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GRADIENT-LIKE NONLINEAR SEMIGROUPS WITH INFINITELY MANY EQUILIBRIA AND APPLICATIONS TO CASCADE SYSTEMS

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Abstract. We consider an autonomous dynamical system coming from a coupled system in cascade where the uncoupled part of the system satisfies that the solutions comes from $-\infty$ and goes to ∞ to equilibrium points, and where the coupled part generates asymptotically a gradient-like nonlinear semigroup. Then, the complete model is proved to be also gradient-like. The interest of this extension comes, for instance, in models where a continuum of equilibrium points holds, and for example a Łojasiewicz–Simon condition is satisfied. Indeed, we illustrate the usefulness of the theory with several examples.

1. Introduction

The study of the structure of invariant sets for infinite-dimensional dynamical systems and its characterization has received a lot of attention in the last

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few decades. Actually, while the finite-dimensional case has been deeply understood, the same goal for the infinite-dimensional case used to be reduced to very particular examples, for instance, a gradient (or gradient-like) structure in the equations (or directly in the attractor), e.g. see [13], [21], [24] and the references therein.

A deeper understanding of structure of attractors for autonomous and non-autonomous dynamical systems that generalize the above comes by some recent developments in [5]–[7] for dynamical systems where the equilibrium points are hyperbolic, and therefore there exists a finite number of them. It involves a new class, called of gradient-like nonlinear semigroups, which roughly speaking means that all complete trajectories in the attractor come from and go to equilibrium points (in particular the attractor is gradient-like) and there is no homoclinic structure among these points.

However, in many situations, the set of equilibrium points is not finite but indeed they form a continuum and the above results cannot be applied. Nevertheless, in that context a new tool can be employed: the study of the asymptotic behaviour in the finite-dimensional setting of a gradient system associated to an analytic function in [15], [16] lead in [22] to the study of the asymptotic behaviour of semilinear parabolic equations when an additional condition, called Łojasiewicz–Simon's inequality, holds. That condition ensures that any solution tends as $t \to \infty$ to a unique stationary solution (for instance see also [10], [25], [14] and the references therein).

Our main goal in this paper is to analyse the behaviour of a coupled system in cascade where the uncoupled part of the system satisfies that the solutions comes from $-\infty$ and goes to ∞ to equilibrium points, and where the coupled part generates asymptotically a gradient-like nonlinear semigroup. This leads to a new gradient-like nonlinear semigroup concept, and in particular allows to describe the attractor of the system as the union of unstable manifolds of all equilibria. The interest of this extension comes for instance in models where a continuum of equilibrium points holds, and for example a Łojasiewicz–Simon condition is satisfied. However, this condition has been applied to the study of the behaviour of a solution only when $t \to \infty$. We present an extension to deal also with the case $t \to -\infty$.

The convergence of solutions to equilibrium can also be proved assuming that the manifold of equilibria is normally hyperbolic (see [19, Theorem 6.1] or [20] for the case when the set of equilibria is normally stable). For this, instead of normal hyperbolicity, here we use the Łojasiewicz–Simon condition.

The structure of the paper is as follows. In Section 2 we state the problem. In Section 3 we give our main result (cf. Theorem 3.4); that is, we give conditions ensuring that the coupled system generates a (new type of) gradient-like nonlinear semigroup, and therefore having a gradient-like attractor. In Section 4 we study the convergence to equilibria of any solution in the attractor, not only when $t \to \infty$ but also for $t \to -\infty$ using the Łojasiewicz–Simon inequality (its proof is revised here, cf. Theorem 4.1). Problems of first and second order in time are analysed. In Section 5 we provide several examples to illustrate the applicability of the theory. Finally, in the Appendix we prove the backwards convergence to equilibrium stated in Section 4.

2. Basic facts and notions

In this section we introduce the basic facts and notions that are needed to state and prove our main results.

Let Z be a metric space with metric $d: Z \times Z \to \mathbb{R}^+$, where $\mathbb{R}^+ := [0, \infty)$. Given a subset $A \subset Z$, the ε-neighbourhood of A is the set

$$\mathcal{O}_\varepsilon(A) = \{z \in Z : d(z, a) < \varepsilon \text{ for some } a \in A\}.$$

Definition 2.1. An evolution process in Z is a two parameter family

$$\{T(t, \tau) : t \geq \tau \in \mathbb{R}\}$$

of continuous maps from Z into itself such that:

(a) $T(\tau, \tau) = I_Z$, with I_Z being the identity in Z,
(b) $T(t, \sigma)T(\sigma, \tau) = T(t, \tau)$, for all $t \geq \sigma \geq \tau$ in $\mathbb{R}$,
(c) The map $\mathcal{P} \times Z \ni (t, \tau, z) \mapsto T(t, \tau)z \in Z$ is continuous, where $\mathcal{P} := \{(t, \tau) \in \mathbb{R}^2 : t \geq \tau\}$.

If $T(t, \tau) = T(t - \tau, 0)$ for all $t \geq \tau \in \mathbb{R}$ the process $\{T(t, \tau) : t \geq \tau \in \mathbb{R}\}$ is called an *autonomous evolution* process and the family $\{T(t) : t \geq 0\}$, defined by $T(t) := T(t, 0)$ for $t \geq 0$, is called a *semigroup.* Clearly

(a') $T(0) = I_Z$, with I_Z being the identity in Z,
(b') $T(t + s) = T(t)T(s)$, for all $t, s \in \mathbb{R}^+$, and
(c') $\mathbb{R}^+ \times Z \ni (t, z) \mapsto T(t)z \in Z$ is continuous.

A continuous map $z: \mathbb{R} \to Z$ is called a *global solution* for the evolution process $\{T(t, \tau) : t \geq \tau\}$ if it satisfies

$$T(t, \tau)z(\tau) = z(t), \quad \text{for all } t \geq \tau \in \mathbb{R}.$$

We note that the term *global solution* is often applied to refer solutions that are defined for all positive times. Here it is always used to indicate solutions defined for all real numbers.

In particular, a global solution for a semigroup $\{T(t) : t \geq 0\}$ is a continuous map $\xi: \mathbb{R} \to Z$ with the property that $T(t)\xi(s) = \xi(t + s)$ for all $s \in \mathbb{R}$ and for all $t \in \mathbb{R}^+$. We say that $\xi: \mathbb{R} \to Z$ is a global solution through $z \in Z$ if it is a global solution and $\xi(0) = z$ and that ξ is a equilibrium point (or stationary solution) when the function $\xi(t) = \xi$ for all $t \in \mathbb{R}$ is a solution. The unstable set of an equilibrium z^* is the set

$$W^u(z^*) = \Big\{ z \in Z : \text{there is a global solution } \zeta \text{ through } z \\ \text{such that } \lim_{t \to -\infty} \operatorname{dist}(\zeta(t), z^*) = 0 \Big\}.$$

The notion of invariance plays a fundamental role in the study of the asymptotic behaviour of semigroups.

Definition 2.2. A subset A of Z is said invariant under the action of the semigroup $\{T(t) : t \geq 0\}$ if $T(t)A = A$ for all $t \geq 0$.

Now we will introduce the notions of attraction and absorption. For that we recall the definitions of Hausdorff semi-distance and distance. For $A, B \subset Z$, the Hausdorff semi-distance from A to B is given by

$$\operatorname{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} d(a, b),$$

and the Hausdorff distance between A and B is defined by

$$d_H(A, B) := \max\{\operatorname{dist}(A, B), \operatorname{dist}(B, A)\}.$$

Definition 2.3. Given two subsets A, B of Z we say that A attracts B under the action of the semigroup $\{T(t) : t \geq 0\}$ if $\operatorname{dist}(T(t)B, A) \xrightarrow{t \to \infty} 0$, and we say that A absorbs B under the action of $\{T(t) : t \geq 0\}$ if there is a $t_B > 0$ such that $T(t)B \subset A$ for all $t \geq t_B$.

With these elements we can introduce the notion of *global attractors*.

Definition 2.4. A subset $\mathcal{A}$ of Z is a *global attractor* for a semigroup $\{T(t) : t \geq 0\}$ if it is compact, invariant and for every bounded subset B of Z we have that $\mathcal{A}$ attracts B under the action of $\{T(t) : t \geq 0\}$.

Next, we will introduce the notions of pullback attractor and generalized gradient-like process (see [6]). In order to do it, we first need the definition of invariance and of isolated global solution.

Definition 2.5. Let $\{T(t, \tau) : t \geq \tau\}$ be an evolution process and $\{\Xi(t) : t \in \mathbb{R}\}$ a family of subsets of Z. We say that $\{\Xi(t) : t \in \mathbb{R}\}$ is invariant under the process $\{T(t, \tau) : t \geq \tau\}$, when

$$T(t, \tau)\Xi(\tau) = \Xi(t) \quad \text{for } t \geq \tau.$$

Definition 2.7. Let $\xi\colon\mathbb{R}\to Z$ be a solution for a nonlinear evolution process $\{T(t,\tau):t\geq\tau\in\mathbb{R}\}$. The set

$$\Gamma:=\{\xi(t):t\in\mathbb{R}\}$$

is called trace of $\xi\colon\mathbb{R}\to Z$. If $\xi\colon\mathbb{R}\to Z$ is a solution and there exists $\delta>0$ such that, any global solution $\zeta\colon\mathbb{R}\to Z$ with $\zeta(\mathbb{R})\subset\mathcal{O}_\delta(\Gamma):=\{z\in Z:\operatorname{dist}(z,\Gamma)<\delta\}$ must satisfy $\zeta(t)=\xi(t)$ for all $t\in\mathbb{R}$, then we say that $\xi\colon\mathbb{R}\to Z$ is an *isolated global solution*. $\boldsymbol{\Xi}=\{\xi_1,\dots,\xi_n\}$ is said a *set of isolated global solutions* if each ξ_i is an isolated global solution and there exists $\delta>0$ such that $\mathcal{O}_\delta(\Gamma_i)\cap\mathcal{O}_\delta(\Gamma_j)=\emptyset$, $1\leq i<j\leq n$, where Γ_i is the trace of $\xi_i\colon\mathbb{R}\to Z$.

Definition 2.8. A pullback attractor for an evolution process $\{T(t,\tau):t\geq\tau\in\mathbb{R}\}$ is an invariant family $\{\mathcal{A}(t):t\in\mathbb{R}\}$ of compact sets with $\overline{\bigcup_{t\leq\tau}\mathcal{A}(t)}$ bounded for each $\tau\in\mathbb{R}$ and such that, for each $t\in\mathbb{R}$ and bounded subset B of Z, we have that

$$\lim_{\tau\to-\infty}\operatorname{dist}(T(t,\tau)B,\mathcal{A}(t))=0.$$

Remark 2.9. It must be pointed out that this is not the most general definition of a pullback attractor. Indeed, the assumption that $\bigcup_{t\leq\tau}\mathcal{A}(t)$ is bounded does not appear in many of the definitions in the literature. However, this suites our purposes in this paper.

Let $\{T(t,\tau):t\geq\tau\in\mathbb{R}\}$ be a nonlinear evolution process with a pullback attractor $\{\mathcal{A}(t):t\in\mathbb{R}\}$ which contains a finite number of isolated global solutions $\boldsymbol{\Xi}=\{\xi_1,\dots,\xi_n\}$. Let Γ_i be the trace of ξ_i.

Definition 2.10 (cf. [11]). Let δ be as in Definition 2.7 and fix $\varepsilon_0\in(0,\delta)$. For $\xi\in\boldsymbol{\Xi}$ and $\varepsilon\in(0,\varepsilon_0)$, an ε-chain from ξ to ξ is a sequence of natural numbers $\ell_i\in\{1,\dots,n\}$, a sequence of real numbers, $\tau_i<\sigma_i<t_i$, and a sequence z_i in Z, $1\leq i\leq k$, such that $z_i\in\mathcal{O}_\varepsilon(\Gamma_{\ell_i})$, $T(\sigma_i,\tau_i)z_i\notin\mathcal{O}_{\varepsilon_0}\Big(\bigcup_{j=1}^n(\Gamma_j)\Big)$ and $T(t_i,\tau_i)z_i\in\mathcal{O}_\varepsilon(\Gamma_{\ell_{i+1}})$, $1\leq i\leq k$, with $\xi=\xi_{\ell_{k+1}}=\xi_{\ell_1}$. We say that $\xi\in\boldsymbol{\Xi}$ is chain recurrent if there is an $\varepsilon_0\in(0,\delta)$ and ε-chain from ξ to ξ for each $\varepsilon\in(0,\varepsilon_0)$.

We are now ready to define gradient-like evolution processes.

