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ON CONVERGENCE AND COMPACTNESS IN PARABOLIC PROBLEMS WITH GLOBALLY LARGE DIFFUSION AND NONLINEAR BOUNDARY CONDITIONS

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Abstract. We establish some abstract convergence and compactness results for families of singularly perturbed semilinear parabolic equations and apply them to reaction-diffusion equations with nonlinear boundary conditions and large diffusion. This refines some previous results of [17].

1. Introduction

Evolution equations with large diffusion were studied in numerous papers, starting with the work [8] by Hale, cf. also [6], [9] and the references contained in [18]. In those papers results like global bounds of solutions, asymptotic spatial homogenization, and existence of attractors and their upper or lower semicontinuity, as the diffusion goes to infinity, are obtained.

In the present paper we study some systems of parabolic equations with (globally) large diffusion from the point of view of Conley index theory.

More specifically, let r and $N \in \mathbb{N}$, $N \geq 2$, Ω be a bounded smooth domain in $\mathbb{R}^N$ and $\Gamma = \partial\Omega$. For each $\varepsilon > 0$, consider the following system of parabolic

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equations

$$(\mathrm{E}_\varepsilon)\quad \begin{cases} u_{i,t} - \mathrm{Div}(d_{i,\varepsilon}(x)\nabla u_i) + (\lambda + V_{i,\varepsilon}(x))u_i = \varphi_{i,\varepsilon}(x,u), & t>0,\ x\in\Omega,\\ d_{i,\varepsilon}(x)\partial_\nu u_i + b_{i,\varepsilon}(x)u = \psi_{i,\varepsilon}(x,u), & t>0,\ x\in\Gamma,\\ & i\in[1\,.\,.\,r]. \end{cases}$$

Here, $\lambda \in \mathbb{R}$ and ν is the exterior normal vector field on $\partial\Omega$. Moreover, for each $i \in [1\,.\,.\,r]$, $d_{i,\varepsilon} \geq m > 0$, $V_{i,\varepsilon}$ and $b_{i,\varepsilon}$, resp. $\varphi_{i,\varepsilon}$ and $\psi_{i,\varepsilon}$, are given functions on Ω and Γ, resp. $\Omega \times \mathbb{R}^r$ and $\Gamma \times \mathbb{R}^r$ satisfying some regularity assumptions. We assume that, for $\varepsilon \to 0$, $\varphi_{i,\varepsilon} \to \varphi_{i,0}$, $\psi_{i,\varepsilon} \to \psi_{i,0}$ (in some sense), $\frac{1}{|\Omega|}\int_\Omega V_{i,\varepsilon}\,dx \to V_{i,0} \in \mathbb{R}$, $\frac{1}{|\Gamma|}\int_\Gamma b_{i,\varepsilon} \to b_{i,0} \in \mathbb{R}$, while $d_{i,\varepsilon} \to \infty$, uniformly on Ω.

Equation (E_ε) can be written abstractly as a semilinear problem

$$\dot{u}_i = -\widetilde{A}_{i,\varepsilon} u + f_{i,\varepsilon}(u), \quad i \in [1\,.\,.\,r] \tag{1.1}$$

generating a local semiflow π_ε on $H^1(\Omega,\mathbb{R}^r)$. Define

$$\mu_i := V_{i,0} + \frac{|\Gamma|}{|\Omega|} b_{i,0} + \lambda, \quad i \in [1\,.\,.\,r]\,.$$

Consider the system

$$(\mathrm{E}_0)\qquad u_{i,t} = -\mu_i u_i + \frac{1}{|\Omega|}\left(\int_\Omega \varphi_{i,0}(x,u)dx + \int_\Gamma \psi_{i,0}(x,\gamma(u))\,d\sigma\right),$$

$i \in [1\,.\,.\,r]$, of ordinary differential equations on the r-dimensional linear subspace $H^1_c(\Omega,\mathbb{R}^r)$ of $H^1(\Omega,\mathbb{R}^r)$ consisting of (equivalence classes) of constant functions. This system generates a (forward time) local semiflow π_0 on $H^1_c(\Omega,\mathbb{R}^r)$.

In the paper [13] the case $r = 1$ in (E_ε) is considered. The authors prove a spectral convergence of the family $(A_{1,\varepsilon})_{\varepsilon>0}$ for $\varepsilon \to 0$. In the paper [17] the author establishes a upper semicontinuity result for global attractors of π_ε, $\varepsilon \geq 0$, under additional dissipativity conditions on the nonlinearities.

In this paper we refine some of the results from [17]. In particular, we prove that, as $\varepsilon \to 0$, the semiflows π_ε converge in a singular sense to the semiflow π_0 and we establish a singular compactness result for the family π_ε, $\varepsilon \geq 0$. As in [5] we then obtain singular Conley index and homology index braid continuation principles for this family of semiflows. In particular, invariant sets of the ODE system (E_0) continue to invariant sets of the PDE system (E_ε) with the same Conley index. This provides useful information about the dynamics of (E_ε) for small $\varepsilon > 0$.

We proceed as in [1] and [5] and keep the presentation of our results at an abstract level. In fact we only assume certain spectral convergence properties on a family of linear operators $(A_{i,\varepsilon})_{\varepsilon>0}$, $i \in [1\,.\,.\,r]$ (see condition (FSpec)) in Section 3). We also make an abstract convergence hypothesis (condition (Conv) in section 4) on a family of nonlinear operators $(f_\varepsilon)_{\varepsilon>0}$.

The paper is organized as follows.

In Section 2 we introduce some notation and collect a few preliminary results.

In Section 3 we introduce condition (FSpec) and obtain linear singular convergence results (cf. Theorems 3.6 and 3.7). We also prove that our abstract condition implies a first singular compactness result (cf. Proposition 3.4).

