \lim_{n\to\infty} d(x_{n+1}, u) = 0. \end{aligned}$$

This implies that $d(u, Tu) = 0$, and since Tu is closed it must be the case that $u \in Tu$. □

Remark 3.3. As $\max\{a, b\} \ge \frac{1}{2}\{a + b\}$ for all $a, b \ge 0$, it follows that multi-valued contraction (2.1) always implies multi-valued H^+-contraction but the converse implication need not be true.

To see this, we observe the following:

Example 3.4. Let $X = \{0, 1/4, 1\}$ and $d: X \times X \to \mathbb{R}$ be a standard metric. Let $T: X \to \mathrm{CB}(X)$ be such that

$$T(x) = \begin{cases} \{0\} & \text{for } x = 0, \\ \{0, 1/4\} & \text{for } x = 1/4, \\ \{0, 1\} & \text{for } x = 1. \end{cases}$$

It is routine to check that multi-valued H^+-contraction condition (1) is satisfied for all $x, y \in X$ and for any $L \in [2/3, 1)$. Further, we see that for every $x \in X$,

$y \in T(x)$ and $\varepsilon > 0$, there exists $z \in T(y)$ such that $d(y,z) \le H^+(T(y),T(x))+\varepsilon$. Indeed,

(i) if $x = 0$, $y \in T(0) = \{0\}$, $\varepsilon > 0$, there exists $z \in T(y) = \{0\}$ such that

$$0 = d(y,z) \le H^+(T(y),T(x)) + \varepsilon,$$

(iia) if $x = 1/4$, $y \in T(x) = T(1/4) = \{0, 1/4\}$, say $y = 0$, $\varepsilon > 0$, there exists $z \in T(y) = \{0\}$ such that

$$0 = d(y,z) < \frac{1}{8} + \varepsilon = H^+(T(y),T(x)) + \varepsilon,$$

(iib) if $x = 1/4$, $y \in T(x) = T(1/4) = \{0, 1/4\}$, say $y = 1/4$, $\varepsilon > 0$, there exists $z(= 1/4) \in T(y) = \{0, 1/4\}$ such that

$$0 = d(y,z) < 0 + \varepsilon = H^+(T(y),T(x)) + \varepsilon,$$

(iiia) if $x = 1$, $y \in T(x) = T(1) = \{0, 1\}$, say $y = 0$, $\varepsilon > 0$, there exists $z \in T(y) = \{0\}$ such that

$$0 = d(y,z) < \frac{1}{2} + \varepsilon = H^+(T(y),T(x)) + \varepsilon,$$

(iiib) if $x = 1$, $y \in T(x) = T(1) = \{0, 1\}$, say $y = 1$, $\varepsilon > 0$, there exists $z(= 1) \in T(y) = \{0, 1\}$ such that

$$0 = d(y,z) < 0 + \varepsilon = H^+(T(y),T(x)) + \varepsilon.$$

Thus the condition (2) is also satisfied. Clearly, 0, 1/4, 1 are fixed points of T. However, we observe that the map T does not satisfy the assumptions of Theorem 2.7. Indeed, for $x = 0$ and $y = 1$ we have

$$H(T(0),T(1)) = H(\{0\},\{0,1\}) = 1 > L\,d(0,1), \quad \text{for all } L \in (0,1).$$

EXAMPLE 3.5. Let $X = [0, 2\sqrt{2}/3] \cup \{1\}$ and $d\colon X \times X \to \mathbb{R}$ be a standard metric. Let $T\colon X \to \mathrm{CB}(X)$ be such that

$$T(x) = \begin{cases} \left[\dfrac{11x}{50(x+1)}, \dfrac{11}{50}\right] & \text{for } x \in \left[0, \dfrac{2\sqrt{2}}{3}\right], \\ \left\{\dfrac{11}{50}\right\} & \text{for } x = 1. \end{cases}$$

Set $L = 0.99$. We discuss the following cases:

Case 1. When $x, y \in [0, 2\sqrt{2}/3]$, $y > x$, we note that

$$\begin{aligned} H^+(Tx,Ty) &= \frac{11}{100} \cdot \frac{y-x}{1+x+y+xy} \\ &\le \frac{11}{100} \cdot \frac{y-x}{1+y-x} < 0.99\frac{y-x}{1+y-x} \le 0.99\,d(x,y). \end{aligned}$$

Case 2. When $x \in [0, 2\sqrt{2}/3]$ and $y = 1$, we note that

$$H^{+}(Tx, Ty) = \frac{11}{100}\left|1 - \frac{x}{1+x}\right| \leq 0.99(1-x)$$

is true if

$$\frac{11}{100} \cdot \frac{1}{1+x} \leq 0.99(1-x)$$

i.e. if $1/9 \leq 1 - x^2$, i.e. if $0 \leq x \leq 2\sqrt{2}/3$.