Definition 2.11. Let Z be a metric space and $\{T(t,\tau):t\geq\tau\}$ be a nonlinear evolution process in Z. Let $\{\mathcal{A}(t):t\in\mathbb{R}\}$ be the pullback attractor for $\{T(t,\tau):t\geq\tau\}$. We say that $\{T(t,\tau):t\geq\tau\}$ is a generalized gradient-like process if the following two hypotheses are satisfied:

(H1) $\{\mathcal{A}(t):t\in\mathbb{R}\}$ contains a finite number of isolated global solutions $\boldsymbol{\Xi}=\{\xi_1,\dots,\xi_n\}$ with the property that any global solution $\xi\colon\mathbb{R}\to\ \mathcal{Z}$

in $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ satisfies

$$\lim_{t\to-\infty} \operatorname{dist}(\xi(t), \xi_i(t)) = 0 \quad \text{and} \quad \lim_{t\to\infty} \operatorname{dist}(\xi(t), \xi_j(t)) = 0,$$

for some $1 \le i, j \le n$.

(H2) $\boldsymbol{\Xi} = \{\xi_1, \ldots, \xi_n\}$ does not contain any chain recurrent isolated solution.

Next we seek to introduce the notion of gradient-like semigroups (see [6]). To that end we first need the definition of homoclinic structure.

Definition 2.12. Let $\{T(t) : t \ge 0\}$ be a semigroup which has a set $\mathcal{E}$ of equilibrium points. A homoclinic structure associated to $\mathcal{E}$ is a finite subset $\{z_1^*, \ldots, z_p^*\}$ of $\mathcal{E}$ together with a set of global solutions $\{\xi_1, \ldots, \xi_p\}$ such that

$$z_j^* \xleftarrow{t\to-\infty} \xi_j(t) \xrightarrow{t\to\infty} z_{j+1}^*, \quad 1 \le j \le p$$

where $z_{p+1}^* := z_1^*$.

Now, for the autonomous case, we recall the definition of a gradient-like semigroup (which indeed is slightly different from that in [6]).

Definition 2.13. Consider a metric space (Z, d) and a nonlinear semigroup $\{T(t) : t \ge 0\}$ in Z, which has a global attractor $\mathcal{A}$ and a set of equilibrium points $\mathcal{E}$ (possible infinite). We say that $\{T(t) : t \ge 0\}$ is a gradient-like semigroup when

(G1) For any bounded global solution $\xi\colon \mathbb{R} \to Z$ for $\{T(t) : t \ge 0\}$, there exist two equilibrium points z_1^* and z_2^* in $\mathcal{E}$ such that

$$\lim_{t\to-\infty} d(\xi(t), z_1^*) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(\xi(t), z_2^*) = 0.$$

(G2) The attractor $\mathcal{A}$ does not contain homoclinic structures.

Remark 2.14. We remark that, in the context of this definition, it is possible that a gradient system is not gradient-like, since the condition (G1) above, may not hold for a general gradient system. In [1], the notion of generalized gradient-like semigroups is proved to coincide with the notion of generalized gradient semigroups (replacing equilibria by isolated invariant sets). It is fairly difficult to find examples of gradient semigroups which do not satisfy $(G1)$ (one example can be found in [17, p. 15]).

Our goal is to describe the asymptotic dynamics of the following partially coupled (autonomous) model:

$$\begin{cases} x' = Ax + g(x), & t > 0, \\ y' = By + f(x, y), & t > 0, \\ x(0) = x_0 \in X, \quad y(0) = y_0 \in Y, \end{cases} \tag{2.1}$$

where X and Y are Banach spaces, the operators $A\colon D(A) \subset X \to X$ and $B\colon D(B) \subset Y \to Y$ generate strongly continuous semigroups of linear operators and f, g are suitable nonlinearities. Let also assume that there exists a vectorial subspace $D(C)$ in $D(A)\times D(B)$ (possibly all $D(A)\times D(B)$) dense in $Z = X\times Y$, and we consider the linear operator $C\colon D(C) \subset Z \to Z$ given by $Cz = C(x,y) := (Ax, By)$ for $z = (x,y) \in D(C)$ and suppose that C generates a C_0-semigroup in Z (or even a singular semigroup). Then define $h\colon Z \to Z$ as $h(z) = h(x,y) := (g(x), f(x,y))$ for $z = (x,y) \in Z$. So, problem (2.1) can be reformulated as

$$\begin{cases} z' = Cz + h(z), & t > 0, \\ z(0) = z_0 \in Z. \end{cases} \tag{2.2}$$

Finally, assume that $f\colon X \times Y \to Y$ and $g\colon X \to X$ are so that the systems (2.2) and

$$\begin{cases} x' = Ax + g(x), & t > 0, \\ x(0) = x_0 \in X, \end{cases}$$

generate, respectively, a nonlinear semigroup $\{T(t) : t \geq 0\}$ in Z with global attractor $\mathcal{A}_{\mathcal{C}}$ and set of equilibrium points $\mathcal{E}_C$, and a nonlinear semigroup $\{S(t) : t \geq 0\}$ in X, which is gradient with Lyapunov functional $E\colon X \to \mathbb{R}$, global attractor $\mathcal{A}$, and set of equilibrium points $\mathcal{E}_A$.

We aim to establish conditions so that the semigroup $\{T(t) : t \geq 0\}$ related to (2.2) is gradient-like in the sense of Definition 2.13. In order to obtain this, we introduce the notion of hyperbolic equilibrium and recall two theorems which will be useful for our results.

Definition 2.15. An equilibrium solution x_0^* for the problem

$$\begin{cases} x' = Dx + m(x), & t > 0, \\ x(0) = x_0 \in X, \end{cases}$$

is hyperbolic when the spectrum of $\widetilde{D} := D + m'(x_0^*)$ does not intersect the imaginary axis, the set $\sigma^+ = \{\lambda \in \sigma(\widetilde{D}) : \operatorname{Re}\lambda > 0\}$ is compact and, if γ is a smooth closed simple curve in $\rho(\widetilde{D}) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ oriented counterclockwise and enclosing σ^+,

$$\mathcal{Q} = \mathcal{Q}(\sigma^+) := \frac{1}{2\pi i}\int_\gamma (\lambda - \widetilde{D})^{-1}\,d\lambda,$$

$\mathcal{Q}e^{\widetilde{D}\,t} = e^{\widetilde{D}\,t}\mathcal{Q}$, $R(\mathcal{Q}) \subset D(\widetilde{D})$ and there are constants $M_1 \geq 1$, $\beta > 0$, such that

$$\begin{aligned} \|e^{\widetilde{D}t}\mathcal{Q}\|_{\mathcal{L}(X)} &\leq M_1 e^{\beta t}, & t \leq 0, \\ \|e^{\widetilde{D}t}(I - \mathcal{Q})\|_{\mathcal{L}(X)} &\leq M_1 e^{-\beta t}, & t \geq 0. \end{aligned}$$

Next we define exponential dichotomy for a linear evolution process and introduce the concept of global hyperbolic solutions, which is the analogous non-autonomous of the concept of hyperbolic equilibrium.

Definition 2.16. We say that a linear evolution process $\{U(t,s) : t \geq s\} \subset \mathcal{L}(X)$ in a Banach space X has exponential dichotomy with exponent ω and constant M if there exists a family of projections $\{Q(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X)$ such that

(a) $Q(t)U(t,s) = U(t,s)Q(s)$, for all $t \geq s$;
(b) The restriction $U(t,s)_{|_{R(Q(s))}}$, $t \geq s$ is an isomorphism from $R(Q(s))$ into $R(Q(t))$ and its inverse is denoted by $U(s,t)\colon R(Q(t)) \to R(Q(s))$;
(c) for some $\omega > 0$

$$\begin{aligned} \|U(t,s)(I - Q(s))\| &\leq Me^{-\omega(t-s)}, \quad t \geq s, \\ \|U(t,s)Q(s)\| &\leq Me^{\omega(t-s)}, \quad\ \ t \leq s. \end{aligned}$$

We will say that a global solution $\zeta\colon \mathbb{R} \to X$ of a nonlinear evolution process $\{T(t,\tau) : t \geq \tau\}$ in X, generated by a semilinear equation

$$\begin{cases} y' = By + f(t,y), & t > \tau, \\ y(\tau) = y_0, \end{cases}$$

is a global hyperbolic solution when the linearization around it, i.e. the linear equation

$$\begin{cases} y' = By + f_y(t,\zeta(t))y, & t > \tau, \\ y(\tau) = y_0, \end{cases}$$

has solution operator with exponential dichotomy.

With the above definitions and notation we can state the following result.

Theorem 2.17 (cf. [8]). *Consider Y – a Banach space, $\{f_\eta\colon \mathbb{R} \times Y \to Y\}_{\eta\in(0,1]}$ – a family of applications with each element having continuous partial derivative with respect to the second variable in $\mathbb{R} \times Y$, $f_0\colon Y \to Y$ – a continuously differentiable map and $B\colon D(B) \subset Y \to Y$ the generator of a strongly continuous semigroup. Consider the following problems*:

$$\begin{cases} y' = By + f_\eta(t,y), & t > \tau, \\ y(\tau) = y_0, \end{cases} \tag{2.3}$$

$$\begin{cases} y' = By + f_0(y), & t > \tau, \\ y(\tau) = y_0. \end{cases} \tag{2.4}$$

Assume that $\{f_\eta\colon \mathbb{R}\times Y \to Y\}_{\eta\in(0,1]}$ and $f_0\colon Y \to Y$ are such that (2.3) *and* (2.4) *generate nonlinear evolution processes $\{T_\eta(t,\tau)\colon t \geq \tau\}_{\eta\in(0,1]}$, and a nonlinear semigroup $\{T_0(t) : t \geq 0\}$, respectively. Suppose also that T_0 has a global attractor* *with a finite number of equilibrium points* $\mathcal{E} = \{y_1^*, \dots, y_n^*\}$, *all of them being hyperbolic.*

If for each $r > 0$ *it holds that*

$$\lim_{\eta\to 0^+} \sup_{t\in\mathbb{R}} \sup_{\|y\|_Y\le r} \{\|f_\eta(t,y) - f_0(y)\|_Y + \|(f_\eta)_y(t,y) - f_0'(y)\|_{\mathcal{L}(Y)}\} = 0,$$

then there exists $\eta_0 \in (0,1]$ *such that for each* $\eta \le \eta_0$ *there exist* $\xi_{i,\eta}^*\colon \mathbb{R} \to Y$, $i = 1, \dots, n$, *global solutions for* (2.3), *that are hyperbolic, and satisfying*

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \|\xi_{i,\eta}^*(t) - y_i^*\|_Y = 0, \quad \textit{for all } i = 1, \dots, n.$$

Theorem 2.18 (cf. [6]). *Consider* Y *a Banach space, and* $\{T_\eta(t,\tau) : t \ge \tau\}$ *a nonlinear evolution process on* Y *with a pullback attractor* $\{\mathcal{A}_\eta(t) : t \in \mathbb{R}\}$ *for* $\eta \in [0,1]$. *Assume that the following conditions hold*:

(a) $\overline{\bigcup_{\eta\in[0,1]} \bigcup_{t\in\mathbb{R}} \mathcal{A}_\eta(t)}$ *is compact.*

(b) $\{T_0(t,\tau) : t \ge \tau\}$ *is an autonomous process, i.e.* $T_0(t,\tau) = S(t-\tau)$ *for all* $t \ge \tau$, *where* $\{S(t) : t \ge 0\}$ *is a semigroup, which additionally is gradient-like with a finite number of equilibrium points* $\mathcal{E} = \{y_{1,0}^*, \dots, y_{n,0}^*\}$.