In Section 4 we introduce an abstract condition (Conv). As in [5] we obtain a singular convergence result (Theorem 4.5), a singular compactness result (Theorem 4.7) a Conley index continuation result (Theorem 4.8) and an index braid continuation result (Theorem 4.10).

In Section 5 we show that, under appropriate hypotheses on the coefficient functions and the nonlinearities involved, the system of parabolic equations (E_ε) gives rise to a family of linear operators $(A_{i,\varepsilon})_{\varepsilon>0}$, $i \in [1\,..\,r]$ satisfying condition (FSpec) and a family $(f_\varepsilon)_{\varepsilon>0}$ of nonlinear operators for which condition (Conv) holds. (cf. Hypothesis 5.5).

2. Preliminaries

Assume H is (a finite or infinite dimensional) real linear space which is complete with respect to the scalar product $\langle \cdot\,, \cdot \rangle_H$ and let $A\colon D(A) \subset H \to H$ be a (densely defined) positive self-adjoint linear operator on $(H, \langle \cdot\,, \cdot \rangle_H)$ with $A^{-1}\colon H \to H$ compact. Let $S = \mathbb{N}$ if H is infinite dimensional and $S = [1\,..\,\ell]$ if $\dim H = \ell < \infty$. Let $(v_j)_{j\in S}$ be an H-orthonormal and H-complete sequence of eigenvectors of A and $(\mu_j)_{j\in S}$ the corresponding sequence of eigenvalues. Then there is a bijection $\nu\colon S \to S$ such that $(\lambda_r)_{r\in S}$, where $\lambda_r = \mu_{\nu(r)}$, $r \in S$, is *nondecreasing*. The sequence $(\lambda_j)_{j\in S}$, called *the repeated sequence of eigenvalues of* A, is uniquely determined by the properties that it is nondecreasing and contains exactly the eigenvalues of A such that the number of occurrences of each eigenvalue μ of A in this sequence is equal to the multiplicity of μ.

The ordering of $(\lambda_r)_{r\in S}$ plays no role in this section and can even be slightly confusing when we discuss product operators. Therefore for the moment we will work with the original unordered sequence $(\mu_i)_{i\in S}$.

For $\alpha \in [0, \infty[$, let $H_\alpha = H_\alpha(A) = D(A^{\alpha/2})$. In particular,

$$H_0 = H.$$

Note that H_α is a Hilbert space under the scalar product

$$\langle u, v\rangle_{H_\alpha} = \langle A^{\alpha/2}u, A^{\alpha/2}v\rangle_H, \quad u,\, v \in H_\alpha.$$

For every $j \in S$, $v_j \in H_\alpha$ and the sequence $(\mu_j^{-\alpha/2} v_j)_{j\in S}$ is H_α-orthonormal and H_α-complete. If H is infinite dimensional and $u \in H_\alpha$ we have

$$\left| u - \sum_{j=1}^{k} \langle u, v_j\rangle_H v_j \right|_{H_\alpha} \to 0 \quad \text{as } k \to \infty \tag{2.1}$$

and so

$$|u|_{H_\alpha}^2 = \sum_{j=1}^{\infty} \mu_j^\alpha |\langle u, v_j\rangle_H|^2. \tag{2.2}$$

If $\dim H = \ell$, then H_α and H are identical as sets and the corresponding norms are equivalent. Moreover, if $u \in H_\alpha = H$ then

$$|u|_{H_\alpha}^2 = \sum_{j=1}^{\ell} \mu_j^\alpha |\langle u, v_j\rangle_H|^2. \tag{2.3}$$

If $\alpha \in \,]0,\infty[$, let $H_{-\alpha} = H_{-\alpha}(A) = H'_\alpha$ be the dual of H_α. (Thus in the finite-dimensional case the set $H_{-\alpha}$ is identical to the dual H' of H.)

$H_{-\alpha}$ is a Hilbert space under the dual norm

$$\langle u, v\rangle_{H_{-\alpha}} = \langle F_\alpha^{-1} v, F_\alpha^{-1} u\rangle_{H_\alpha}, \quad u,\, v \in H_{-\alpha},$$

where $F_\alpha\colon H_\alpha \to H_{-\alpha}$, $u \mapsto \langle\,\cdot\,, u\rangle_{H_\alpha}$, is the Fréchet–Riesz isomorphism.

Define the map $\psi = \psi_{H,\alpha}\colon H = H_0 \to H_{-\alpha}$ by $\psi(u) = y$, where $y\colon H_\alpha \to \mathbb{K}$ is defined by

$$y(v) = \langle v, u\rangle_H, \quad v \in H_\alpha.$$

The map ψ is injective (and bijective if H is finite-dimensional) so that we can (and will) identify elements $u \in H$ with $\psi(u) \in H_{-\alpha}$.

With this identification, the sequence $(\mu_j^{\alpha/2} v_j)_{j\in S}$ is $H_{-\alpha}$-orthonormal and $H_{-\alpha}$-complete. If H is infinite dimensional and $u \in H_{-\alpha}$ then

$$\Big| u - \sum_{j=1}^{k} u(v_j) v_j \Big|_{H_{-\alpha}} \to 0 \quad \text{as } k \to \infty \tag{2.4}$$

and so

$$|u|_{H_{-\alpha}}^2 = \sum_{j=1}^{\infty} \mu_j^{-\alpha} |u(v_j)|^2. \tag{2.5}$$

If $\dim H = \ell$ and $u \in H_{-\alpha} = H'$ then

$$|u|_{H_{-\alpha}}^2 = \sum_{j=1}^{\ell} \mu_j^{-\alpha} |u(v_j)|^2. \tag{2.6}$$

For $\alpha \in \,]0,\infty[$ there is a unique continuous extension $\widetilde{A}^{-1} = \widetilde{A}_\alpha^{-1}\colon H_{-\alpha} \to H_{2-\alpha}$ of $A^{-1}\colon H \to H_2$. The map $\widetilde{A}^{-1}$ is a bijective linear isometry. Let $\widetilde{A} = \widetilde{A}_\alpha\colon H_{2-\alpha} \to H_{-\alpha}$ be the inverse of $\widetilde{A}^{-1}$. Then $\widetilde{A}$ is a positive densely defined self-adjoint operator on $H_{-\alpha}$. Moreover, for $\beta \in [0,\infty]$ the β-fractional power space $H_\beta(\widetilde{A})$ of $\widetilde{A}$ is isometric (as a Hilbert space) to $H_{\beta-\alpha} = H_{\beta-\alpha}(A)$. If H is finite-dimensional then, due to our identifications, $\widetilde{A}_\alpha^{-1} = A^{-1}$ and $\widetilde{A}_\alpha = A$.