To check the condition (2), we consider the following cases:

Case (i). For any $x \in [0, \frac{2\sqrt{2}}{3}]$, $y \in Tx = [\frac{11x}{50(x+1)}, \frac{11}{50}]$ and $\varepsilon > 0$ there exists $z(= y) \in Ty = [\frac{11y}{50(y+1)}, \frac{11}{50}]$ such that

$$0 = d(y, z) \leq \frac{11}{100} \cdot \frac{y - x}{1 + x + y + xy} + \varepsilon = H^{+}(T(y), T(x)) + \varepsilon.$$

Note that

$$\frac{11y}{50(y+1)} \leq \frac{11y}{50} \leq y \leq \frac{11}{50}$$

i.e. $y \in Ty$.

Case (ii). For $x = 1$, $y \in Tx = \{\frac{11}{50}\}$ i.e. $y = \frac{11}{50}$ and $\varepsilon > 0$, there exists $z(= \frac{792}{6100}) \in Ty = [\frac{121}{3050}, \frac{11}{50}]$ such that

$$d(y, z) = \frac{11}{122} < \frac{11}{122} + \varepsilon = H^{+}(T(y), T(x)) + \varepsilon.$$

This proves the condition (2). Thus, all the requirements of Theorem 3.1 are satisfied and $0 \in T0$ is the unique fixed point of T. However, we note that when $y = 1$ and $x \to 2\sqrt{2}/3$ from the left, then

$$H(Tx, Ty) = \frac{11x}{50(1+x)} > 1 - x.$$

Thus, T does not satisfy the assumptions of Theorem 2.7.

Proposition 3.6. *Suppose X and $\mathrm{CB}(X)$ are as in the preceding theorem, and let $T_i: X \to \mathrm{CB}(X)$, $i = 1, 2$, be two H^{+}-type multi-valued contraction mappings with Lipschitz constant $L < 1$. Then if $\mathrm{Fix}(T_1)$ and $\mathrm{Fix}(T_2)$ denote the respective fixed point sets of T_1 and T_2,*

$$H^{+}(\mathrm{Fix}(T_1), \mathrm{Fix}(T_2)) \leq \frac{1}{1 - \sqrt{L}} \sup_{x \in X} H^{+}(T_1 x, T_2 x).$$

Proof. Let $\varepsilon > 0$ be given. Select $x_0 \in Fix(T_1)$, and then select $x_1 \in T_2 x_0$. From (2) it follows that we can choose $x_2 \in T_2 x_1$ such that

$$d(x_1, x_2) \leq H^{+}(T_2 x_0, T_2 x_1) + \varepsilon.$$

Now define $\{x_n\}$ inductively so that $x_{n+1} \in T_2(x_n)$ and

$$d(x_n, x_{n+1}) \leq H^{+}(T_2 x_{n-1}, T_2 x_n) + \varepsilon. \tag{3.4}$$

Set $\varepsilon = \big(\frac{1}{\sqrt{L}} - 1\big) H^+(T_2x_{n-1}, T_2x_n)$. Then from (3.4), it follows that

$$d(x_n, x_{n+1}) \le H^+(T_2x_{n-1}, T_2x_n) + \left(\frac{1}{\sqrt{L}} - 1)\, H^+(T_2x_{n-1}, T_2x_n\right)$$
$$= \frac{1}{\sqrt{L}} H^+(T_2x_{n-1}, T_2x_n).$$

Thus, we have

$$\sqrt{L}\, d(x_n, x_{n+1}) \le H^+(T_2x_{n-1}, T_2x_n). \tag{3.5}$$

Now applying (1) for T_2 we have

$$\sqrt{L}\, d(x_n, x_{n+1}) \le L\, d(x_{n-1}, x_n) = (\sqrt{L})^2\, d(x_{n-1}, x_n).$$

Hence, for all $n \in \mathbb{N}$ we have

$$d(x_n, x_{n+1}) \le \sqrt{L}\, d(x_{n-1}, x_n).$$

Repeating the same argument n-times we get

$$d(x_n, x_{n+1}) \le L^{n/2}\, d(x_0, x_1).$$

This implies that $\{x_n\}$ is a Cauchy sequence with limit, say z. Since T_2 is continuous, we have

$$\lim_{n\to\infty} H(T_2x_n, T_2z) = 0.$$

Also, since $x_{n+1} \in T_2(x_n)$ it must be the case that $z \in T_2z$; that is, $z \in \mathrm{Fix}(T_2)$. Furthermore, using (3.5) we have

$$d(x_0, z) \le \sum_{n=0}^{\infty} d(x_{n+1}, x_n) \le (1 + \sqrt{L} + (\sqrt{L})^2 + \dots)d(x_1, x_0)$$
$$\le \frac{1}{1-\sqrt{L}}\, (H^+(T_2x_0, T_1x_0) + \varepsilon).$$

Reversing the roles of T_1 and T_2 and repeating the argument as above leads to the conclusion that, for each $y_0 \in \mathrm{Fix}(T_2)$, there exist $y_1 \in T_1y_0$ and $w \in \mathrm{Fix}(T_1)$ such that

$$d(y_0, w) \le \frac{1}{1-\sqrt{L}}\, (H^+(T_1y_0, T_2y_0) + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows. □