(c) *For each* $\eta \in (0,1]$, $\{T_\eta(t,\tau) : t \ge \tau \in \mathbb{R}\}$ *has* n *global isolated solutions* $\xi_{i,\eta}^*\colon \mathbb{R} \to Y$ $i = 1, \dots, n$, *such that, if* $\Gamma_{i,\eta}$ *is the trace of* $\xi_{i,\eta}^*\colon \mathbb{R} \to Y$, *for* $i = 1, \dots, n$ *and* $\eta \in (0,1]$, *then one has that*

$$\lim_{\eta\to 0^+} \sup_{1\le i\le} d_H(y_{i,0}^*, \Gamma_{i,\eta}) = 0.$$

(d) *For each* $T > 0$ *and compact set* $K \subset Y$, *it holds*

$$\lim_{\eta\to 0^+} \sup_{\tau\in\mathbb{R}} \sup_{t\in[0,T]} \sup_{y\in K} \|T_\eta(t+\tau,\tau)y - T_0(t+\tau,\tau)y\|_Y = 0.$$

(e) *There exist* $\delta > 0$ *and* $\eta_1 \in (0,1]$ *such that if* $\xi_\eta\colon \mathbb{R} \to Y$ *is a bounded solution of* $\{T_\eta(t,\tau) : t \ge \tau \in \mathbb{R}\}$ *with* $\eta \le \eta_1$ *so that there exists* $t_0 \in \mathbb{R}$ *and some* $i \in \{1, \dots, n\}$ *with* $\sup_{t\le t_0} \operatorname{dist}(\xi_\eta(t), \Gamma_{i,\eta}) < \delta$ (*resp.* $\sup_{t\ge t_0} \operatorname{dist}(\xi_\eta(t), \Gamma_{i,\eta}) < \delta$), *then* $\lim_{t\to-\infty} \|\xi_\eta(t) - \xi_{i,\eta}^*(t)\|_Y = 0$ (*resp.* $\lim_{t\to\infty} \|\xi_\eta(t) - \xi_{i,\eta}^*(t)\|_Y = 0$).

Then, there exists $\eta_0 \in (0,\eta_1]$ *such that, for each* $\eta \in (0,\eta_0]$, $\{T_\eta(t,\tau) : t \ge \tau \in \mathbb{R}\}$ *is a non-autonomous gradient-like evolution process.*

3. Gradient like cascade systems

Combining Theorems 2.17 and 2.18, we will be able to establish our first result, concerning the asymptotic behaviour of system (2.1). But firstly let us give one last definition.

Definition 3.1. Let $f:\mathbb{R}\times Y\to Y$ be an application such that the semilinear problem:

$$\begin{cases} y' = By + f(t,y), & t>\tau, \\ y(\tau) = y_0, \end{cases} \tag{3.1}$$

generates a nonlinear evolution process $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ which has a pullback attractor $\{\mathcal{A}(t) : t\in\mathbb{R}\}$.

Denote, for any $\nu>0$ and $\mu>0$, the applications $f_\nu, f^\mu:\mathbb{R}\times Y\to Y$ as

$$f_\nu(t,y) = \begin{cases} f(t,y) & \text{if } t\leq -\nu, \\ f(-\nu,y) & \text{if } t>-\nu, \end{cases}$$

and

$$f^\mu(t,y) = \begin{cases} f(t,y) & \text{if } t\geq \mu, \\ f(\mu,y) & \text{if } t<\mu. \end{cases}$$

We will say that f is *compatible from the left* (resp. *from the right*) with respect to the system (3.1) if there exists ν_0 (resp. μ_0) such that for all $\nu\geq\nu_0$ (resp. $\mu\geq\mu_0$) the problem

$$\begin{cases} y' = By + f_\nu(t,y), & t>\tau, \\ y(\tau) = y_0, \end{cases}$$

(resp. the problem

$$\begin{cases} y' = By + f^\mu(t,y), & t>\tau, \\ y(\tau) = y_0, \end{cases}\Bigg)$$

generates a nonlinear evolution process $\{T_\nu(t,\tau) : t\geq\tau\}$ which has a pullback attractor $\{\mathcal{A}_\nu(t) : t\in\mathbb{R}\}$ with $\bigcup_{\nu\geq\nu_0}\bigcup_{t\in\mathbb{R}}\mathcal{A}_\nu(t)$ relatively compact (resp. generates a nonlinear evolution process $\{T^\mu(t,\tau) : t\geq\tau\}$ which has a pullback attractor $\{\mathcal{A}^\mu(t) : t\in\mathbb{R}\}$ with $\bigcup_{\mu\geq\mu_0}\bigcup_{t\in\mathbb{R}}\mathcal{A}^\mu(t)$ relatively compact).

Lemma 3.2. *With the above notation, suppose that the uncoupled equation of system* (2.1), $x' = Ax + g(x)$, *generates a nonlinear semigroup* $\{S(t) : t\in\mathbb{R}\}$ *which has a global attractor* $\mathcal{A}$, *set of equilibria* $\mathcal{E}$, *and such that any global solution in the attractor converges to equilibrium points when* $t\to\pm\infty$. *Assume also that for any equilibrium point* $x^*\in\mathcal{E}$ *the autonomous semilinear problem*

$$\begin{cases} y' = By + f(x^*,y), & t>\tau, \\ y(\tau) = y_0 \end{cases}$$

defines a semigroup $\{S_{x^*}(t) : t\geq 0\}$ *which is gradient-like, has global attractor* $\mathcal{A}_{x^*}$ *and a finite number of equilibrium points* $\mathcal{E}_{x^*}$, *all of them hyperbolic.*

Let $\xi = (\varphi, \psi): \mathbb{R} \to X \times Y$ *be a (global) solution in the global attractor related to* (2.2)*, and denote* $f(t, y) := f(\varphi(t), y)$ *for* $(t, y) \in \mathbb{R} \times Y$*. Assume that the non-autonomous problem*:

$$
\begin{cases} y' = By + f(t, y), & t > \tau, \\ y(\tau) = y_0, \end{cases} \tag{3.2}
$$

generates a nonlinear evolution process $\{S(t, \tau) : t \geq \tau\}$ *which has a pullback attractor* $\{\mathcal{A}(t) : t \in \mathbb{R}\}$*, and that* f *is compatible from the left and from the right with respect to the system* (3.2)*. According to the above assumption, there exist equilibrium points* $x_-^*, x_+^* \in \mathcal{E}$ *such that*

$$
\lim_{t\to-\infty} \|\varphi(t) - x_-^*\|_X = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\varphi(t) - x_+^*\|_X = 0.
$$

If for each $r > 0$*, it holds that*

$$
\lim_{t\to-\infty} \sup_{\|y\|_Y \leq r} \{\|f(t, y) - f(x_-^*, y)\|_Y + \|f_y(t, y) - f_y(x_-^*, y)\|_{\mathcal{L}(Y)}\} = 0 \tag{3.3}
$$

and

$$
\lim_{t\to\infty} \sup_{\|y\|_Y \leq r} \{\|f(t, y) - f(x_+^*, y)\|_Y + \|f_y(t, y) - f_y(x_+^*, y)\|_{\mathcal{L}(Y)}\} = 0, \tag{3.4}
$$

then there exist equilibrium points $y_-^* \in \mathcal{E}_{x_-^*}$ *and* $y_+^* \in \mathcal{E}_{x_+^*}$ *such that*

$$
\lim_{t\to-\infty} \|\xi(t) - z_-^*\|_{X\times Y} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\xi(t) - z_+^*\|_{X\times Y} = 0,
$$

where $z_-^* = (x_-^*, y_-^*)$ *and* $z_+^* = (x_+^*, y_+^*)$.

Proof. We will only deal with the case $t \to -\infty$, since the case $t \to \infty$ can be treated analogously.

For each $\nu > 0$, define the map $f_\nu: \mathbb{R} \times Y \to Y$ as before. Now we consider the family of semilinear problems

$$
\begin{cases} y' = By + f_\nu(t, y), & t > \tau, \\ y(\tau) = y_0. \end{cases} \tag{3.5}
$$

From the compatibility assumption we know that f_ν is such that there exists a family of evolution processes $\{T_\nu(t, \tau) : t \geq \tau\}$ with $\nu \in (\nu_0, \infty)$, for some $\nu_0 > 0$, such that each of them has a pullback attractor $\{\mathcal{A}_\nu(t) : t \in \mathbb{R}\}$ with $\bigcup_{\nu\geq\nu_0} \bigcup_{t\in\mathbb{R}} \mathcal{A}_\nu(t)$ relatively compact on Y.

It is not difficult to see that for $t \leq -\nu$ it holds that $T_\nu(t, \tau) = S(t, \tau)$ for all $\tau \leq t$. We also have that if $\zeta: \mathbb{R} \to Y$ is a global solution for $\{S(t, \tau) : t \geq \tau\}$ then $\zeta(t) = S(t, \tau)\zeta(\tau) = T_\nu(t, \tau)\zeta(\tau)$ provided that $\tau \leq t \leq -\nu$. So, defining for each $\nu \geq \nu_0$, $\zeta_\nu: \mathbb{R} \to Y$ as

$$
\zeta_\nu(t) = \begin{cases} \zeta(t) & \text{if } t \leq -\nu, \\ T_\nu(t, -\nu)\zeta(-\nu) & \text{if } t > -\nu, \end{cases}
$$

then $\zeta_\nu\colon \mathbb{R} \to Y$ is a global solution for $\{T_\nu(t,\tau) : t \geq \tau\}$.

Now, using (3.3), one has that

$$\lim_{\nu\to\infty} \sup_{t\in\mathbb{R}} \sup_{\|y\|_Y\leq r} \{\|f_\nu(t,y) - f(x^*_-,y)\|_Y + \|(f_\nu)_y(t,y) - f_y(x^*_-,y)\|_{\mathcal{L}(Y)}\} = 0. \tag{3.6}$$

Therefore, from Theorem 2.17 we deduce that there exists $\nu_1 > 0$ such that for each $\nu \geq \nu_1$ problem (3.5) has n global hyperbolic solutions $\xi^*_{i,\nu}\colon \mathbb{R} \to Y$, $i = 1,\ldots,n$ which satisfy

$$\lim_{\nu\to\infty} \sup_{t\in\mathbb{R}} \|\xi^*_{i,\nu}(t) - y^*_{i,-}\|_Y = 0, \quad \text{for all } i = 1,\ldots,n, \tag{3.7}$$

where $\mathcal{E}_{x^*_-} = \{y^*_{1,-},\ldots,y^*_{n,-}\}$.

From (3.6) we deduce that for each $T > 0$ and each compact set $K \subset Y$, it holds

$$\lim_{\nu\to\infty} \sup_{\tau\in\mathbb{R}} \sup_{t\in[0,T]} \sup_{y\in K} \|T_\nu(t+\tau,\tau)y - S_{x^*_-}(t+\tau,\tau)y\|_Y = 0.$$

Thus, from the above and the hyperbolicity of the solutions $\xi^*_{i,\nu}\colon \mathbb{R} \to Y$, $i = 1,\ldots,n$, the assumptions in Theorem 2.18 are fulfilled. Therefore, there exists $\nu_2 \geq \nu_1$ such that for each $\nu \geq \nu_2$, $\{T_\nu(t,\tau) : t \geq \tau\}$ is a non-autonomous gradient-like process.

Finally, consider $\psi : \mathbb{R} \to Y$ as in the statement. This is a bounded solution of $\{S(t,\tau) : t \geq \tau\}$. This implies that the functions $\psi_\nu\colon \mathbb{R} \to Y$ associated to $\psi\colon \mathbb{R} \to Y$ are solutions for $\{T_\nu(t,\tau) : t \geq \tau\}$ and they are in $\{\mathcal{A}_\nu(t) : t \in \mathbb{R}\}$, as long as they are bounded because they coincide with ψ for $t \leq -\nu$. But in $\{\mathcal{A}_\nu(t) : t \in \mathbb{R}\}$ there exists a (unique) value $i_0 \in \{1,\ldots,n\}$ such that for ν big enough it holds that

$$\lim_{t\to-\infty} \|\psi(t) - \xi^*_{i_0,\nu}(t)\|_Y = \lim_{t\to-\infty} \|\psi_\nu(t) - \xi^*_{i_0,\nu}(t)\|_Y = 0.$$

This, jointly with (3.7), means, as desired, that

$$\lim_{t\to-\infty} \|\psi(t) - y^*_{i_0,-}\|_Y = 0. \qquad \square$$

Remark 3.3. We note that, if the problem $x' = Ax + g(x)$ generates a nonlinear *gradient* (or just *gradient-like*) semigroup $\{S(t) : t \in \mathbb{R}\}$ which has a global attractor $\mathcal{A}$ and a finite set of equilibria $\mathcal{E}$, then the condition "any global solution in the attractor converges to equilibrium points when $t \to \pm\infty$", that we have used in the previous lemma, is automatically satisfied.

Under the same assumptions of the above lemma we can prove now our main result in this section, which is that the semilinear problem (2.2) is gradient-like.

Theorem 3.4. *Under the assumptions in Lemma* 3.2*, the semigroup* $\{T(t) : t \geq 0\}$*, associated to problem* (2.2)*, is gradient-like.*

Proof. Firstly observe that Lemma 3.2 means that $\{T(t) : t \geq 0\}$ fulfills condition (G1) in Definition 2.13.