The linear semigroup $e^{-t\widetilde{A}}\colon H_{-\alpha} \to H_{-\alpha}$, $t \in [0,\infty[$, is an extension of the semigroup $e^{-tA}\colon H \to H$, $t \in [0,\infty[$, i.e.

$$e^{-t\widetilde{A}}\psi(u) = \psi(e^{-tA}u), \quad t \in [0,\infty[\,,\ u \in H. \tag{2.7}$$

Using this it is easily proved that

$$(e^{-t\widetilde{A}}u)(h) = u(e^{-tA}h), \quad t \in [0,\infty[\,,\ u \in H_{-\alpha},\ h \in H_\alpha. \tag{2.8}$$

In fact, if $u = \psi(v)$ for some $v \in H$, then

$$\begin{aligned}(e^{-t\widetilde{A}}u)(h) &= (e^{-t\widetilde{A}}\psi(v))(h) = \psi(e^{-tA}v)(h)\\ &= \langle h, e^{-tA}v\rangle_H = \langle e^{-tA}h, v\rangle_H = \psi(v)(e^{-tA}h) = u(e^{-tA}h).\end{aligned}$$

Now the general case follows by a density argument. For every $j \in S$ and $t \in [0,\infty[$,

$$e^{-tA}v_j = e^{-t\widetilde{A}}v_j = e^{-t\mu_j}v_j.$$

Therefore, if H is infinite dimensional, then for every $u \in H$, every $\beta \in [0,\infty[$ and every $t \in]0,\infty[$

$$\left|e^{-tA}u - \sum_{j=1}^{k} e^{-t\mu_j}\langle u, v_j\rangle_H v_j\right|_{H_\beta} \to 0 \quad \text{as } k \to \infty. \tag{2.9}$$

Moreover, for every $u \in H_{-\alpha}$, every $\beta \in [0,\infty[$ and every $t \in]0,\infty[$

$$\left|e^{-t\widetilde{A}}u - \sum_{j=1}^{k} e^{-t\mu_j}u(v_j)v_j\right|_{H_\beta} \to 0 \quad \text{as } k \to \infty. \tag{2.10}$$

Now assume that $r \in \mathbb{N}$ and for each $i \in [1\,..\,r]$ let $(H_{(i)}, \langle\cdot\,,\cdot\rangle_{H_{(i)}})$ be a Hilbert space and let $A_i\colon D(A_i) \subset H_{(i)} \to H_{(i)}$ be a (densely defined) positive self-adjoint linear operator on $(H_{(i)}, \langle\cdot\,,\cdot\rangle_{H_{(i)}})$ with $A_i^{-1}\colon H_{(i)} \to H_{(i)}$ compact. Then the product operator $A = \bigtimes_{i=1}^r A_i\colon D(A) = \bigtimes_{i=1}^r D(A_i) \to H = \bigtimes_{i=1}^r H_{(i)}$, $u = (u_1,\dots,u_r) \mapsto (A_1u_1,\dots,A_ru_r)$ is a (densely defined) positive self-adjoint linear operator on the product Hilbert space $(H, \langle\cdot\,,\cdot\rangle_H)$ with $A^{-1}\colon H \to H$ compact. Here,

$$\langle u, u'\rangle_H = \sum_{i=1}^r \langle u_i, u_i'\rangle_{H_{(i)}}, \quad u = (u_1,\dots,u_r),\ u' = (u_1',\dots,u_r') \in H.$$

For each $\alpha \in \mathbb{R}$ let $H_\alpha = H_\alpha(A)$ and for $i \in [1\,..\,r]$ let $H_{(i),\alpha} = H_\alpha(A_i)$. Then, for $\alpha \in [0,\infty[$, H_α is identical (as a set and as a Hilbert space) to the product $\bigtimes_{i=1}^r H_{(i),\alpha}$. In particular,

$$|u|_{H_\alpha}^2 = \sum_{i=1}^r |u_i|_{H_{(i),\alpha}}^2, \quad u = (u_1,\dots,u_r) \in H_\alpha. \tag{2.11}$$

For each $i \in [1..r]$ let $\mathbf{e}_i: H_{(i)} \to H$ be the imbedding $u_i \mapsto (\underbrace{0,\dots,0,u_i}_{i},0,\dots,0)$.

Then, for $\alpha \in]0,\infty[$ and $k \in [1..r]$, the map $\Lambda_{(k)} = \Lambda_{(k),\alpha}: H_{-\alpha} \to H_{(k),-\alpha}$, $u \mapsto u_k$ is defined by

$$u_k: H_{(k),\alpha} \to \mathbb{R}, \quad h_k \mapsto u(\mathbf{e}_k(h_k)),\ h_k \in H_{(k),\alpha}.$$

The map $\Lambda = \Lambda_\alpha: H_{-\alpha} \to \bigtimes_{i=1}^r H_{(i),-\alpha}$, $u \mapsto (\Lambda_{(1)}(u),\dots,\Lambda_{(r)}(u))$, is a bijective linear isometry, i.e,

$$(2.12) \qquad |u|^2_{H_{-\alpha}} = \sum_{i=1}^r |u_i|^2_{H_{(i),-\alpha}}, \quad u \in H_{-\alpha},\ u_i = \Lambda_{(i)}(u),\ i \in [1..r]\,.$$

Using this map, we identify $H_{-\alpha}$ with the product space $\bigtimes_{i=1}^r H_{(i),-\alpha}$.