THEOREM 3.7. *Suppose X and $\mathrm{CB}(X)$ are as in the preceding theorem, and let $T_i\colon X \to \mathrm{CB}(X)$, $i = 1, 2, \dots$ be a sequence of H^+-type multi-valued contraction mappings with Lipschitz constant $L < 1$. If $\lim_{n\to\infty} H^+(T_nx, T_0x) = 0$ uniformly for $x \in X$, then*

$$\lim_{n\to\infty} H^+(\mathrm{Fix}(T_n), \mathrm{Fix}(T_0)) = 0.$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} H^+(T_n x, T_0 x) = 0$ uniformly for $x \in X$, it is possible to choose $N \in \mathbb{N}$, so that for $n \geq N$,

$$\sup_{x\in X} H^+(T_n x, T_0 x) < (1 - \sqrt{L})\varepsilon.$$

By Proposition 3.6, $H^+(\mathrm{Fix}(T_n), \mathrm{Fix}(T_0)) < \varepsilon$ for all $n \geq N$. Hence the conclusion follows. $\square$

4. H^+-type nonexpansive mappings

In this section, first we introduce the class of H^+-type nonexpansive mappings. Then we apply the main result of preceding section to obtain fixed points of H^+-type nonexpansive mappings in its natural terrain; i.e. Banach space satisfying Opial's condition.

Definition 4.1. Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $T\colon X \to \mathcal{C}B(X)$ is called H^+-nonexpansive if

(1′) $H^+(Tx, Ty) \leq \|x - y\|$ for every $x, y \in X$,
(2′) for every $x \in X$, $y \in T(x)$ and $\varepsilon > 0$, there exists $z \in T(y)$ such that

$$\|y - z\| \leq H^+(T(y), T(x)) + \varepsilon.$$

In the following K is a nonempty convex weakly compact subset of a Banach space X. X is said to satisfy Opial's condition if for each x_0 in X and each sequence $\{x_n\}$ converging weakly to x_0 (i.e. $x_n \rightharpoonup x_0$), the inequality

$$\liminf_{n\to\infty} \|x_n - x\| > \liminf_{n\to\infty} \|x_n - x_0\|$$

holds for all $x \neq x_0$.

We will say that a mapping $T\colon X \to 2^X$ is *demiclosed* if

$$x_n \rightharpoonup x \quad \text{and} \quad y_n \in Tx_n \to y \Rightarrow y \in Tx.$$

Proposition 4.2. *Let $T\colon K \to \mathrm{C}(X)$ be H^+-type multi-valued nonexpansive mapping and let X satisfy Opial's condition. Then $I - T$ is demiclosed.*

Proof. Since the domain of $I - T$ is weakly compact it is enough to prove that the graph of $I - T$ is sequentially closed. Let $(x_n, y_n) \in \mathrm{G}(I - T)$ where $\mathrm{G}(I - T)$ denotes the graph of $I - T$ such that

$$x_n \rightharpoonup x \quad \text{and} \quad y_n \to y.$$

Then $x \in K$ and we have to prove that $y \in (I - T)x$. Since $y_n \in x_n - Tx_n$, $y_n = x_n - z_n$ for some $z_n \in Tx_n$.

By (2′), for $z_n \in Tx_n$ and $\varepsilon > 0$, we can choose $z_n' \in Tx$ such that

$$\|z_n - z_n'\| \leq H^+(Tx_n, Tx) + \varepsilon.$$

Since T is nonexpansive, the above inequality yields

$$\|z_n - z'_n\| \le \|x_n - x\| + \varepsilon. \tag{4.1}$$

Since $\varepsilon > 0$ is arbitrary, so on letting $\varepsilon \to 0$ and taking $\liminf$ on both sides of (4.1), we have

$$\liminf_{n\to\infty} \|x_n - x\| \ge \liminf_{n\to\infty} |z_n - z'_n\| \ge \liminf_{n\to\infty} \|x_n - y_n - z'_n\|. \tag{4.2}$$

But Tx is compact and $y_n \to y$. Hence there exists a subsequence of $\{z'_n\}$, again denoted by $\{z'_n\}$, converging to $z \in Tx$. Hence, from (4.2) we get

$$\liminf_{n\to\infty} \|x_n - x\| \ge \liminf_{n\to\infty} \|x_n - y - z\|. \tag{4.3}$$

By Opial's condition we have $y + z = x$. Thus $y = x - z \in x - Tx$. □

Let K be a nonempty convex subset of a Banach space X. Let $T : K \to \mathcal{C}(X)$ be a multi-valued mapping. For a fixed $x_0 \in K$ and any $x \in K$, we define the segment $[x, x_0]$ by $[x, x_0] = \{y \in K : y = \lambda x + (1 - \lambda)x_0, 0 \le \lambda \le 1\}$. We call T to be x_0-redundant if $Ty = Tx$ for all $y \in [x, x_0]$.