So, it remains to check that (G2) holds. We will proceed by contradiction. Assume that there exist a finite number of global solutions $\{\xi_i = (\xi_i^{(1)}, \xi_i^{(2)})\colon R \to X\times Y : i = 1, \ldots, k\}$ and equilibrium points $\{z_i^* = (x_i^{(1)*}, y_i^{(2)*}) : i = 1, \ldots, k\} \subset \mathcal{E}$ which are part of a homoclinic structure in the attractor of $\{T(t) : t \geq 0\}$. Each $\xi_i^{(1)}\colon \mathbb{R} \to X$ is a global solution of $x' = Ax + g(x)$ and each $\xi_i^{(2)}\colon \mathbb{R} \to Y$ solves the non-autonomous equation $y' = By + f(\xi_i^{(1)}(t), y)$. Now we distinguish two cases.

Case 1. Assume that there exists some $i_0 \in \{1, \ldots, k\}$ such that $\xi_{i_0}^{(1)}\colon \mathbb{R} \to X$ is a non-constant solution of $x' = Ax + g(x)$. Then it yields that $\xi_{i_0}^{(1)}\colon \mathbb{R} \to X$ belongs to some homoclinic structure in the attractor $\mathcal{A}$ of $x' = Ax + g(x)$. But this contradicts the fact that $x' = Ax + g(x)$ is a gradient system.

Case 2. For every $i \in \{1, \ldots, k\}$ the solutions $\xi_i^{(1)}\colon \mathbb{R} \to X$ are equilibrium points. Of course in this case all the solutions $x_i^{(1)*}$ are forced to coincide in a same fixed element, say $x_{i_0}^{(1)*}$, i.e. $\xi_i^{(1)}(t) = x_{i_0}^{(1)*}$ for all $i = 1, \ldots, k$ and all $t \in \mathbb{R}$. Then, all maps $\xi_i^{(2)}\colon \mathbb{R} \to Y$ solve the same equation, namely $y' = By + f(x_{i_0}^{(1)*}, y)$ and jointly with the equilibrium points $\{y_{\ell_1}^{(2)*}, \ldots, y_{\ell_k}^{(2)*}\}$, are part of a homoclinic structure in the attractor related to the problem $y' = By + f(x_{i_0}^{(1)*}, y)$. But this contradicts the assumption of $y' = By + f(x_{i_0}^{(1)*}, y)$ being gradient-like. The proof is complete. $\square$

Corollary 3.5. *Under the assumptions of Lemma 3.2, the attractor $\mathcal{A}_{\mathcal{C}}$ of the semigroup $\{T(t) : t \geq 0\}$ is gradient-like, i.e. it is given by the union of unstable manifolds of its equilibrium points.*

Proof. From Lemma 3.2 we deduce that any bounded solution of (2.2) converges when $t \to -\infty$ to an equilibrium point, whence the result follows. $\square$

Corollary 3.6. *Under the assumptions of Lemma* 3.2*, if the set $\mathcal{E}$ of equilibrium points of $x' = Ax + g(x)$ is finite, then the semigroup $\{T(t) : t \geq 0\}$ is gradient in the sense of* [13]*.*

Proof. Indeed, by results from [1], both concepts, gradient and gradient-like, are the same, and the corollary follows. $\square$

4. Łojasiewicz–Simon inequality

The aim of this section is to present a sufficient condition such that a global solution in the attractor converges to equilibrium points when times goes to $\pm\infty$.

This was an assumption in Theorem 3.4. A well-known condition that guarantees this convergence is the Łojasiewicz–Simon inequality for gradient systems, which can be formulated in an abstract way in Hilbert spaces. To our knowledge, this has been applied to analyse the behaviour of the problem when $t \to \infty$ (see for instance [10], [25], [14]). In Section 6 we present an extension of this result to deal also with the case $t \to -\infty$. Here, we give another proof of the Łojasiewicz–Simon inequality, which will be used in Section 6 to establish the convergence of solutions when $t \to -\infty$.

We point out briefly the importance of this kind of condition. In the case of a finite number of equilibrium points, if the semigroup is gradient with relatively compact orbits, then the omega-limit of a solution (which is connected) is a singleton. However for the case of a continuum of equilibrium points, the convergence of a solution when $t \to \pm\infty$ to equilibrium points may not hold (see [17], [18] for a counter example).

The analysis we will carry out in this section involves conditions such that the results of previous section can be applied. In particular, we will study abstract problems of first and second order in time.

Consider $V = (V, (\,\cdot\,,\,\cdot\,)_V)$ and $H = (H, (\,\cdot\,,\,\cdot\,)_H)$ real Hilbert spaces with V dense in H and with compact injection. We will identify H with its topological dual H', so we have the chain of dense and compact injections

$$V \hookrightarrow H \hookrightarrow V'.$$

We establish now a slight variant of the well-known Łojasiewicz–Simon inequality, e.g. cf. [14], [23]. The proof is similar to the previous ones but, since we use a different projector, we give the details.

Theorem 4.1 (Łojasiewicz–Simon inequality). *Consider $G \in C^2(V;\mathbb{R})$ and denote $M := G' = \nabla G\colon V \to V'$. Let $\varphi \in V$ be a solution of $M(u) = 0$ such that the following conditions are satisfied:*

(a) *The linearization $L := M'(\varphi) \in \mathcal{L}(V, V')$ of M in φ is given as $L = \Lambda + B$, where $\Lambda\colon V \to V'$ is an isomorphism and $B : V \to V'$ is a compact operator.*

(b) *If $\ker(L)$ ($\dim(\ker(L)) < \infty$ for L has compact resolvent) is non-trivial, we assume that the map $L\colon V \subset V' \to V'$ has non-empty resolvent set and, denote by $\Pi\colon H \to H$ the projection defined by*

$$\Pi u := \frac{1}{2\pi i}\int_\gamma (\lambda - L)^{-1} u \, d\lambda,$$

where $\gamma\colon [0,1] \to \rho(L)$ is a closed, simple and smooth contour of $\lambda_0 = 0$ with $\int_\gamma (1/\zeta)\, d\zeta = 2\pi i$, and $d := \dim(R(\Pi)) \in \mathbb{N}$, ($\dim(R(\Pi)) < \infty$ for L has compact resolvent) then we assume that there exists an open set

$U \subset \mathbb{R}^d$ and a homeomorphism $h: U \to h(U)$, such that $\varphi \in h(U) \subset M^{-1}(0)$.

Then, the Łojasiewicz–Simon inequality is satisfied at φ, i.e. there exist $\sigma > 0$ and $c > 0$ such that for all $u \in V$ with $\|u - \varphi\|_V < \sigma$ it holds

$$|G(u) - G(\varphi)| \leq c\|M(u)\|^2_{V'}.$$

Proof. *Case 1.* Assume that $\ker(L) = \{0\}$. In this case, $L: V \to V'$ is an isomorphism.

Using Taylor expansion of second order in G around φ we obtain for all u that

$$G(u) - G(\varphi) = \langle G''(\varphi)(u - \varphi), u - \varphi\rangle + o(\|u - \varphi\|^2_V),$$

This yields

$$|G(u) - G(\varphi)| \leq C_1\|u - \varphi\|^2_V, \tag{4.1}$$

for u close enough to φ.

On the other hand, using again Taylor expansion of first order for $M = G'$ around φ we have

$$M(u) = M(\varphi) + L(u - \varphi) + o(\|u - \varphi\|_{V'}) = L(u - \varphi) + o(\|u - \varphi\|_V),$$

which yields

$$u - \varphi = L^{-1}[M(u)] + o(\|u - \varphi\|_V).$$

Then, for values u close enough to φ we deduce $\|u - \varphi\|_V \leq C_2\|M(u)\|_{V'}$, for some constant $C_2 > 0$. From this and (4.1), there exists $\sigma > 0$ such that $\|u - \varphi\|_V < \sigma$ implies that $|G(u) - G(\varphi)| \leq C\|M(u)\|^2_{V'}$, for some constant $C > 0$, which concludes the proof in this case.

Case 2. Assume that $\ker(L) \neq \{0\}$. Making a change of variables, we can consider $\varphi = 0$ and $G(\varphi) = 0$. In fact, we can define $G_0: V \to \mathbb{R}$ by $G_0(u) := G(u + \varphi) - G(\varphi)$ and observe that the result for G_0 is equivalent to the result for G.

Let $\mathcal{L}: V \to V'$ be the linear operator given by

$$\mathcal{L}u := \Pi u + Lu, \quad u \in V.$$

Claim 2.1. *$\mathcal{L}$ is injective.*

Indeed, let $u_0 \in V$ be such that $0 = \mathcal{L}u_0 = \Pi u_0 + Lu_0$. As L commute with Π, it holds that $0 = \Pi^2 u_0 + \Pi L u_0 = \Pi u_0 + L\Pi u_0$. Let $n_0 \in \mathbb{N}$ be the least positive integer such that $R(\Pi) = N(L^{n_0})$. From this and the fact that $\Pi u_0 = -Lu_0$ we have that $L^{n_0-1}\Pi u_0 = -L^{n_0}\Pi u_0 = 0$ and $L^{n_0-1}\Pi u_0 = 0$. By induction $\Pi u_0 = 0$. Consequently, $Lu_0 = -\Pi u_0 = 0$ and $u_0 \in N \subset R(\Pi)$. Hence $u_0 = \Pi u_0 = 0$.

Using the decomposition of L given in the assumption (a) we have

$$\Lambda^{-1}\mathcal{L} = I_V + \Lambda^{-1}(\Pi + B),$$

with $\Lambda^{-1}\mathcal{L}: V \to V$ injective. As $\Lambda^{-1}(\Pi + B): V \to V$ is compact, from Fredholm Alternative follows that $\Lambda^{-1}\mathcal{L}: V \to V$ is also surjective, and therefore an isomorphism.

Consider now the map $\mathcal{N}: V \to V'$ given by

$$\mathcal{N}u := \Pi u + M(u), \quad u \in V.$$

From assumptions about G we deduce that $\mathcal{N}$ is C^1 with $\mathcal{N}'(0) = \Pi + M'(0) = \mathcal{L}$. Then, by the Inverse Function Theorem, there exist an open set $W_1(0) \subset V$ with $0 \in W_1(0)$ and an open set $W_2(0) \subset V'$ with $0 \in W_2(0)$, such that $\mathcal{N}$ is a diffeomorphism from $W_1(0)$ onto $W_2(0)$. So, there exists a C^1 map $\Psi: W_2(0) \to W_1(0)$ which is the inverse of $\mathcal{N}: W_1(0) \to W_2(0)$.

Taking smaller open sets $W_1(0)$ and $W_2(0)$ if necessary, and using the Mean Value inequality we may assume that

$$\begin{aligned} &\|\Psi(g_1) - \Psi(g_2)\|_V \leq C_1\|g_1 - g_2\|_{V'} &&\text{for all } g_1, g_2 \in W_2(0), \\ &\|M(u) - M(v)\|_{V'} \leq C_2\|u - v\|_V &&\text{for all } u, v \in W_1(0), \end{aligned} \tag{4.2}$$

for certain constants C_1 and $C_2 > 0$.

Now, we consider an orthonormal basis $\varphi_1, \ldots, \varphi_d$, for $R(\Pi) = N(L^{n_0})$, relative to the inner product of H and we define the map $f: \mathbb{R}^d \to V'$ by

$$f(\xi) := \sum_{j=1}^{d} \xi_j \varphi_j, \quad \text{for } \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d.$$

It is clear that f is an isomorphism between $\mathbb{R}^d$ and $N(L^{n_0})$. Then

$$\widetilde{W}_2(0) := f^{-1}(W_2(0)) = \{\xi \in \mathbb{R}^d : f(\xi) \in W_2(0)\}$$

is an open set of $\mathbb{R}^d$ which contain 0.

Consider then the real function $\Gamma: \widetilde{W}_2(0) \to \mathbb{R}$ given by

$$\Gamma(\xi) := (G \circ \Psi \circ f)(\xi), \quad \xi \in \widetilde{W}_2(0).$$

Clearly, Γ is C^1 in $\widetilde{W}_2(0)$.

Now we define $\widetilde{W}_1(0) := \{u \in W_1(0) : \Pi u \in W_2(0)\}$. It is not difficult to see, using that $\Pi^{-1}(W_2(0))$ is open in H and the continuous inclusion $V \hookrightarrow H$, that $\widetilde{W}_1(0)$ is open in V.

Then, for each $u \in \widetilde{W}_1(0)$ consider $\xi \in \widetilde{W}_2(0)$ such that $f(\xi) = \Pi u \in W_2(0)$.