Now let $\psi = \psi_{H,\alpha}$ and for each $i \in [1..r]$ let $\psi_i = \psi_{H_{(i)},\alpha}$. Then

$$(2.13) \qquad \Lambda_{(i)}(\psi(u)) = \psi_i(u_i), \quad i \in [1..r]\,,\ u = (u_1,\dots,u_r) \in H.$$

Now let $\widetilde{A}: H_{2-\alpha} \to H_{-\alpha}$ be the extension of A and for $i \in [1..r]$ let $\widetilde{A}_i: H_{(i),2-\alpha} \to H_{(i),-\alpha}$ be the extension of A_i. Then, for $t \in [0,\infty[$, $i \in [1..r]$ and $u \in H_{-\alpha}$

$$(2.14) \qquad \Lambda_{(i)}(e^{-t\widetilde{A}}u) = e^{-t\widetilde{A}_i}\Lambda_{(i)}(u).$$

We prove (2.14) first for u of the form $u = \psi(h)$, where $h = (h_1,\dots,h_r) \in H$. Since $e^{-t\widetilde{A}}\psi(h) = \psi(e^{-tA}h)$, $e^{-t\widetilde{A}_i}\psi_i(h_i) = \psi_i(e^{-tA_i}h_i)$ and $e^{-tA}h = (e^{-tA_1}h_1, \dots, e^{-tA_r}h_r)$, we have by (2.13),

$$\begin{aligned}\Lambda_{(i)}(e^{-t\widetilde{A}}\psi(h)) &= \Lambda_{(i)}(\psi(e^{-tA}h)) = \psi_i(e^{-tA_i}h_i)\\ &= e^{-t\widetilde{A}_i}\psi_i(h_i) = e^{-t\widetilde{A}_i}\Lambda_{(i)}(\psi(h)).\end{aligned}$$

Now a simple density argument completes the proof for general u.

For $t \in]0,\infty[$, $\beta \in [0,\infty[$ and $u \in H_{-\alpha}$, we have that $e^{-t\widetilde{A}}u$ lies in H_β. This follows from (2.10) and means precisely that there is a $w = (w_1,\dots,w_r) \in H_\beta \subset H$ such that $e^{-t\widetilde{A}}u = \psi(w)$. Analogously, for every $i \in [1..r]$ there is an $h_i \in H_{(i),\beta}$ with $e^{-t\widetilde{A}_i}\Lambda_i(u) = \psi_i(h_i)$. Now (2.13) and (2.14) imply that $\psi_i(w_i) = \psi_i(h_i)$, so $w_i = h_i$ for all $i \in [1..r]$. In particular, by (2.11),

$$(2.15) \qquad |e^{-t\widetilde{A}}u|^2_{H_\beta} = \sum_{i=1}^r |e^{-t\widetilde{A}_i}u_i|^2_{H_{(i),\beta}}, \\ t \in]0,\infty[\,,\ u \in H_{-\alpha},\ (u_1,\dots,u_r) = \Lambda(u),\ \beta \in [0,\infty[\,.$$

Now suppose α and $\gamma \in]0,\infty[$ are such that $\gamma + \alpha < 2$ and let $f: H_\gamma \to H_{-\alpha}$ be a locally Lipschitzian map. Thus $f: H_{\gamma+\alpha}(\widetilde{A}) \to H_0(\widetilde{A})$ is locally Lipschitzian so for every $a \in H_\gamma$ there is a $\omega_a \in \,]0,\infty]$ and a unique, maximally defined solution $u = u_{(a)}\colon [0,\omega_a[\, \to H_\gamma$ of the equation

$$\dot{u} = -\widetilde{A}u + f(u) \tag{2.16}$$

with $u(0) = a$. By definition, this means that u is continuous into H_γ and

$$\psi(u(t)) = e^{-t\widetilde{A}}\psi(a) + \int_0^t e^{-(t-s)\widetilde{A}} f(u(s))\,ds, \quad t \in [0,\omega_a[\,. \tag{2.17}$$

Let $D(\pi)$ be the set of all $(t,a) \in [0,\infty[\, \times H_\gamma$ with $t \in [0,\omega_a[$ and $\pi\colon D(\pi) \to H_\gamma$ be the map $(t,a) \mapsto u_{(a)}(t)$. π is the local semiflow generated by equation (2.16). We write $a\pi t$ instead of $\pi(t,a)$. By (2.13) and (2.14), $u = u_{(a)}$ if and only if for each $i \in [1\,.\,.\,r]$ u_i is continuous into $H_{(i),\gamma}$ and

$$\psi_i(u_i(t)) = e^{-t\widetilde{A}_i}\psi(a_i) + \int_0^t e^{-(t-s)\widetilde{A}_i} f_i(u(s))\,ds, \quad t \in [0,\omega_a[\,. \tag{2.18}$$

Here, u_i is the ith component function of u, a_i is the ith component of a and $f_i = \Lambda_i \circ f$ is the ith component of f. Thus we regard the following system

$$\dot{u}_i = -\widetilde{A}_i u_i + f_i(u), \quad i \in [1\,.\,.\,r] \tag{2.19}$$

as an alternative form of equation (2.16). By (2.8), formula (2.18) is equivalent to the validity of the statement

$$\langle u_i(t), h_i\rangle_{H_{(i)}} = \langle a_i, e^{-tA_i}h_i\rangle_{H_{(i)}} + \int_0^t f_i(u(s))(e^{-(t-s)A_i}h_i)\,ds, \tag{2.20}$$

$t \in [0,\omega_a[$, for every $h_i \in H_{(i),\alpha}$.