Theorem 4.3. *Let X be a Banach space which satisfies Opial's condition, K is a nonempty convex weakly compact subset of X and let $T: K \to \mathrm{C}(K)$ be a H^+-type multi-valued nonexpansive mapping. If there exists $x_0 \in K$ such that T is x_0-redundant, then T has a fixed point in K.*

Proof. Let $\{k_n\}$ be a sequence of real numbers such that $0 < k_n < 1$ and $k_n \to 1$ as $n \to \infty$. Define

$$T_n x = k_n T x + (1 - k_n)x_0 \quad \text{for all } x \in K \text{ and } n \in \mathbb{N}. \tag{4.4}$$

By Proposition 2.2 (i) and (ii), for any $x, y \in K$ and $n \in \mathbb{N}$ we have

$$H^+(T_n(x)), T_n(y)) = k_n H^+(T(x), T(y)) \le k_n\|x - y\|.$$

Now let $\varepsilon > 0$ be given. By$(2')$, corresponding to any y in $T(x)$ i.e. in turn, for any $y' = k_n y + (1-k_n)x_0$ in $T_n(x)$, there exists $z \in T(y)$ and, in turn, there exists $z' = k_n z + (1 - k_n)x_0$ in $T_n(y)$, and hence, in $T_n(y') = k_n T(y') + (1 - k_n)x_0 = k_n T(k_n y + (1 - k_n)x_0) + (1 - k_n)x_0 = k_n T(y) + (1 - k_n)x_0 = T_n(y)$ such that

$$\|y - z\| \le H^+(T(y), T(x)) + \varepsilon.$$

Thus, for all $n \in \mathbb{N}$, this yields

$$\begin{aligned} \|y' - z'\| &= k_n\|y - z\| \le k_n(H^+(T(y), T(x)) + \varepsilon) \\ &= H^+(T_n(y), T_n(x)) + k_n\varepsilon < H^+(T_n(y'), T_n(x)) + \varepsilon. \end{aligned}$$

Hence T_n is a H^+-type multivalued k_n-contraction mapping for all $n \in \mathbb{N}$. Also, since K is a complete metric space, therefore it follows from Theorem 3.2, that for each $n \in \mathbb{N}$, there exists $x_n \in K$ such that $x_n \in T_n(x_n)$. Since K is weakly compact, there exists a subsequence of $\{x_n\}$, again denoted by $\{x_n\}$, converging weakly to $x \in K$. From (4.4), there exists $z_n \in Tx_n$ such that

$$x_n = k_n z_n + (1 - k_n)x_0 \quad \text{for all } n \in \mathbb{N}.$$

It then follows that

$$\|x_n - z_n\| = (1 - k_n)\|x_0 - z_n\|.$$

Hence $y_n = x_n - z_n \in (I - T)x_n$ and $y_n \to 0$ as $n \to \infty$. This means that $(x_n, y_n) \in \mathrm{G}(I - T)$ and $x_n \rightharpoonup x$, $y_n \to 0$. So by demiclosedness of $(I - T), 0 \in (I - T)x$ i.e. $x \in Tx$. $\square$

5. Existence theorem for nonconvex integral inclusions

In this section, we shall consider a nonconvex integral inclusion and prove a Filippov type existence theorem by using an appropriate norm on the space of selection of the multifunction and a H^+-type contraction for set-valued maps.

Let $I := [0, T]$, $T > 0$ and $\mathcal{L}(I)$ denote the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with the norm $\|\cdot\|$. Let $\mathcal{P}(X)$ denote the family of all nonempty subsets of X and $\mathcal{B}(X)$ the family of all Borel subsets of X.

Throughout this section, let $\mathrm{C}(I, X)$ denote the Banach space of all continuous functions $x(\,\cdot\,)\colon I \to X$ endowed with the norm $\|x(\,\cdot\,)\|_C = \sup_{t \in I} \|x(t)\|$. Consider the following integral inclusion

$$x(t) = \lambda(t) + \int_0^t [a(t, s)\, g(t, u(s)) + f(t, s, u(s))]\, ds, \tag{5.1}$$

$$u(t) \in F(t, V(x)(t)) \quad \text{a.e. } (I := [0, T]), \tag{5.2}$$

where $\lambda(\,\cdot\,)\colon I \to X$, $g(\,\cdot\,,\,\cdot\,)\colon I \times X \to X$, $F(\,\cdot\,,\,\cdot\,)\colon I \times X \to \mathcal{P}(X)$, $f(\,\cdot\,,\,\cdot\,,\,\cdot\,)\colon I \times I \times X \to X$, $V\colon \mathrm{C}(I, X) \to \mathrm{C}(I, X)$, $a(\,\cdot\,,\,\cdot\,)\colon I \times I \to \mathbb{R} = (-\infty, \infty)$ are given mappings. In the sequel, we also use the following: For any $x \in X$, $\lambda \in C(I, X), \sigma \in L^1(I, E)$, we define the set-valued maps $M_{\lambda,\sigma}(t) := F(t, V(x_{\sigma,\lambda})(t))$, $t \in I, T_\lambda(\sigma) := \{\psi(\cdot) \in L^1(I, E) : \psi(t) \in M_{\lambda,\sigma}(t) \ \ a.e.\ (I)\}$.

In order to study problem (5.1)–(5.2) we introduce the following assumption.