Claim 2.2 *Estimate for $G(u) - \Gamma(\xi)$.* There exists a constant $C > 0$, independent of u and ξ, such that

$$|G(u) - \Gamma(\xi)| \leq C\|M(u)\|_{V'}^2. \tag{4.3}$$

Indeed,

$$\begin{aligned}|G(u)-\Gamma(\xi)| &= |G(u)-G(\Psi(f(\xi)))| = \left|\int_0^1 \frac{d}{dt}\{G(u+t[\Psi(f(\xi))-u])\}\,dt\right| \\ &\le \int_0^1 \|M(u+t[\Psi(f(\xi))-u])\|_{V'}\|\Psi(f(\xi))-u\|_V\,dt \\ &= \|\Psi(f(\xi))-u\|_V \int_0^1 \|M(u+t[\Psi(f(\xi))-u])\|_{V'}\,dt.\end{aligned}$$

Using (4.2) we have that

$$\begin{aligned}\|M(u+t[\Psi(f(\xi))-u])\|_{V'} &= \|M(u+t[\Psi(f(\xi))-u])-M(u)+M(u)\|_{V'} \\ &\le \|M(u)\|_{V'}+C_2\|t[\Psi(f(\xi))-u]\|_{V'}.\end{aligned}$$

Applying this to the above inequality, we deduce that

$$|G(u)-\Gamma(\xi)| \le \|\Psi(f(\xi))-u\|_V \int_0^1 \{\|M(u)\|_{V'}+tC_2\|[\Psi(f(\xi))-u]\|_{V'}\}\,dt. \tag{4.4}$$

Now, by (4.2) we have

$$\|\Psi(f(\xi))-u\|_V = \|\Psi(f(\xi))-\Psi(\Pi u+M(u))\|_V \le C_1\|M(u)\|_{V'}. \tag{4.5}$$

This, jointly with (4.4), leads to (4.3).

Observe that, provided that (4.3) holds, the proof will complete if we conclude that $\Gamma(\xi)=0$ for any ξ small enough.

For each $k=1,\dots,d$ from the chain rule we have

$$\frac{\partial\Gamma}{\partial\xi_k}(\xi) = \langle M(\Psi(f(\xi))),\Psi'(f(\psi))\varphi_k\rangle = (M(\Psi(f(\xi))),\Psi'(f(\psi))\varphi_k)_H, \tag{4.6}$$

where the last equality is due to the fact that $M(\Psi(f(\xi))) = f(\xi)-\Pi(\Psi(f(\xi))) \in N(L^{n_0}) \subset H \hookrightarrow V'$.

As $M(\Psi(f(\xi))) \in N(L^{n_0})$ one has

$$M(\Psi(f(\xi))) = \sum_{k=1}^d (M(\Psi(f(\xi))),\varphi_k)_H\varphi_k.$$

This, jointly with (4.6), implies that

$$\begin{aligned}&\left\|\sum_{k=1}^d \frac{\partial\Gamma}{\partial\xi_k}(\xi)\varphi_k - M(\Psi(f(\xi)))\right\|_{V'} \\ &\qquad= \left\|\sum_{k=1}^d [(M(\xi(f(\xi))),\Psi'(f(\xi))\varphi_k-\varphi_k)_H]\varphi_k\right\|_{V'} \\ &\qquad\le C_3\|M(\Psi(f(\xi)))\|_{V'}\sum_{k=1}^d \|\Psi'(f(\xi))\varphi_k-\varphi_k\|_V,\end{aligned}$$

for some constant $C_3 > 0$ depending only on the embedding $V \hookrightarrow H$ and on the choice of the elements $\{\varphi_1, \ldots, \varphi_d\}$ of $N(L^{n_0})$.

On the other hand, since $\Psi'(0) = \mathcal{L}^{-1}$ and $\mathcal{L}\varphi_k = \Pi\varphi_k + L\varphi_k = \varphi_k$, the above inequality yields

$$\bigg\| \sum_{k=1}^{d} \frac{\partial \Gamma}{\partial \xi_k}(\xi)\varphi_k - M(\Psi(f(\xi))) \bigg\|_{V'} \\ \leq C_3 \|M(\Psi(f(\xi)))\|_{V'} \|\Psi'(f(\xi)) - \Psi'(0)\|_{\mathcal{L}(V',V)} \sum_{k=1}^{d} \|\varphi_k\|_V.$$

From the continuity of $\Psi' \colon V' \to \mathcal{L}(V', V)$, we deduce the existence of a continuous function $\rho \colon \widetilde{W}_2(0) \to [0, \infty)$ with $\rho(0) = 0$ such that

$$\bigg\| \sum_{k=1}^{d} \frac{\partial \Gamma}{\partial \xi_k}(\xi)\varphi_k - M(\Psi(f(\xi))) \bigg\|_{V'} \leq C_3 \rho(\xi) \|M(\Psi(f(\xi)))\|_{V'}. \tag{4.7}$$

Therefore, taking a smaller open set $\widetilde{W}_2(0)$, if necessary, we may assume that

$$\|M(\Psi(f(\xi)))\|_{V'} \leq C \|\nabla\Gamma(\xi)\|, \tag{4.8}$$

for all $\xi \in \widetilde{W}_2(0)$ where $C > 0$ is some constant.

On the other hand, from the continuity of Ψ' and since $N(L^{n_0})$ was finite-dimensional, (4.7) also gives

$$\|\nabla\Gamma(\xi)\| \leq C_4 \|M(\Psi(f(\xi)))\|_{V'},$$

for certain constant $C_4 > 0$. Combining this and the above manipulations, it yields to

$$\begin{aligned} \|\nabla\Gamma(\xi)\| &\leq C_4 \|M(\Psi(f(\xi)))\|_{V'} \leq C_4 \|M(\Psi(f(\xi))) - M(u)\|_{V'} + C_4 \|M(u)\|_{V'} \\ &\leq C_5 \|\Psi(f(\xi)) - u\|_V + C_4 \|M(u)\|_{V'} \leq C_6 \|M(u)\|_{V'}. \end{aligned}$$

where we have used (4.5). Hence,

$$\|\nabla\Gamma(\xi)\| \leq C_6 \|M(u)\|_{V'}, \quad u \in \widetilde{W}_1(0),\ f(\xi) = \Pi u. \tag{4.9}$$

Claim 2.3. *The following equality holds*:

$$\{u \in \widetilde{W}_1(0) : M(u) = 0\} = \Psi(\{f(\xi) : \xi \in \widetilde{W}_2(0) \, and \, \nabla\Gamma(\xi) = 0\}). \tag{4.10}$$

Indeed, consider $u_0 \in \widetilde{W}_1(0)$ with $M(u_0) = 0$. Then $u_0 = \Psi(\mathcal{N}(u_0)) = \Psi(\Pi u_0)$ and, if $\xi_0 \in \widetilde{W}_2(0)$ is such that $f(\xi_0) = \Pi u_0$, then $u_0 = \Psi(f(\xi_0))$. Since (4.9) implies $\nabla\Gamma(\xi_0) = 0$, the inclusion

$$\{u \in \widetilde{W}_1(0) : M(u) = 0\} \subset \Psi(\{f(\xi) : \xi \in \widetilde{W}_2(0) \text{ and } \nabla\Gamma(\xi) = 0\})$$

holds.

For the opposite, consider $u_0 = \Psi(f(\xi_0)) \in W_1(0)$ with $\xi_0 \in \widetilde{W}_2(0)$ and $\nabla\Gamma(\xi_0) = 0$. Then, from (4.8) we have $M(u_0) = 0$, whence

$$\Pi u_0 = \Pi\Psi(f(\xi_0)) = \Pi\Psi(f(\xi_0)) + M(\Psi(f(\xi_0))) = \mathcal{N}(\Psi(f(\xi_0))) = f(\xi_0) \in W_2(0).$$

Therefore $u_0 \in \widetilde{W}_1(0)$ and the inclusion follows.

CLAIM 2.4. $\Gamma(\xi) = 0$ *for* ξ *small enough.* Now we take into account assumption (b).

Since $\{u \in \widetilde{W}_1(0) : M(u) = 0\} = \widetilde{W}_1(0) \cap M^{-1}(0)$ is an open set of $M^{-1}(0)$ with the topology induced by V, it implies that

$$\widetilde{U} = h^{-1}(\{u \in \widetilde{W}_1(0) : M(u) = 0\})$$

is an open set of $\mathbb{R}^d$ and

$$h(\widetilde{U}) = \{u \in \widetilde{W}_1(0) : M(u) = 0\} \subset \Psi(\{f(\xi) : \xi \in \widetilde{W}_2(0)\}) = (\Psi \circ f)(\widetilde{W}_2(0)).$$

Observe that $h(\widetilde{U})$ and $(\Psi \circ f)(\widetilde{W}_2(0))$ are topological manifolds of the same dimension (d), whence $h(\widetilde{U})$ must be an open subset of $(\Psi \circ f)(\widetilde{W}_2(0))$ (cf. Brouwer Domain Invariance Theorem in [12]). Hence, taking a smaller set $\widetilde{W}_2(0)$ if necessary, we may assume the following equalities

$$\{u \in \widetilde{W}_1(0) : M(u) = 0\} = h(\widetilde{U}) = (\Psi \circ f)(\widetilde{W}_2(0)).$$

This and (4.10) gives us

$$\Psi(\{f(\xi) : \xi \in \widetilde{W}_2(0) \text{ and } \nabla\Gamma(\xi) = 0\}) = \Psi(\{f(\xi) : \xi \in \widetilde{W}_2(0)\}),$$

whence $\nabla\Gamma(\xi) = 0$ for any $\xi \in \widetilde{W}_2(0)$. Without lost of generality, we may assume that $\widetilde{W}_2(0)$ is convex, so we deduce that Γ is constant in $\widetilde{W}_2(0)$ and, since $\Gamma(0) = 0$, it must be that $\Gamma(\xi) = 0$ for any $\xi \in \widetilde{W}_2(0)$, which concludes the proof taking (4.3) into account. □

REMARK 4.2. We observe that, we have used a spectral projection Π, in the proof of Theorem 4.1. This was done because it is not clear that the orthogonal projection onto the kernel of the operator $L: V \to V'$ commutes with $L: V \to V'$ (which may not be symmetric).

Under the assumptions of Theorem 4.1 we can prove the forwards and backwards convergence to equilibria. The forwards proof is standard, we include in the Appendix a proof of backwards convergence for the convenience of the reader.

Theorem 4.3. *Assume that the assumptions of Theorem* 4.1 *are fulfilled for any element of* $\mathcal{E} = \{z^* \in V : M(z^*) = 0\}$. *Consider a global solution* $u: \mathbb{R} \to V$ *of the problem*:

$$\begin{cases} u' + M(u) = 0, \\ u(0) = u_0 \in V, \end{cases} \tag{4.11}$$

such that the orbit $\gamma_u(u_0) = \{u(t) \in V : t \in \mathbb{R}\}$ *is relatively compact in* V. *Then, there are* $\varphi, \psi \in \mathcal{E}$ *such that*

$$\lim_{t\to-\infty} \|u(t) - \varphi\|_V = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|u(t) - \psi\|_V = 0.$$

Next we adapt, following [14], the previous results to deal with an abstract problem of second-order in time.

Consider $-A: D(A) \subset H \to H$ a sectorial operator in H, with compact resolvent.

Let $f: V \to H$ be a smooth enough map, $\beta > 0$ and consider the following semilinear problem of second order

$$\begin{cases} u'' + \beta u' = Au + f(u), \\ u(0) = u_0 \in V, \\ u'(0) = u_1 \in H. \end{cases} \tag{4.12}$$

We assume that there exists $G \in C^2(V; \mathbb{R})$ with $M := G': V \to V'$ satisfying for $u \in D(A)$

$$-M(u) = Au + f(u).$$

Problem (4.12) can be rewritten as a semilinear problem of first order (without lost of generality we assume $\beta = 1$ making if necessary the change of variables $t = s/\beta$):

$$\begin{cases} z' = Cz + f_0(z), \\ z(0) = z_0 \in Z, \end{cases} \tag{4.13}$$

where $z \in Z := V \times H$, equipped with the inner product

$$((v_1, v_2), (w_1, w_2))_Z := (v_1, w_1)_V + (v_2, w_2)_H,$$

$C: D(C) \subset Z \to Z$ being the linear operator with $D(C) := D(A) \times V$ given by

$$C := \begin{pmatrix} 0 & I \\ A & -I \end{pmatrix},$$

I the identity in V, and $f_0: Z \to Z$ defined by $f_0(z) := f_0(u, v) := (0, f(u))$, for $z = (u, v) \in Z = V \times H$.

With the above notation and assumptions, we have that the linear operator C generates a C_0 semigroup of contractions in Z and (4.13) has an associated nonlinear semigroup $\{S(t) : t \geq 0\}$ in Z as the solution operator.