Now assume that, for each $i \in [1\,.\,.\,r]$ $H_{(i)}$ has finite dimension ℓ_i and $S_i = [1\,.\,.\,\ell_i]$. Let $(v_{i,j})_{j\in S_i}$ be an $H_{(i)}$-orthonormal and $H_{(i)}$-complete sequence of eigenvectors of A_i and $(\mu_{i,j})_{j\in S_i}$ the corresponding sequence of eigenvalues. By linearity it is enough to have (2.20) for each basis vector $v_{i,j}$. Thus we obtain that formula (2.18) is equivalent to formula

$$\begin{aligned}\langle u_i(t), v_{i,j}\rangle_{H_{(i)}} &= e^{-t\mu_{i,j}}\langle a_i, v_{i,j}\rangle_{H_{(i)}} \\ &\quad + \int_0^t e^{-(t-s)\mu_{i,j}} f_i(u(s))(v_{i,j})\,ds, \quad j \in S_i,\ t \in [0,\omega_a[\,.\end{aligned} \tag{2.21}$$

Now it follows from (2.18) and (2.21) that system (2.19) is just the following system

$$\dot{u}_i = \sum_{j=1}^{\ell_i} \big(-\mu_{i,j}\langle u_i, v_{i,j}\rangle_{H_{(i)}} + f_i(u)(v_{i,j})\big)v_{i,j}, \quad i \in [1\,.\,.\,r] \tag{2.22}$$

of ordinary differential equations.

3. Singular convergence of linear semiflows

We will now introduce a basic abstract spectral convergence condition.

Definition 3.1. Given $\varepsilon_0 > 0$ we say that the family

$$(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$$

satisfies condition (FSpec) if the following properties are satisfied:

(1) for every $\varepsilon \in [0,\varepsilon_0]$, $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon})$ is a Hilbert space and $A_\varepsilon\colon D(A_\varepsilon) \subset H^\varepsilon \to H^\varepsilon$ is a densely defined positive self-adjoint linear operator on $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon})$ with ${A_\varepsilon}^{-1}\colon H^\varepsilon \to H^\varepsilon$ compact. For $\alpha \in \mathbb{R}$ write $H^\varepsilon_\alpha := H_\alpha(A_\varepsilon)$. In particular, $H^\varepsilon_0 = H^\varepsilon$;
(2) H^0 is ℓ-dimensional with $\ell \in \mathbb{N}$ while H^ε is infinite dimensional for $\varepsilon \in]0,\varepsilon_0]$.
(3) for each $\varepsilon \in]0,\varepsilon_0]$, H^0 is a linear subspace of H^ε and H^0_1 is a linear subspace of H^ε_1;
(4) there exists a constant $C \in]1,\infty[$ such that

$$|u|_{H^\varepsilon_1} \le C|u|_{H^0_1} \quad \text{and} \quad |u|_{H^0_1} \le C|u|_{H^\varepsilon_1}$$

for all $u \in H^0_1$ and all $\varepsilon \in]0,\varepsilon_0]$;
(5) for every $\varepsilon \in]0,\varepsilon_0]$ let $(\lambda_{\varepsilon,j})_j$ be the repeated sequence of eigenvalues of A_ε and $(w_{\varepsilon,j})_j$ be a corresponding H^ε-orthonormal sequence of eigenfunctions. Furthermore, let $(\lambda_{0,j})_{j\in[1..\ell]}$ be the repeated sequence of eigenvalues of A_0.

Whenever $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ then
(a) $\lambda_{\varepsilon_n,j} \to \lambda_{0,j}$ as $n \to \infty$, for all $j \in [1\,.\,.\,\ell]$.
(b) $\lambda_{\varepsilon_n,j} \to \infty$ as $n \to \infty$, for all $j > \ell$.

Moreover, there is a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ and there is an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_{j\in[1..\ell]}$ of A_0 corresponding to $(\lambda_{0,j})_{j\in[1..\ell]}$ such that
(c) $|w_{\varepsilon_{n_k},j} - w_{0,j}|_{H^{\varepsilon_{n_k}}_1} \to 0$ as $k \to \infty$, for all $j \in [1\,.\,.\,\ell]$;
(d) $\langle u, w_{\varepsilon_{n_k},j}\rangle_{H^{\varepsilon_{n_k}}} \to \langle u, w_{0,j}\rangle_{H^0}$ as $k \to \infty$, for all $u \in H^0$ and all $j \in [1\,.\,.\,\ell]$.

Such a sequence $(w_{0,j})_{j\in[1..\ell]}$ is called *adapted* to the sequence $(n_k)_k$.

Remark 3.2. Condition (FSpec) differs from condition (Spec) introduced in [5] in that here H_0 is finite dimensional, the convergence statements involving the eigenvalues $\lambda_{\varepsilon_n,j}$ (and eigenfunctions $w_{\varepsilon_{n_k},j}$) hold only for $j \in [1\,.\,.\,\ell]$ and the other eigenvalues diverge off to infinity.