Hypothesis 5.1. *Let $F(\,\cdot\,,\,\cdot\,)\colon I \times X \to \mathcal{P}(X)$ be a set-valued map with nonempty closed values that verify*:

- (H$_1$) *The set-valued map $F(\,\cdot\,,\,\cdot\,)$ is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.*
- (H$_2$) *There exists $L(\,\cdot\,) \in L^1(I, \mathbb{R}_+)$ such that, for almost all $t \in I, F(t,\,\cdot\,)$ is $L(t)$-Lipschitz in the sense that*:

$$H^+(F(t, x), F(t, y)) \le L(t)\, \|x - y\| \quad \textit{for all } x, y \in X, \tag{C1}$$

and for any $x, y \in X$, $w \in F(t,x)$ *and* $\varepsilon > 0$, *there exists* $z \in F(t,y)$ *such that*:

$$\|w - z\| \le H^+(F(t,x), F(t,y)) + \varepsilon \tag{C2}$$

and $T_\lambda(\,\cdot\,)$ *satisfies the condition: For any* $\sigma \in L^1(I,E)$, $\sigma_1 \in T_\lambda(\sigma)$ *and any given* $\varepsilon > 0$ *there exists* $\sigma_2 \in T_\lambda(\sigma_1)$ *such that*:

$$\|\sigma_1 - \sigma_2\|_1 \le H^+(T_\lambda(\sigma), T_\lambda(\sigma_1)) + \varepsilon \quad \text{for almost all } t \in I. \tag{C3}$$

(H_3) *The mappings* $f: I \times I \times X \to X$, $g, \lambda: I \times X \to X$ *are continuous,* $V: \mathrm{C}(I,X) \to \mathrm{C}(I,X)$ *and there exist the constants* $M_1, M_2, M_3 > 0$ *such that*:

$$\begin{aligned}
\|f(t,s,u_1) - f(t,s,u_2)\| &\le M_1\|u_1 - u_2\|, && \text{for all } u_1, u_2 \in X,\\
\|g(s,u_1) - g(s,u_2)\| &\le M_2\|u_1 - u_2\|, && \text{for all } u_1, u_2 \in X,\\
\|V(x_1)(t) - V(x_2)(t)\| &\le M_3\|x_1(t) - x_2(t)\|, && \text{for all } t \in I,\\
& && \text{and all } x_1, x_2 \in \mathrm{C}(I,X).
\end{aligned}$$

(H_4) *Let* $a: I \times I \to \mathbb{R}$ *be continuous and satisfy the uniform Hölder's continuity condition in the first and second arguments with the exponent* ρ; *i.e. there exists a positive number* b *such that*

$$|a(t_1,s_1) - a(t_2,s_2)| \le b(|t_1 - t_2|^\rho + |s_1 - s_2|^\rho)$$

for all $t_1, t_2, s_1, s_2 \in I$ *and* $|a(t,s)| \le 2bT + |a(0,0)| = M_4$ *for all* $t, s \in I$ *and* $0 < \rho \le 1$.

Note that the system (5.1)–(5.2) includes a large variety of differential inclusions and control systems including those defined by partial differential equations.

Assume that U be an open bounded subset of $\mathbb{R}^n$ (or Y, a subset of X homeomorphic to $\mathbb{R}^n$) and $U_T = U \times (0,T]$ for some fixed $T > 0$. We say that the partial differential operator $\frac{\partial}{\partial t} + L$ is parabolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x,t)\xi_i\xi_j \ge \theta|\xi|^2$$

for all $(x,t) \in U_T$, $\xi \in \mathbb{R}^n$. The letter L denotes for each time t a second order partial differential operator, having either the divergence form

$$Lu = -\sum_{i,j=1}^n (a^{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^n b^i(x,t)u_{x_i} + \mathrm{C}(x,t)u$$

or else the nondivergence form

$$Lu = -\sum_{i,j=1}^n a^{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^n b^i(x,t)u_{x_i} + \mathrm{C}(x,t)u,$$

for given coefficients a^{ij}, b^i, c $(i, j = 1, \dots, n)$.

A family$\{G(t) : t \in \mathbb{R}_+ = [0, \infty)\}$ of bounded linear operators from X into X is a C_0-semigroup (also called linear semigroup of class (C_0)) on X if

(i) $G(0) =$ the identity operator, and $G(t + s) = G(t)G(s)$ for all

(ii) $G(\,\cdot\,)$ is strongly continuous in $t \in \mathbb{R}_+$;

(iii) $\|G(t)\| \le Me^{\omega t}$ for some $M > 0$, real ω and $t \in \mathbb{R}_+$.

Example 5.2. Set $f(t, \tau, u) = G(t - \tau)u$, $g(\tau, u(\tau)) = 0$, $V(x) = x$, $\lambda(t) = G(t)x_0$ where $\{G(t)\}_{t \ge 0}$ is a C_0-semigroup with an infinitesimal generator$A$. Then a solution of system (5.1)–(5.2) represents a mild solution of

$$x'(t) \in Ax(t) + F(t, x(t)), \quad x(0) = x_0. \tag{5.3}$$

In particular, this problem includes control systems governed by parabolic partial differential equations as a special case. When $A = 0$, the relation (5.3) reduces to classical differential inclusions

$$x'(t) \in F(t, x(t)), \quad x(0) = x_0. \tag{5.4}$$

Denote

$$\Phi(u)(t) = \int_0^t [a(t, \tau)g(\tau, u(\tau)) + f(t, \tau, u(\tau))]\, d\tau, \quad t \in I. \tag{5.5}$$