With this, the following result holds (again, a proof of the backwards convergence to equilibrium is included in the Appendix).

Theorem 4.4. *Assume that the hypotheses of Theorem* 4.1 *are fulfilled for any* $\varphi \in \mathcal{E}_A := \{\varphi \in D(A) : -A\varphi = f(\varphi)\}$. *Then, if* $u\colon \mathbb{R} \to V$ *is a global* (*strong*) *solution for* (4.12) *with its orbit* $\{(u(t), u'(t)) : t \in \mathbb{R}\}$ *relatively compact in* $V \times H$, *and with* $\|M'(u(t))u'(t)\|_{V'} \le a\|u'(t)\|_{V'}$ *for all* $t \in \mathbb{R}$ *and some* $a > 0$, *then there exist* φ *and* $\psi \in \mathcal{E}_A$ *such that*:

$$\begin{aligned} \lim_{t\to-\infty} \|u(t) - \varphi\|_V &= 0, & \lim_{t\to-\infty} \|u'(t)\|_H &= 0, \\ \lim_{t\to\infty} \|u(t) - \psi\|_V &= 0, & \lim_{t\to\infty} \|u'(t)\|_H &= 0. \end{aligned}$$

5. Examples

In this section, we present some examples for which the results in Section 4 can be applied. Observe that we can couple any of these examples with a gradient-like problem such that we may take advantage of Theorem 3.4 to describe the attractor of the semigroup associated to a coupled system in cascade. In this way, we also present some examples of coupled system (2.1) to illustrate the results in Section 3.

There are a large variety of examples of the form

$$x' = Ax + f(x), \qquad x(0) = x_0 \in X$$

arising in partial differential equations such that X is a Hilbert space, for each $x_0 \in X$ the solution $x(t, x_0)$ of the above initial value problem is defined for all $t \ge 0$ and the semigroup $\{S(t) : t \ge 0\}$ defined by $S(t)x_0 := x(t, x_0)$, $t \ge 0$, $x_0 \in X$ has a global attractor, is gradient and for which either the set of equilibria of the uncoupled equation is finite or, otherwise, the Łojasiewicz–Simon inequality is satisfied. With this, $\{T(t) : t \ge 0\}$ (as in Lemma 3.2, see also equation (2.2) is gradient-like in the sense of Definition 2.13.

Our first example is a simple system with one-sided coupling for which we can apply the abstract theory developed here to conclude the gradient-like structure. We note that, in this example, the nonlinearity is not assumed to be analytic (C^2 is enough) enhancing the applicability of Theorem 4.1.

For the other examples, we can assume that, either the set of equilibria for the uncoupled equation is finite or, otherwise, the nonlinear term in it is analytic to use the Theorem 4.1 (for the analytic case, see [23]).

Example 5.1. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a $C^2(\mathbb{R})$ function such that $g(s) \ne 0$ for all $s \in \mathbb{R}$ and consider the function $G\colon \mathbb{R}^2 \to \mathbb{R}$ given by $G(s,t) := t^2 g(s)$. It follows that $G \in C^2(\mathbb{R}^2, \mathbb{R})$ with $M(s,t) := G'(s,t) = (t^2 g'(s), 2tg(s))$ and

$$G''(s,t) = \begin{pmatrix} t^2 g''(s) & 2tg'(s) \\ 2tg'(s) & 2g(s) \end{pmatrix} \quad \text{for all } (s,t) \in \mathbb{R}^2,$$

particularly, $G''(s,t)$ is a symmetric matrix for all s and t, so the orthogonal projection coincides with the spectral one.

It is clear that $M^{-1}(0) = \mathbb{R} \times \{0\}$ and therefore, since

$$G''(s,0) = \begin{pmatrix} 0 & 0 \\ 0 & 2g(s) \end{pmatrix},$$

we have that $\dim \ker(G''(x^*)) = 1$ for each $x^* \in M^{-1}(0)$.

Now, the map $h\colon \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $h(s) := (s,0)$ is a homeomorphism between $\mathbb{R}$ and $M^{-1}(0)$. Thus, the map G satisfies hypothesis (b) in Theorem 4.1.

Let Y be a Banach space and $f\colon \mathbb{R}^2 \times Y \to Y$ a smooth map such that for every bounded set $B \subset \mathbb{R}^2$ the restriction of f to $B \times Y$ is bounded.

Also, let $-A\colon D(A) \subset Y \to Y$ be a sectorial operator such that the analytic semigroup generated for it, $\{e^{At} : t \geq 0\}$, satisfies $\|e^{At}\|_{\mathcal{L}(Y)} \leq Me^{-\delta t}$ for every $t > 0$ and some constants $M \geq 1$ and $\delta > 0$.

Under these assumptions, by using the results from [3] on existence of pullback attractors, it is not difficult to see that, for any bounded solution $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ of $x' + M(x) = 0$, the map $f_\varphi\colon \mathbb{R} \times Y \to Y$, given by $f_\varphi(t,y) := f(\varphi(t), y)$, is compatible from the left and from the right respect to the system

$$\begin{cases} y' = Ay + f_\varphi(t,y), & t > \tau, \\ y(\tau) = y_0 \in Y. \end{cases}$$

If for every equilibrium point $x^* \in M^{-1}(0)$, the equilibria of the problem $y' = Ay + f(x^*, y)$ $t > 0$ are all hyperbolic, then

$$\begin{cases} x' + M(x) = 0, & t > 0, \\ y' = Ay + f(x,y), & t > 0, \\ x(0) = x_0 \in \mathbb{R}^2, \quad y(0) = y_0 \in Y, \end{cases}$$

is gradient-like.

The next examples are applications of Theorem 4.1. More exactly, we consider applications to differential systems and to PDE, in particular, the reaction-diffusion equation, the damped wave equation and the Cahn–Hilliard equation.

Example 5.2. Let $\phi \in C^2(\mathbb{R}^n, \mathbb{R})$ be such that $\phi(u) \to \infty$ as $\|u\|_{\mathbb{R}^n} \to \infty$ and consider the ordinary differential equations:

$$x' = -\nabla\phi(x), \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{5.1}$$

or, for $\beta > 0$,

$$x'' + \beta x' = -\nabla\phi(x), \qquad x(0) = x_0 \in \mathbb{R}^n, \qquad x'(0) = v_0 \in \mathbb{R}^n. \tag{5.2}$$

Then, (5.1) and (5.2) have global attractors $\mathcal{A}_1$ in $\mathbb{R}^n$ and $\mathcal{A}_2$ in $\mathbb{R}^n \times \mathbb{R}^n$, respectively. If $\mathcal{E} = \{x \in \mathbb{R}^n : \nabla\phi(x) = 0\}$, it follows from the Łojasiewicz–Simon condition (which we assume) and from the results in Section 4 that

$$\mathcal{A}_1 = \bigcup_{x\in\mathcal{E}} W^u(x) \quad \text{and} \quad \mathcal{A}_2 = \bigcup_{x\in\mathcal{E}} W^u \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

We remark that we are not assuming that the set of equilibria $\mathcal{E}$ is finite and without the Łojasiewicz–Simon condition we only have that

$$\mathcal{A}_1 = W^u(\mathcal{E}) \quad \text{and} \quad \mathcal{A}_2 = W^u \begin{pmatrix} \mathcal{E} \\ 0 \end{pmatrix}.$$

Example 5.3. Let $f \in C^2(\mathbb{R},\mathbb{R})$ be such that

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \le -\delta < 0 \tag{5.3}$$

and

$$|f''(u)| \le c(1 + |u|^p) \tag{5.4}$$

with $p \in (0,\infty)$ to be specified. Consider the initial boundary value problems

$$\begin{aligned} u_t &= \Delta u + f(u), && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0, && \text{in } \partial\Omega, \\ u(0) &= u_0 \in H^1(\Omega), \end{aligned} \tag{5.5}$$

with $p + 2 \le (n+2)/(n-2)$ and $n \le 6$ or, for $\beta > 0$,

$$\begin{aligned} u_{tt} + \beta u_t &= \Delta u + f(u), && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0, && \text{in } \partial\Omega, \\ u(0) &= u_0 \in H^1(\Omega), \quad u_t(0) = v_0 \in L^2(\Omega), \end{aligned} \tag{5.6}$$

with $p + 2 \le n/(n-2)$ and $n \le 4$.

Then, it is well-known that (5.5) and (5.6) have global attractors $\mathcal{A}_3$ in $H^1(\Omega)$ and $\mathcal{A}_4$ in $H^1(\Omega)\times L^2(\Omega)$ respectively (see [3]). If $\widetilde{\mathcal{E}} = \{u \in H^2(\Omega) : \Delta u + f(u) = 0 \text{ and } \partial u/\partial n = 0\}$, it follows from the Łojasiewicz–Simon condition (which we assume) and from the results in Section 4 that

$$\mathcal{A}_3 = \bigcup_{u\in\widetilde{\mathcal{E}}} W^u(u) \quad \text{and} \quad \mathcal{A}_4 = \bigcup_{u\in\widetilde{\mathcal{E}}} W^u \begin{pmatrix} u \\ 0 \end{pmatrix}.$$

We remark that we are not assuming that the set of equilibria $\widetilde{\mathcal{E}}$ is finite.

Example 5.4. Let $f \in C^2(\mathbb{R},\mathbb{R})$ be such that (5.3) and (5.4) with $p+2 \leq (n+2)/(n-2)$, $n \leq 6$ hold. Consider the initial boundary value problem

$$\begin{aligned} u_t &= -\Delta(\Delta u + f(u)), \quad && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= \frac{\partial \Delta u}{\partial n} = 0, && \text{in } \partial\Omega, \\ u(0) &= u_0 \in H^1(\Omega). \end{aligned} \tag{5.7}$$

Then (5.7) has a global attractor $\mathcal{A}_5$ in $H^1(\Omega)$ (see [4]). If $\overline{\mathcal{E}} = \{u \in H^4(\Omega) : \Delta u + f(u) = 0,\ \partial u/\partial n = 0 \text{ and } \partial \Delta u/\partial n = 0\}$, it follows from the Łojasiewicz–Simon condition (which we assume) and from the results in Section 4 that

$$\mathcal{A}_5 = \bigcup_{u \in \overline{\mathcal{E}}} W^u(u).$$

We remark that the set of equilibria $\bar{\mathcal{E}}$ is always infinite.

Finally, we consider the situation of cascade system (2.1). The following examples are applications of Theorem 3.4.

Example 5.5. Let $\phi \in C^2(\mathbb{R}^n,\mathbb{R})$ and $f \in C^1(\mathbb{R},\mathbb{R})$ be such that $\phi(u) \to \infty$ as $\|u\|_{\mathbb{R}^n} \to \infty$, $f(s) + x_0 \cdot x^* = 0 \Rightarrow f'(s) + x_0 \cdot x^* \neq 0$ for all $x^* \in \{x \in \mathbb{R}^n : \nabla\phi(x) = 0\}$ and f satisfies (5.3). Consider the ordinary differential equations:

$$\begin{aligned} x' &= -\nabla\phi(x), \\ s' &= f(s) + x_0 \cdot x, \\ s(0) &= s_0 \in \mathbb{R}, \quad x(0) = x_0 \in \mathbb{R}^n, \end{aligned} \tag{5.8}$$

or, for $\beta > 0$,

$$\begin{aligned} x'' + \beta x' &= -\nabla\phi(x), \\ s'' + \beta s' &= f(s) + x_0 \cdot x, \\ s(0) = s_0 \in \mathbb{R}, \quad x(0) &= x_0 \in \mathbb{R}^n, \quad x'(0) = v_0 \in \mathbb{R}^n. \end{aligned} \tag{5.9}$$

Then, (5.8) and (5.9) have global attractors $\mathcal{A}_6$ in $\mathbb{R}^n \times \mathbb{R}$ and $\mathcal{A}_7$ in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, respectively. If $\hat{\mathcal{E}} = \{(x^*, s) \in \mathbb{R}^n \times \mathbb{R} : \nabla\phi(x^*) = 0 \text{ and } f(s) + x_0 \cdot x^* = 0\}$, it follows from the Łojasiewicz–Simon condition (which we assume for $x' = -\nabla\phi(x)$ or $x'' + \beta x' = -\nabla\phi(x)$) and from the results in Sections 3 and 4 that

$$\mathcal{A}_6 = \bigcup_{z \in \hat{\mathcal{E}}} W^u(z) \quad \text{and} \quad \mathcal{A}_7 = \bigcup_{z \in \hat{\mathcal{E}}} W^u \begin{pmatrix} z \\ 0 \end{pmatrix}.$$

In addition, all solutions in $\mathcal{A}_6$ (or $\mathcal{A}_7$) are forwards asymptotic to equilibria and $\mathcal{A}_6$ (or $\mathcal{A}_7$) does not contain homoclinic structures.