Remark 3.3. Note that, for α, $t \in]0,\infty[$ and $\lambda \in [0,\infty[$

$$\lambda^\alpha e^{-\lambda t} \le C(\alpha)t^{-\alpha} \quad \text{with } C(\alpha) = (\alpha/e)^\alpha.$$

Let $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition (FSpec). Let $\alpha \in [0,\infty[$, $\varepsilon \in]0,\varepsilon_0]$ and $r \in]0,\infty[$. Using the above estimate, we obtain for every $u \in H^\varepsilon_{-\alpha}$

$$
\begin{aligned}
|e^{-\widetilde{A}_\varepsilon r}u|^2_{H^\varepsilon_1} &= \sum_{j=1}^{\infty} \lambda_{\varepsilon,j}^{\alpha+1}(e^{-\lambda_{\varepsilon,j}r})^2 \lambda_{\varepsilon,j}^{-\alpha}|u(w_{\varepsilon,j})|^2 \\
&= \sum_{j=1}^{\infty} ((\lambda_{\varepsilon,j})^{(\alpha+1)/2} e^{-\lambda_{\varepsilon,j}r})^2 \lambda_{\varepsilon,j}^{-\alpha}|u(w_{\varepsilon,j})|^2 \\
&\le (C((\alpha+1)/2)^2 r^{-(\alpha+1)})|u|^2_{H^\varepsilon_{-\alpha}}.
\end{aligned}
$$

Consequently, we obtain for every $u \in H^\varepsilon_{-\alpha}$

$$
|e^{-\widetilde{A}_\varepsilon r}u|_{H^\varepsilon_1} \le C_0 r^{-(\alpha+1)/2}|u|_{H^\varepsilon_{-\alpha}}, \tag{3.1}
$$

where $C_0 = C((\alpha+1)/2)$. Moreover, we obtain for every $u \in H^0_{-\alpha}$

$$
\begin{aligned}
|e^{-\widetilde{A}_0 r}u|^2_{H^0_1} &= \sum_{j=1}^{\ell} \lambda_{0,j}^{\alpha+1}(e^{-\lambda_{0,j}r})^2 \lambda_{0,j}^{-\alpha}|\langle u, w_{0,j}\rangle_{H^0}|^2 \\
&= \sum_{j=1}^{\ell} ((\lambda_{0,j})^{(\alpha+1)/2} e^{-\lambda_{0,j}r})^2 \lambda_{0,j}^{-\alpha}|\langle u, w_{0,j}\rangle_{H^0}|^2 \\
&\le (C((\alpha+1)/2)^2 r^{-(\alpha+1)})|u|^2_{H^0_{-\alpha}}.
\end{aligned}
$$

Consequently, we obtain for every $u \in H^0$

$$
|e^{-\widetilde{A}_0 r}u|_{H^0_1} \le C_0 r^{-(\alpha+1)/2}|u|_{H^0_{-\alpha}}. \tag{3.2}
$$

We shall need these estimates in the results to follow.

It turns out that Condition (FSpec) implies an abstract asymptotic compactness property:

Proposition 3.4. *Suppose the family* $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies condition* (FSpec)*. Then the following statement holds:*

(3.3) *Whenever* $(\varepsilon_n)_n$ *is a sequence in* $]0,\varepsilon_0]$ *with* $\varepsilon_n \to 0$ *and* $(\xi_n)_n$ *is a sequence with* $\xi_n \in H^{\varepsilon_n}_1$ *for every* $n \in \mathbb{N}$ *and* $\sup_{n\in\mathbb{N}} |\xi_n|_{H^{\varepsilon_n}_1} < \infty$, *then there exist a* $v \in H^0_1$ *and a sequence* $(n_k)_k$ *in* $\mathbb{N}$ *with* $n_k \to \infty$ *as* $k \to \infty$ *such that* $|\xi_{n_k} - v|_{H^{\varepsilon_{n_k}}} \to 0$ *as* $k \to \infty$.

Proof. Let $(\varepsilon_n)_n$ be a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and $(\xi_n)_n$ be a sequence with $\xi_n \in H^{\varepsilon_n}_1$ for every $n \in \mathbb{N}$ and

$$
\sup_{n\in\mathbb{N}} |\xi_n|_{H^{\varepsilon_n}_1} \le C,
$$

for some $C \in]0,\infty[$. For each $n \in \mathbb{N}$, we have

$$
|\xi_n|^2_{H^{\varepsilon_n}_1} = \sum_{j=1}^{\infty} \lambda_{\varepsilon_n,j}|\langle \xi_n, w_{\varepsilon_n,j}\rangle|^2.
$$

In particular, there exist a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ and a sequence $(\zeta_j)_j$ such that for each $j \in \mathbb{N}$

$$\langle \xi_{n_k}, w_{\varepsilon_{n_k},j} \rangle \to \zeta_j \quad \text{as } k \to \infty.$$

Taking a further subsequence, if necessary, and using condition (FSpec) we may also assume that there exists an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_{j\in[1..\ell]}$ corresponding to $(\lambda_{0,j})_{j\in[1..\ell]}$ and adapted to $(n_k)_k$. For each $k \in \mathbb{N}$ define

$$v_k := \sum_{j=1}^{\ell} \zeta_j w_{\varepsilon_{n_k},j}.$$

We claim that

$$|\xi_{n_k} - v_k|_{H^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty. \tag{3.4}$$

Indeed for each $k \in \mathbb{N}$ we have

$$\begin{aligned} |\xi_{n_k} - v_k|^2_{H^{\varepsilon_{n_k}}} &= \sum_{j=1}^{\infty} |\langle \xi_{n_k} - v_k, w_{\varepsilon_{n_k},j} \rangle|^2 \\ &= \sum_{j=1}^{\ell} |\langle \xi_{n_k} - v_k, w_{\varepsilon_{n_k},j} \rangle|^2 + \sum_{j=\ell+1}^{\infty} |\langle \xi_{n_k} - v_k, w_{\varepsilon_{n_k},j} \rangle|^2. \end{aligned}$$

For each $j \in [1\,..\,\ell]$ we have

$$\langle \xi_{n_k} - v_k, w_{\varepsilon_{n_k},j} \rangle = \langle \xi_{n_k}, w_{\varepsilon_{n_k},j} \rangle - \langle v_k, w_{\varepsilon_{n_k},j} \rangle = \langle \xi_{n_k}, w_{\varepsilon_{n_k},j} \rangle - \zeta_j \to 0$$

as $k \to \infty$. Therefore

$$\sum_{j=1}^{\ell} |\langle \xi_{n_k} - v_k, w_{\varepsilon_{n_k},j} \rangle|^2 \to 0 \quad \text{as } k \to \infty. \tag{3.5}$$