Then the integral inclusion system (5.1)–(5.2) reduces to the form

$$x(t) = \lambda(t) + \Phi(u)(t), u(t) \in F(t, V(x)(t)) \quad \text{a.e. } (I), \tag{5.6}$$

which may be written in more compact form as

$$u(t) \in F(t, V(\lambda + \Phi(u))(t)) \quad \text{a.e. } (I).$$

Now we recall the following:

Definition 5.3. A pair of functions (x, u) is called a solution pair of integral inclusion system (5.6), if $x(\,\cdot\,) \in \mathrm{C}(I, X)$, $u(\,\cdot\,) \in L^1(I, X)$ and satisfy relation (5.6).

For our further discussion, we denote by $S(\lambda)$ the solution set of (5.1)–(5.2).

Notice that the integral operator in (5.5) plays a key role in the proofs of our main results.

For given $\alpha \in \mathbb{R}$ we denote by $L^1(I, X)$ the Banach space of all Bochner integrable functions $u(\,\cdot\,): I \to X$ endowed with the norm

$$\|u(\,\cdot\,)\|_1 = \int_0^T e^{-\alpha(M_4M_2+M_1)M_3m(t)}\|u(t)\|\, dt,$$

where $m(t) = \int_0^t L(s)\, ds$, $t \in I$.

Theorem 5.4. *Let Hypothesis* 5.1 *be satisfied,* $\lambda(\,\cdot\,,\,\cdot\,), \mu(\,\cdot\,,\,\cdot\,) \in \mathrm{C}(I\times X, X)$ *and let* $u(\,\cdot\,) \in L^1(I,X)$ *be such that*

$$d(v(t), F(t, V(y)(t)) \leq p(t) \quad a.e.\ (I),$$

where $p(\,\cdot\,) \in L^1(I,\mathbb{R}_+)$ *and* $y(t) = \mu(t,u(t)) + \Phi(u)(t)$*, for all* $t \in I$*. Then for every* $\alpha > 1$*,* $0 < h < 1$*, there exist* $\nu \in \mathbb{N}$ *and* $x(\,\cdot\,) \in S(\lambda)$ *such that, for every* $t \in I$*,*

$$\begin{aligned}\|x(t) - y(t)\| &\leq \|\lambda - \mu\|_C \left[1 + \frac{e^{\alpha(M_4M_2+M_1)M_3m(T)}}{\sqrt{\alpha}(\sqrt{\alpha}-1)}\right] \\ &+ \frac{\sqrt{\alpha}}{(\sqrt{\alpha}-1)}(M_4M_2+M_1)e^{\alpha(M_4M_2+M_1)M_3m(T)}\int_0^T e^{-\alpha(M_4M_2+M_1)M_3m(t)}p(t)\,dt.\end{aligned}$$

Proof. For $\lambda \in \mathrm{C}(I,X)$ and $u \in L^1(I,X)$ define

$$x_{u,\lambda}(t) = \lambda(t) + \int_0^t [a(t,s)\,g(t,u(s)) + f(t,s,u(s))]\,ds.$$

Let us consider that $\lambda \in \mathrm{C}(I,X)$, $\sigma \in L^1(I,X)$ and define the set-valued maps

$$M_{\lambda,\sigma}(t) := F(t, V(x_{\sigma,\lambda})(t)), \quad t \in I, \tag{5.7}$$

$$T_\lambda(\sigma) := \{\psi(\,\cdot\,) \in L^1(I,X) : \psi(t) \in M_{\lambda,\ \sigma}(t) \text{ a.e. } (I)\}. \tag{5.8}$$

Further, in view of condition (C3) of Hypothesis 5.1(H_2), $T_\lambda(\,\cdot\,)$ satisfies the condition: For any $\sigma \in L^p(I,E)$, $\sigma_1 \in T_\lambda(\sigma)$ and any given $\varepsilon > 0$ there exists $\sigma_2 \in T_\lambda(\sigma_1)$ such that

$$\|\sigma_1 - \sigma_2\|_1 \leq H^+(T_\lambda(\sigma), T_\lambda(\sigma_1)) + \varepsilon. \tag{5.9}$$

Now we claim that $T_\lambda(\sigma)$ is nonempty and closed for every $\sigma \in L^1(I,X)$.

The set-valued map $M_{\lambda,\sigma}(\,\cdot\,)$ is measurable. For example the map $t \to F(t, V(x_{\sigma,\lambda})(t)$ can be approximated by step functions and so we can apply Theorem III.40 in [2]. Since the values of F are closed, with the measurable selection theorem we infer that $M_{\lambda,\sigma}(\,\cdot\,)$ is nonempty.

Also, the set $T_\lambda(\sigma)$ is closed. Indeed, if $\psi_n \in T_\lambda(\,\cdot\,)$ and $\|\psi_n - \psi\|_1 \to 0$, then there exists a subsequence ψ_{n_k} such that $\psi_{n_k}(t) \to \psi(t)$ for almost every $t \in I$ and we find that $\psi \in T_\lambda(\sigma)$.