The last example corresponds to the limiting problem of reaction-diffusion problems in a dumbbell domain (see [2]).

Example 5.6. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and $P_0, P_1 \in \bar{\Omega}$. Assume that $f, g \in C^2(\mathbb{R}, \mathbb{R})$, f satisfies (5.4) with $p + 2 \le (n+2)/(n-2)$ and that both f and g satisfy (5.3). Consider the initial boundary value problem:

$$\begin{aligned} u_t &= \Delta u + f(u), && \text{in } \Omega,\\ \frac{\partial u}{\partial n} &= 0, && \text{in } \partial\Omega,\\ u(0) &= u_0 \in C(\overline{\Omega}), &&\\ v_t &= v_{xx} + g(v), && \text{in } (0,1),\\ v(t,0) &= u(t,P_0), \quad v_x(t,1) = 0, &&\\ v(0,\,\cdot\,) &= v_0 \in H^1(0,1). && \end{aligned} \tag{5.10}$$

Then, the initial boundary value problem (5.10) defines a semigroup $\{S(t) : t \geq 0\}$ in $C(\overline{\Omega}) \times H^1(0,1)$ which has a global attractor $\mathcal{A}_8$.

Assume that for all $c \in [\alpha, \omega]$ with $\alpha = \inf\{\xi : f(\xi) = 0\}$ and $\omega = \sup\{\xi : f(\xi) = 0\}$, the solutions of

$$\begin{aligned} v_{xx} + g(v + c) &= 0, \quad \text{in } (0,1),\\ v(0) = v_x(1) &= 0, \end{aligned} \tag{5.11}$$

are hyperbolic (which may be accomplished, for example, if $|\alpha|$ and $|\omega|$ are suitably small as in [9]). If the Łojasiewicz–Simon condition holds for the equation in Ω, it follows from Theorem 3.4 that the solutions in $\mathcal{A}_8$ are backwards and forwards asymptotic to equilibria.

6. Appendix

This appendix is dedicated to the proof of Theorems 4.3 and 4.4 in the case $t \to -\infty$. To do that, we will make use the following two technical lemmas, which helpful in the proof of convergence to an equilibrium point when the Łojasiewicz–Simon inequality holds.

Lemma 6.1. *Let $u\colon (-\infty, 0] \to H$ be differentiable. Assume that there exist constants $a > 0$ and $\gamma > 0$ such that for some $T > 0$ and all $t \in [-T, 0]$,*

$$\int_{-\infty}^{t} \|u'(s)\|_H^2 \, ds \le a e^{\gamma t}. \tag{6.1}$$

Then, for $-T \le \tau \le t \le 0$,

$$\|u(t) - u(\tau)\|_H \le b\sqrt{a} e^{\gamma t/2},$$

where $b = e^{\gamma/2}/(e^{\gamma/2} - 1)$.

Proof. Consider $-T \le \tau \le t \le 0$. First assume that $t-\tau \le 1$. Then, from Hölder inequality and (6.1), one has

$$\|u(t)-u(\tau)\|_H = \bigg\| \int_\tau^t u'(s)\,ds \bigg\|_H \le \sqrt{t-\tau}\bigg(\int_\tau^t \|u'(s)\|_H^2\,ds\bigg)^{1/2} \le \sqrt{a}e^{\gamma t/2}.$$

Now, if $t-\tau > 1$, let n be the least natural number such that $(t-\tau)/n \le 1$. Then, from above we deduce

$$\begin{aligned}\|u(t)-u(\tau)\|_H &\le \int_\tau^{t-n+1} \|u'(s)\|_H\,ds + \sum_{k=0}^{n-2}\int_{t-k-1}^{t-k} \|u'(s)\|_H\,ds \\ &\le \sqrt{a}e^{\gamma/2(t-n+1)} + \sum_{k=0}^{n-2}\sqrt{a}e^{\gamma(t-k)/2} \le \sqrt{a}e^{\gamma t/2}\bigg[\frac{e^{\gamma/2}}{e^{\gamma/2}-1}\bigg],\end{aligned}$$

which concludes the proof. □

The next result is analogous to the above lemma for $\mathbb{R}^+$ instead of $\mathbb{R}^-$. Its proof can be found in [14, Lemma 2.2].

Lemma 6.2. *Let $u\colon[0,\infty) \to H$ be differentiable. Assume that there exist constants $a>0$ and $\gamma>0$ such that for some $T>0$ and all $t\in[0,T]$,*

$$\int_t^\infty \|u'(s)\|_H^2\,ds \le ae^{-\gamma t}.$$

Then, for $0 \le t \le \tau \le T$,

$$\|u(t)-u(\tau)\|_H \le b\sqrt{a}e^{-\gamma t/2},$$

where $b = e^{\gamma/2}/(e^{\gamma/2}-1)$.

Proof of Theorem 4.3. We only sketch the proof of the case $t \to -\infty$, since it is similar to the case $t\to\infty$ which is proven in [14, Theorem 1.1].

Observe that for any global (strong) solution $\xi\colon\mathbb{R}\to V$ of (4.11), the following inequality holds for all $t\in\mathbb{R}$:

$$\frac{d}{dt}(G\circ\xi)(t) = \langle M(\xi(t)),\xi'(t)\rangle = -\|\xi'(t)\|_H^2 = -\|M(\xi(t))\|_H^2 \le 0.$$

Therefore, the function $\mathbb{R}\ni t\mapsto G(\xi(t))\in\mathbb{R}$ is non-increasing. From this and the fact that $\frac{d}{dt}(G\circ\xi)(t) = -\|\xi'(t)\|_H^2$, we conclude that $G\colon V\to\mathbb{R}$ is a Lyapunov functional for the nonlinear semigroup in V associated to (4.11). Then, the α-limit of u_0 relative to a global solution u, that is,

$$\alpha_u(u_0) := \Big\{\varphi\in V : \exists\,(t_n)_{n\in\mathbb{N}},\ t_n\to-\infty \text{ such that } \lim_{n\to\infty}\|u(t_n)-\varphi\|_V = 0\Big\},$$

is a subset of $\mathcal{E}$, which is nonempty from the relatively compactness of $\gamma_u(u_0)$.

Consider a point $\varphi\in\alpha_u(u_0)$. Then, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $t_n\to-\infty$ such that $\lim_{n\to\infty}\|u(t_n)-\varphi\|_V = 0$. Hence, $\lim_{n\to\infty} G(u(t_n)) = G(\varphi)$

and from the monotonicity of $\mathbb{R} \ni t \mapsto G(u(t)) \in \mathbb{R}$ it holds that $\lim_{t\to-\infty} G(u(t)) = G(\varphi)$ with $G(u(t)) \le G(\varphi)$ for all $t \in \mathbb{R}$.

As before we have that

$$\text{(6.2)} \quad \frac{d}{dt}[(G\circ u)(t) - G(\varphi)] = \langle M(u(t)), u'(t)\rangle = -\|u'(t)\|_H^2 = -\|M(u(t))\|_H^2.$$

So, for a fixed value $t < 0$, integrating on $(-\infty, t)$, we have

$$\text{(6.3)} \qquad \int_{-\infty}^{t} \|u'(s)\|_H^2\, ds = [G(\varphi) - G(u(t))].$$

On the other hand, for each natural number j we take $n_j \in \mathbb{N}$ strictly increasing such that

$$\text{(6.4)} \qquad \|u(t_{n_j}) - \varphi\|_V < \frac{1}{j} \quad \text{and} \quad [G(\varphi) - G(u(t_{n_j}))]^{1/2} < \frac{1}{j}.$$

Choose $\sigma > 0$ as in Theorem 4.1 associated to the equilibrium point φ and j_0 such that $1/j < \sigma$ if $j \ge j_0$. Define for $j \ge j_0$

$$\bar{t}_j := \inf\{\tau < t_{n_j} : \|u(t) - \varphi\|_V < \sigma \text{ for all } t \in [\tau, t_{n_j}]\}.$$

From the Łojasiewicz–Simon inequality and the continuous injection of H in V' we deduce the existence of a constant $\gamma > 0$ depending on φ such that (6.2) implies, for $t \in [\bar{t}_j, t_{n_j}]$, that

$$\frac{d}{dt}[(G\circ u)(t) - G(\varphi)] \le \gamma[G(u(t)) - G(\varphi)].$$

This yields

$$\frac{d}{dt}\{e^{-\gamma t}[(G\circ u)(t) - G(\varphi)]\} \le 0.$$

Integrating over $(\bar{t}_j, t_{n_j})$ we obtain

$$e^{-\gamma t_{n_j}}[G(u(t_{n_j})) - G(\varphi)] - e^{-\gamma t}[G(u(t)) - G(\varphi)] \le 0,$$

which, combined with (6.3), implies that

$$\text{(6.5)} \qquad \int_{-\infty}^{t} \|u'(s)\|_H^2\, ds \le e^{\gamma t}\{e^{-\gamma t_{n_j}}[G(\varphi) - G(u(t_{n_j}))]\},$$

for all $t \in (\bar{t}_j, t_{n_j})$, $j \ge j_0$.

Claim. *There exists a value $j_1 \ge j_0$ such that $\bar{t}_{j_1} = -\infty$.*

If $G(\varphi) = G(u(t_{n_j}))$ for some j the claim is clearly true. On the other hand, if $G(\varphi) > G(u(t_{n_j}))$ and $\bar{t}_j > -\infty$, for all $j \ge j_0$, then, taking into account (6.5), we may apply Lemma 6.1 with $\tau = \bar{t}_j$, $t = t_{n_j}$ and $a = e^{-\gamma t_{n_j}}[G(\varphi) - G(u(t_{n_j}))]$ and conclude that

$$\|u(\bar{t}_j) - u(t_{n_j})\|_H < b e^{\gamma/2(\bar{t}_j - t_{n_j})}\sqrt{[G(\varphi) - G(u(t_{n_j}))]} < \frac{b}{j},$$

for all $j \ge j_0$, where $b > 0$ is a constant depending on γ.

From (6.4) and the continuous embedding $V \hookrightarrow H$, it holds that $\|u(t_{n_j}) - \varphi\|_H < C_0/j$ for all $j \geq j_0$ and some constant C_0.

From above we have that, for all $j \geq j_0$,

$$\|u(\overline{t}_j) - \varphi\|_H \leq \|u(\overline{t}_j) - u(t_{n_j})\|_H + \|u(t_{n_j}) - \varphi\|_H < \frac{b}{j} + \frac{C_0}{j},$$

hence $\lim_{j\to\infty} \|u(\overline{t}_j) - \varphi\|_H = 0$. This also means, from the relatively compactness of $\gamma_u(u_0)$ in V that for a subsequence (relabelled the same) $(u(\overline{t}_j))_{j\in\mathbb{N}}$ converges to φ in V. In particular, there exists a value j^* such that

$$\|u(\overline{t}_{j^*}) - \varphi\|_V < \sigma,$$

which is a contradiction with the definition of $\overline{t}_{j^*}$. Therefore, there must exists a $j_1 \geq j_0$ such that $\overline{t}_j = -\infty$ for all $j \geq j_1$ and the claim is proved.

Now, from (6.5),

$$\int_{-\infty}^{t} \|u'(s)\|_H^2\, ds \leq e^{\gamma t}\{e^{-\gamma t_{n_j}}[G(\varphi) - G(u(t_{n_j}))]\} \quad \text{for all } t \in (-\infty, t_{n_j}),\ j \geq j_1.$$

Lemma 6.1 implies that, for any $j \geq j_1$ and $t \in (-\infty, t_{n_j})$,

$$\|u(t) - u(t_{n_j})\|_H < be^{\gamma(t-t_{n_j})/2}\sqrt{[G(\varphi) - G(u(t_{n_j}))]} < \frac{b}{j}.$$

Therefore,

$$\|u(t) - \varphi\|_H \leq \|u(t) - u(t_{n_j})\|_H + \|u(t_{n_j}) - \varphi\|_H < \frac{b}{j} + \frac{C_0}{j}.$$

From this we have that $\lim_{t\to-\infty} \|u(t) - \varphi\|_H = 0$. The relatively compactness of $\gamma_u(u_0)$ in V implies that $\lim_{t\to-\infty} \|u(t) - \varphi\|_V = 0$, completing the proof. □

Proof of Theorem 4.4. As in the previous section, we will pay only attention to the case $t \to -\infty$, being the case $t \to \infty$ analogous.