Moreover,

$$\begin{aligned} \sum_{j=\ell+1}^{\infty} |\langle \xi_{n_k} - v_k, w_{\varepsilon_{n_k},j} \rangle|^2 &= \sum_{j=\ell+1}^{\infty} |\langle \xi_{n_k}, w_{\varepsilon_{n_k},j} \rangle|^2 \\ &= \frac{1}{\lambda_{\varepsilon_{n_k},\ell+1}} \sum_{j=\ell+1}^{\infty} \lambda_{\varepsilon_{n_k},\ell+1} |\langle \xi_{n_k}, w_{\varepsilon_{n_k},j} \rangle|^2 \\ &\le \frac{1}{\lambda_{\varepsilon_{n_k},\ell+1}} \sum_{j=\ell+1}^{\infty} \lambda_{\varepsilon_{n_k},j} |\langle \xi_{n_k}, w_{\varepsilon_{n_k},j} \rangle|^2 \le \frac{C^2}{\lambda_{\varepsilon_{n_k},\ell+1}}. \end{aligned}$$

Condition (FSpec) now implies

$$\sum_{j=\ell+1}^{\infty} |\langle \xi_{n_k} - v_k, w_{\varepsilon_{n_k},j} \rangle|^2 \to 0 \quad \text{as } k \to \infty. \tag{3.6}$$

Now (3.5) and (3.6) imply (3.4). Define

$$v := \sum_{j=1}^{\ell} \zeta_j w_{0,j}.$$

Then

$$|v_k - v|_{H^{\varepsilon_{n_k}}} \leq \sum_{j=1}^{\ell} |\zeta_j| \cdot |w_{\varepsilon_{n_k},j} - w_{0,j}|_{H^{\varepsilon_{n_k}}} \to 0, \quad k \to \infty \tag{3.7}$$

(3.4) and (3.7) imply the assertion of the proposition. □

REMARK 3.5. Assertion (3.3) is called condition (Comp) in [5]. Thus condition (FSpec), unlike condition (Spec), automatically implies condition (Comp).

We now prove our first linear convergence result.

THEOREM 3.6. *Let* $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfy condition* (FSpec). *Suppose* $(\varepsilon_n)_n$ *is a sequence in* $]0,\varepsilon_0]$ *with* $\varepsilon_n \to 0$. *Let* $u_0 \in H_1^0$ *and* $(u_n)_n$ *be a sequence such that, for every* $n \in \mathbb{N}$, $u_n \in H_1^{\varepsilon_n}$ *and*

$$|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \textit{as } n \to \infty.$$

Then

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_n}} u_n - e^{-tA_0} u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \textit{as } n \to \infty.$$

PROOF. Since $\lambda_{\varepsilon,j} > 0$ for all $\varepsilon \in]0,\varepsilon_0]$ and for all $j \in \mathbb{N}$, we have

$$|e^{-tA_\varepsilon} v|^2_{H_1^\varepsilon} = \sum_{j=1}^{\infty} (e^{-t\lambda_{\varepsilon,j}})^2 \lambda_{\varepsilon,j} |\langle v, w_{\varepsilon,j}\rangle_{H^\varepsilon}|^2 \leq \sum_{j=1}^{\infty} \lambda_{\varepsilon,j} |\langle v, w_{\varepsilon,j}\rangle_{H^\varepsilon}|^2 = |v|^2_{H_1^\varepsilon},$$

for all $v \in H_1^\varepsilon$, $\varepsilon \in]0,\varepsilon_0]$ and $t \in [0,\infty[$. Thus we obtain, for all $n \in \mathbb{N}$ and all $t \in [0,\infty[$,

$$\begin{aligned} |e^{-tA_{\varepsilon_n}} u_n - e^{-tA_0} u_0|_{H_1^{\varepsilon_n}} &\leq |e^{-tA_{\varepsilon_n}} (u_n - u_0)|_{H_1^{\varepsilon_n}} + |e^{-tA_{\varepsilon_n}} u_0 - e^{-tA_0} u_0|_{H_1^{\varepsilon_n}} \\ &\leq |u_n - u_0|_{H_1^{\varepsilon_n}} + |e^{-tA_{\varepsilon_n}} u_0 - e^{-tA_0} u_0|_{H_1^{\varepsilon_n}}. \end{aligned}$$

Therefore we only have to prove that

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_n}} u_0 - e^{-tA_0} u_0|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

Suppose (3.8) is not true. Then there are a $\delta_0 > 0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_{n_k}}} u_0 - e^{-tA_0} u_0|_{H_1^{\varepsilon_{n_k}}} \geq \delta_0 \quad \text{for all } k \in \mathbb{N}. \tag{3.9}$$

Taking a further subsequence, if necessary, and using condition (FSpec) we may also assume that there exists an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_{j\in[1..\ell]}$ corresponding to $(\lambda_{0,j})_{j\in[1..\ell]}$ and adapted to $(n_k)_k$.

For each $k \in \mathbb{N}$, let $P_k\colon H^{\varepsilon_{n_k}} \to H^{\varepsilon_{n_k}}$ be the $H^{\varepsilon_{n_k}}$-orthogonal projection of $H^{\varepsilon_{n_k}}$ onto the span of $\{ w_{\varepsilon_{n_k},1}, \ldots, w_{\varepsilon_{n_k},\ell} \}$.