Let $\sigma_1, \sigma_2 \in L^1(I,X)$ be given. Let $\psi_1 \in T_\lambda(\sigma_1)$ and let $\delta > 0$. Consider the following set-valued map:

$$\begin{aligned}&\mathcal{G}(t) := M_{\lambda,\sigma_2}(t) \\ &\cap \left\{z \in X : \|\psi_1(t) - z\| \leq M_3(M_4M_2+M_1)L(t)\int_0^t \|\sigma_1(s) - \sigma_2(s)\|\,ds + \delta\right\}.\end{aligned}$$

Then

$$\begin{aligned}
d(\psi_1, M_{\lambda,\sigma_2}(t)) &\le \rho(F(t, V(x_{\sigma_1,\lambda})(t)), F(t, V(x_{\sigma_2,\lambda})(t))) + \varepsilon \\
&\le L(t)\|V(x_{\sigma_1,\lambda})(t)) - V(x_{\sigma_2,\lambda})(t))\| + \varepsilon \\
&\le M_3 L(t)\|x_{\sigma_1,\lambda}(t) - x_{\sigma_2,\lambda}(t)\| + \varepsilon \\
&\le M_3 L(t)\bigg[\int_0^t |a(t,s)|\|g(t,\sigma_1(s)) - g(t,\sigma_2(s))\|\,ds \\
&\qquad + \int_0^t \|f(t,s,\sigma_1(s)) - f(t,s,\sigma_2(s))\|ds\bigg] + \varepsilon \\
&\le M_3 L(t)\bigg[(M_4 M_2 + M_1)\int_0^t \|\sigma_1(s) - \sigma_2(s)\|ds\bigg] + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, letting $\varepsilon \to 0$, we have that $\mathcal{G}(\cdot)$ is nonempty bounded and has closed values. Further, by Proposition III.4 in [2], $\mathcal{G}(\cdot)$ is measurable.

Let $\psi_2(\,\cdot\,)$ be a measurable selector of $\mathcal{G}(\,\cdot\,)$. It follows that $\psi_2 \in T_\lambda(\sigma_2)$ and

$$\begin{aligned}
\|\psi_1 - \psi_2\|_1 &= \int_0^T e^{-\alpha(M_4M_2+M_1)M_3m(t)}\|\psi_1(t) - \psi_2(t)\,\|dt \\
&\le \int_0^T e^{-\alpha(M_4M_2+M_1)M_3m(t)} M_3 L(t)\bigg[(M_4M_2+M_1)\int_0^t \|\sigma_1(s) - \sigma_2(s)\|\,ds\bigg]\,dt \\
&\quad + \delta\int_0^T e^{-\alpha(M_4M_2+M_1)M_3m(t)}\,dt \\
&\le \frac{1}{\alpha}\|\sigma_1 - \sigma_2\|_1 + \delta\int_0^T e^{-\alpha(M_4M_2+M_1)M_3m(t)}\,dt.
\end{aligned}$$

Since δ is arbitrary, so letting $\delta \to 0$ we deduce from the above inequality that

$$d(\psi_1, T_\lambda(\sigma_2)) \le \frac{1}{\alpha}\|\sigma_1 - \sigma_2\|_1.$$

Thus, we have

$$\rho(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) = \sup_{\psi_1 \in T_\lambda(\sigma_1)} d(\psi_1, T_\lambda(\sigma_2)) \le \frac{1}{\alpha}\|\sigma_1 - \sigma_2\|_1. \tag{5.10}$$

Now replacing $\sigma_1(\,\cdot\,)$ with $\sigma_2(\,\cdot\,)$, we obtain

$$H^+(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) \le \frac{1}{\alpha}\|\sigma_1 - \sigma_2\|_1. \tag{5.11}$$

Now adding (5.10) and (5.11) and dividing by 2, we obtain

$$H^+(T_\lambda(\sigma_1), T_\lambda(\sigma_2)) \le \frac{1}{\alpha}\|\sigma_1 - \sigma_2\|_1.$$

Hence we conclude that $T_\lambda(\,\cdot\,)$ is a contraction on $L^1(I,X)$. Next, we consider the following set-valued maps

$$\begin{aligned}
\widetilde{F}(t,x) &:= F(t,x) + p(t),\\
\widetilde{M}_{\lambda,\sigma}(t) &:= \widetilde{F}(t, V(x_{\sigma,\lambda})(t)), \quad t \in I,\\
\widetilde{T}_\lambda(\sigma) &:= \{\psi(\,\cdot\,) \in L^1(I,X);\, \psi(t) \in \widetilde{M}_{\lambda,\sigma}(t) \text{ a.e. } (I)\}.
\end{aligned}$$

It is obvious that $\widetilde{F}(\,\cdot\,,\,\cdot\,)$ satisfies Hypothesis 5.1.

Let $\phi \in T_\lambda(\sigma)$, $\delta > 0$ and define

$$\mathcal{G}_1(t) := \widetilde{M}_{\lambda,\sigma(t)} \cap \{z \in X : \|\phi(t) - z\| \le M_3 L(t)\|\lambda - \mu\|_C + p(t) + \delta\}.$$

Using the same argument as used for the set valued map $\mathcal{G}(\,\cdot\,)$, we deduce that $\mathcal{G}_1(\,\cdot\,)$ is measurable with nonempty closed values.