Define the continuous functional $E: V \times H \to \mathbb{R}$ by

$$E(u_0, u_1) := \frac{1}{2}\|u_1\|_H^2 + G(u_0).$$

It is not difficult to check that E is a Lyapunov functional for the semigroup $\{S(t) : t \geq 0\}$.

From the relatively compactness of the orbit of $u : \mathbb{R} \to V$ we have that the α-limit of u,

$$\alpha(u) = \Big\{(z, w) \in V \times H : \exists\, t_n \to -\infty,\ \lim_{n\to\infty} \|u(t_n) - z\|_V = 0 \\ \text{and } \lim_{n\to\infty} \|u'(t_n) - w\|_H = 0\Big\}$$

is nonempty, compact, invariant, and attracts the solution (u, u') in $V \times H$ when $t \to -\infty$, Moreover, being $S(\,\cdot\,)$ a gradient semigroup, $\alpha(u)$ is contained in $\mathcal{E}^* = \{(z^*, w^*) \in V \times H : S(t)(z^*, w^*) = (z^*, w^*) \text{ for all } t \geq 0\}$.

As far as equilibrium points of problem (4.13) are also strong solutions, then any $(z^*, w^*) \in \mathcal{E}^*$ must satisfy $w^* = 0$ and $M(z^*) = 0$, i.e. $\mathcal{E}^* \subset \mathcal{E}_A \times \{0\}$. Moreover, the equality $\mathcal{E}^* = \mathcal{E}_A \times \{0\}$ holds since the other inclusion $\mathcal{E}^* \supset \mathcal{E}_A \times \{0\}$ is always true. In particular, this implies that $\lim_{t\to-\infty} \|u'(t)\|_H = 0$.

Now we prove that there exists a unique $\varphi \in \mathcal{E}_A$ such that $\alpha(u) = \{(\varphi, 0)\}$, which will conclude the proof.

Consider a positive value $\varepsilon > 0$, which will be fixed later on. Define for $t \leq 0$

$$E_\varepsilon(t) := E(u(t),\ u'(t)) + \varepsilon(M(u(t)), u'(t))_{V'}.$$

Since a strong solution satisfies (4.12) in a classic sense, we can derive and obtain

$$\begin{aligned} E'_\varepsilon(t) =& -\|u'(t)\|^2_H + \varepsilon[(M'(u(t))u'(t), u'(t))_{V'} + (M(u(t)), u''(t))_{V'}] \\ \leq& -\|u'(t)\|^2_H + \varepsilon[\|M'(u(t))u'(t)\|_{V'}\|u'(t)\|_{V'} \\ & - \|M(u(t))\|^2_{V'} + \|M(u(t))\|_{V'}\|u'(t)\|_{V'}]. \end{aligned}$$

From the relatively compactness of the orbit of u, the hypothesis

$$\|M'(u(t))u'(t)\|_{V'} \leq a\|u'(t)\|_{V'} \quad \text{for all } t \in \mathbb{R}$$

and the continuity of the injection $H \hookrightarrow V'$, one can estimate the right hand side above by

$$\begin{aligned} -\|u'(t)\|^2_H + \varepsilon\bigg[C\|u'(t)\|^2_H - \|M(u(t))\|^2_{V'} + \frac{1}{2}\|M(u(t))\|^2_{V'} + \frac{C}{2}\|u'(t)\|^2_H\bigg] \\ = \bigg[\varepsilon\bigg(C + \frac{C}{2}\bigg) - 1\bigg]\|u'(t)\|^2_H - \frac{\varepsilon}{2}\|M(u(t))\|^2_{V'}, \end{aligned}$$

for some constant $C > 0$ depending on u. Therefore, we conclude

$$E'_\varepsilon(t) \leq \bigg[\varepsilon\bigg(C + \frac{C}{2}\bigg) - 1\bigg]\|u'(t)\|^2_H - \frac{\varepsilon}{2}\|M(u(t))\|^2_{V'}. \tag{6.6}$$

So, fixing $\varepsilon > 0$ small enough, we obtain that for a constant $C_2 > 0$ and all $t \leq 0$, it holds

$$\frac{d}{dt}[E_\varepsilon(t) - G(\varphi)] \leq -C_2[\|u'(t)\|^2_H + \|M(u(t))\|^2_{V'}]. \tag{6.7}$$

In particular, $E_\varepsilon\colon (-\infty, 0] \to \mathbb{R}$ is non-increasing. By the relatively compactness of the orbit of u, E_ε is bounded from below, thus it exists the limit $\lim_{t\to-\infty} E_\varepsilon(t)$.

Now, take any point $(\varphi, 0) \in \alpha(u)$, so there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to -\infty$ and $\lim_{n\to\infty} \|u(t_n) - \varphi\|_V = 0$ and, from above, it also holds

$$\lim_{n\to\infty} E_\varepsilon(t_n) = \lim_{n\to\infty} \bigg[\frac{1}{2}\|u'(t_n)\|^2_H + G(u(t_n)) + \varepsilon(M(u(t_n)), u'(t_n))_{V'}\bigg] = G(\varphi).$$

Actually, we deduce that $\lim_{t\to-\infty} E_\varepsilon(t) = G(\varphi)$ with $E_\varepsilon(t) \le G(\varphi)$ for all $t \le 0$.

On the other hand, from (6.7) there exists some constant $\widetilde{C} > 0$ such that

$$E'_\varepsilon(t) \le -\frac{1}{\widetilde{C}}\|u'(t)\|_H^2,$$

whence integrating over $(-\infty, t)$, for $t \in (-\infty, 0)$, it turns out

$$\int_{-\infty}^t \|u'(s)\|_H^2\,ds \le \widetilde{C}[G(\varphi) - E_\varepsilon(t)]. \tag{6.8}$$

Now we take a sequence $(n_j)_{j\in\mathbb{N}}$ as in Theorem 4.3 with

$$\|u(t_{n_j}) - \varphi\|_V < \frac{1}{j} \quad \text{and} \quad [G(\varphi) - E_\varepsilon(t_{n_j})]^{1/2} < \frac{1}{j}.$$

Analogously, take $\sigma > 0$ associated to the equilibrium point φ according to the statement of Theorem 4.1; fix $j_0 \in \mathbb{N}$ such that $1/j_0 < \sigma$ and define for $j \ge j_0$

$$\bar{t}_j = \inf\{\tau < t_{n_j} : \|u(t) - \varphi\|_V < \sigma \text{ for all } t \in [\tau, t_{n_j}]\}.$$

Then, for each $j \ge j_0$ and $t \in [\bar{t}_j, t_{n_j}]$, the Łojasiewicz–Simon inequality gives

$$\begin{aligned} E_\varepsilon(t) - G(\varphi) &= \frac{1}{2}\|u'(t)\|_H^2 + \varepsilon(M(u(t)), u'(t))_{V'} + [G(u(t)) - G(\varphi)] \\ &\ge \frac{1}{2}\|u'(t)\|_H^2 + \varepsilon(M(u(t)), u'(t))_{V'} - c\|M(u(t))\|_{V'}^2, \end{aligned} \tag{6.9}$$

which, jointly with Young inequality with some $\delta > 0$, implies

$$E_\varepsilon(t) - G(\varphi) \ge \frac{1}{2}\|u'(t)\|_H^2 - c\|M(u(t))\|_{V'}^2 - \frac{\varepsilon}{2\delta}\|M(u(t))\|_{V'}^2 - \frac{\varepsilon\delta}{2}\|u'(t)\|_{V'}^2.$$

Taking a value $\delta > 0$ big enough, we obtain the existence of some constant $C_1 > 0$ such that

$$E_\varepsilon(t) - G(\varphi) \ge -C_1[\|u'(t)\|_H^2 + \|M(u(t))\|_{V'}^2], \quad \text{for all } t \in [\bar{t}_j, t_{n_j}]. \tag{6.10}$$

This, combined with (6.7), implies that for $t \in [\bar{t}_j, t_{n_j}]$

$$\frac{d}{dt}[E_\varepsilon(t) - G(\varphi)] \le \gamma[E_\varepsilon(t) - G(\varphi)],$$

with $\gamma := C_2/C_1$. Now we just have to argue in the same way as in the proof of Theorem 4.3, to obtain that for all $t \in [\bar{t}_j, t_{n_j}]$

$$[G(\varphi) - E_\varepsilon(t)] \le [G(\varphi) - E_\varepsilon(t_{n_j})]e^{\gamma(t - t_{n_j})}.$$

This, combined with (6.8), implies that for $t \in [\bar{t}_j, t_{n_j}]$, it holds that

$$\int_{-\infty}^t \|u'(s)\|_H^2 ds \le \widetilde{C}\left[G(\varphi) - E_\varepsilon(t_{n_j})\right] e^{\gamma(t - t_{n_j})}.$$

Now we can apply Lemma 6.1 and conclude the proof similarly as done for Theorem 4.3. □

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References

[1] E.R. Aragao-Costa, T. Caraballo, A.N. Carvalho and J.A. Langa, *Stability of gradient semigroups under perturbations*, Nonlinearity **24** (2011), 2099–2117.

[2] J.M. Arrieta, A.N. Carvalho and G. Lozada-Cruz, *Dynamics in dumbbell domains* II. *The limiting problem*, J. Differential Equations **247** (2009), 174–202.

[3] T. Caraballo, A.N. Carvalho, J.A. Langa and L.F. Rivero, *Existence of pullback attractors for pullback asymptotically compact processes*, Nonlinear Anal. **72** (2010), 1967–1976.

[4] A.N. Carvalho and T. Dlotko, *Dynamics of the viscous Cahn–Hilliard Equation*, J. Math. Anal. Appl. **344** (2008), 703–325.

[5] A.N. Carvalho and J.A. Langa, *Non-autonomous perturbation of autonomous semilinear differential equations: Continuity of local stable and unstable manifolds*, J. Differential Equations **233** (2007), 622–653.

[6] ———, *An extension of the concept of gradient semigroups which is stable under perturbation*, J. Differential Equations **246** (2009), 2646–2668.

[7] A.N. Carvalho, J.A. Langa and J.C. Robinson, *Lower semicontinuity of attractors for non-autonomous dynamical systems*, Ergodic Theory Dynam. Systems **29** (2009), 1765–1780.

[8] A.N. Carvalho, J.A. Langa, J.C. Robinson and A. Suárez, *Characterization of non-autonomous attractors of a perturbed infinite-dimensional gradient system*, J. Differential Equations **236** (2007), 570–603.

[9] N. Chafee and E.F. Infante, *A bifurcation problem for a nonlinear partial differential equation of parabolic type*, Appl. Anal. **4** (1974), 17–37.

[10] R. Chill, *On the Łojasiewicz–Simon gradient inequality*, J. Funct. Anal. **201** (2003), 572–601.

[11] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics, vol. 38, Amer. Math. Soc., Providence, R.I., 1978.

[12] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1965.

[13] J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, Math. Surveys Mongr., vol. 25, Amer. Math. Soc., 1989.

[14] A. Haraux and M.A. Jendoubi, *On the convergence of global and bounded solutions of some evolution equations*, J. Evol. Equ. **7** (2007), 449–470.

[15] S. Łojasiewicz, *Une propiété topologique des sous-ensembles analytiques réels*, Colloques Internationaux du C.N.R.S., Les Équations aux Dérivées Partielles (Paris, 1962), vol. 117, 1963, pp. 87–89.

[16] ———, *Ensembles semi-analytiques*, I.H.E.S. notes (1965).

[17] J. Palis and W. de Melo, *Geometric Theory of Dynamical Systems. An Introduction*, Springer–Verlag, New York–Berlin, 1982.

[18] P. Poláčik and F. Simondon, *Nonconvergent bounded solutions of semilinear heat equations on arbitrary domains*, J. Differential Equations **186** (2002), 586–610.

[19] J. Prüss, G. Simonett and R. Zacher, *On convergence of solutions to equilibria for quasilinear parabolic problems*, J. Differential Equations **246** (2009), 3902–3931.

[20] ______, *On normal stability for nonlinear parabolic equations*, Discrete Contin. Dyn. Syst. Supplement (2009), 612–621.

[21] J.C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge Univ. Press, Cambridge, UK, 2001.

[22] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math. **118** (1983), 525–571.

[23] ______, *Theorems on regularity and singularity of energy minimizing maps*, Lect. in Math., ETH Zürich, Birkhäuser, 1996.

[24] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed., Springer–Verlag, Berlin, 1996.

[25] S. Zheng, *Nonlinear Evolution Equations*, Chapman & Hall/CRC, Boca Raton, FL, 2004.

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