Next, we prove the following estimate:

$$\begin{aligned}
(5.12)\quad H^+(T_\lambda(\sigma), \widetilde{T}_\mu(\sigma)) \le \frac{1}{\alpha(M_4M_2 + M_1)}&\|\lambda - \mu\|_C \\
&+ \int_0^T e^{-\alpha(M_4M_2+M_1)M_3 m(t)} p(t)\,dt.
\end{aligned}$$

Let $\psi(\,\cdot\,) \in T_\mu(\sigma)$. Then

$$\begin{aligned}
\|\phi - \psi\|_1 &\le \int_0^T e^{-\alpha(M_4M_2+M_1)M_3 m(t)} \|\phi(t) - \psi(t)\|\,dt\\
&\le \int_0^T e^{-\alpha(M_4M_2+M_1)M_3 m(t)} [M_3 L(t)\|\lambda - \mu\|_C + p(t) + \delta]\,dt\\
&= \|\lambda - \mu\|_C \int_0^T e^{-\alpha(M_4M_2+M_1)M_3 m(t)} M_3 L(t)\,dt\\
&\quad + \int_0^T e^{-\alpha(M_4M_2+M_1)M_3 m(t)} p(t)\,dt + \delta \int_0^T e^{-\alpha(M_4M_2+M_1)M_3 m(t)}\,dt\\
&\le \frac{1}{\alpha(M_4M_2+M_1)}\|\lambda - \mu\|_C + \int_0^T e^{-\alpha(M_4M_2+M_1)M_3 m(t)} p(t)\,dt\\
&\quad + \delta \int_0^T e^{-\alpha(M_4M_2+M_1)M_3 m(t)}\,dt.
\end{aligned}$$

As δ is arbitrary, we obtain (5.12).

Now applying Proposition 3.6 we obtain

$$\begin{aligned}
H^+(\mathrm{Fix}(T_\lambda), \mathrm{Fix}(\widetilde{T}_\mu)) \le \frac{1}{\sqrt{\alpha}(\sqrt{\alpha} - 1)(M_4M_2 + M_1)}&\|\lambda - \mu\|_C\\
&+ \frac{\sqrt{\alpha}}{\sqrt{\alpha} - 1} \int_0^T e^{-\alpha(M_4M_2+M_1)M_3 m(t)} p(t)\,dt.
\end{aligned}$$

Since $v(\,\cdot\,) \in \mathrm{Fix}(\widetilde{T}_\mu)$, it follows that there exists $\nu \in \mathbb{N}$ and $u(\,\cdot\,) \in \mathrm{Fix}(T_\mu)$ such that

$$
\begin{aligned}
\|v-u\|_1 \le{} & \frac{1}{\sqrt{\alpha}(\sqrt{\alpha}-1)(M_4M_2+M_1)}\|\lambda-\mu\|_C \\
& + \frac{\sqrt{\alpha}}{\sqrt{\alpha}-1}\int_0^T e^{-\alpha(M_4M_2+M_1)M_3m(t)}p(t)\,dt.
\end{aligned}
\tag{5.13}
$$

We define

$$
x = \lambda(t) + \int_0^t [a(t,s)\,g(t,u(s)) + f(t,s,u(s))]\,ds.
$$

Then one has the following inequality:

$$
\begin{aligned}
\|x(t)-y(t)\| &\le \|\lambda(t)-\mu(t)\| + (M_4M_2+M_1)\int_0^t \|u(s)-v(s)\|\,ds \\
&\le \|\lambda-\mu\|_C + (M_4M_2+M_1)e^{\alpha(M_4M_2+M_1)M_3m(T)}\|u-v\|_1.
\end{aligned}
$$

Combining the last inequality with (5.13), we obtain

$$
\begin{aligned}
\|x(t)-y(t)\| \le{} & \|\lambda-\mu\|_C\bigg[1+\frac{e^{\alpha(M_4M_2+M_1)M_3m(T)}}{\sqrt{\alpha}(\sqrt{\alpha}-1)}\bigg] \\
& + \frac{\sqrt{\alpha}}{(\sqrt{\alpha}-1)}(M_4M_2+M_1)e^{\alpha(M_4M_2+M_1)M_3m(T)}\int_0^T e^{-\alpha(M_4M_2+M_1)M_3m(t)}p(t)\,dt.
\end{aligned}
$$

This completes the proof. □

Remarks 5.5. (a) If $a(t,\tau) \equiv 0$, Theorem 5.4 complements the result in [4] obtained for mild solutions of the semilinear differential inclusion (5.3).

(b) If $a(t,\tau) = 0$, $f(t,\tau,u) = \mathcal{G}(t-\tau)u$, $V(x) = x$, $\lambda(t) = \mathcal{G}(t)x_0$ where $\{\mathcal{G}(t)\}_{t\ge 0}$ is a C_0-semigroup with an infinitesimal generator A, Theorem 5.4 complements the result in [3] obtained for mild solutions of the semilinear differential inclusion (5.3).

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