Let $t \in [0,\infty[$ be arbitrary. Then for each $k \in \mathbb{N}$ we have

$$|e^{-tA_{\varepsilon_{n_k}}} u_0 - e^{-tA_0} u_0|_{H_1^{\varepsilon_{n_k}}} \\ \le |P_k e^{-tA_{\varepsilon_{n_k}}} u_0 - e^{-tA_0} u_0|_{H_1^{\varepsilon_{n_k}}} + |(I-P_k) e^{-tA_{\varepsilon_{n_k}}} u_0|_{H_1^{\varepsilon_{n_k}}}.$$

Notice that

$$|P_k u_0 - u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty. \tag{3.10}$$

Indeed, for each $k \in \mathbb{N}$ we have

$$\begin{aligned} |P_k u_0 - u_0|_{H_1^{\varepsilon_{n_k}}} &= \bigg| \sum_{i=1}^{\ell} \langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} w_{\varepsilon_{n_k},i} - \sum_{i=1}^{\ell} \langle u_0, w_{0,i}\rangle_{H^0} w_{0,i} \bigg|_{H_1^{\varepsilon_{n_k}}} \\ &\le \sum_{i=1}^{\ell} |\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}}| \, |w_{\varepsilon_{n_k},i} - w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\ &\quad + \sum_{i=1}^{\ell} |\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} - \langle u_0, w_{0,i}\rangle_{H^0}| \, |w_{0,i}|_{H_1^{\varepsilon_{n_k}}}. \end{aligned}$$

Condition (FSpec) now imply (3.10). Since

$$|(I-P_k) e^{-tA_{\varepsilon_{n_k}}} u_0|_{H_1^{\varepsilon_{n_k}}} = |e^{-tA_{\varepsilon_{n_k}}} (I-P_k) u_0|_{H_1^{\varepsilon_{n_k}}} \le |(I-P_k) u_0|_{H_1^{\varepsilon_{n_k}}},$$

it follows from (3.10) that

$$\sup_{t\in[0,\infty[} |(I-P_k) e^{-tA_{\varepsilon_{n_k}}} u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty. \tag{3.11}$$

We further have

$$\begin{aligned} &|P_k e^{-tA_{\varepsilon_{n_k}}} u_0 - e^{-tA_0} u_0|_{H_1^{\varepsilon_{n_k}}} \\ &\le \sum_{i=1}^{\ell} |e^{-t\lambda_{\varepsilon_{n_k},i}} \langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} w_{\varepsilon_{n_k},i} - e^{-t\lambda_{0,i}} \langle u_0, w_{0,i}\rangle_{H^0} w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\ &\le \sum_{i=1}^{\ell} |e^{-t\lambda_{\varepsilon_{n_k},i}} \langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} (w_{\varepsilon_{n_k},i} - w_{0,i})|_{H_1^{\varepsilon_{n_k}}} \\ &\quad + \sum_{i=1}^{\ell} |e^{-t\lambda_{\varepsilon_{n_k},i}} \langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} w_{0,i} - e^{-t\lambda_{0,i}} \langle u_0, w_{0,i}\rangle_{H^0} w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\ &\le \sum_{i=1}^{\ell} |\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}}| \, |w_{\varepsilon_{n_k},i} - w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\ &\quad + C \sum_{i=1}^{\ell} |e^{-t\lambda_{\varepsilon_{n_k},i}} \langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} - e^{-t\lambda_{0,i}} \langle u_0, w_{0,i}\rangle_{H^0}| \, |w_{0,i}|_{H_1^0}. \end{aligned}$$

Since for every $i \in [1\,..\,\ell]$,

$$\sup_{t\in[0,\infty[} |e^{-t\lambda_{\varepsilon_{n_k},i}} - e^{-t\lambda_{0,i}}| \to 0 \quad \text{as } k \to \infty,$$

it follows that

$$\sup_{t\in[0,\infty[} |P_k e^{-tA_{\varepsilon_{n_k}}} u_0 - e^{-tA_0} u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty. \tag{3.12}$$

Formulas (3.11) and (3.12) imply that

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_{n_k}}} u_0 - e^{-tA_0} u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty,$$

but this contradicts (3.9). The proof is complete. □

We can also prove a second, more technical, linear convergence result.

Theorem 3.7. *Let* $(H^\varepsilon, \langle \cdot\,, \cdot \rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfy condition* (FSpec)*. Suppose* $(\varepsilon_n)_n$ *is a sequence in* $]0,\varepsilon_0]$ *with* $\varepsilon_n \to 0$*. Let* $\alpha \in [0,\infty[$, $u_0 \in H^0$ *be arbitrary and let* $(u_n)_n$ *and* $(v_n)_n$ *be sequences such that* u_n *and* $v_n \in H^{\varepsilon_n}_{-\alpha}$ *for* $n \in \mathbb{N}$*. Suppose that*

1. $|u_n - v_n|_{H^{\varepsilon_n}_{-\alpha}} \to 0$ *as* $n \to \infty$.
2. *whenever* $(n_k)_k$ *is a sequence in* $\mathbb{N}$ *with* $n_k \to \infty$ *as* $k \to \infty$ *and whenever* $(w_{0,j})_{j\in[1..\ell]}$ *is adapted to* $(n_k)_k$*, then* $v_{n_k}(w_{\varepsilon_{n_k},j}) \to \langle u_0, w_{0,j}\rangle_{H^0}$ *as* $k \to \infty$ *for all* $j \in [1\,..\,\ell]$.
3. $\sup_{n\in\mathbb{N}} |v_n|_{H^{\varepsilon_n}_{-\alpha}} < \infty$.

For every $\varepsilon \in\,]0,\varepsilon_0]$*, let* $\widetilde{A}_\varepsilon = \widetilde{A}_{\varepsilon,-\alpha}\colon H^\varepsilon_{2-\alpha} \to H^\varepsilon_{-\alpha}$ *be the extension of* A_ε *to* $H^\varepsilon_{-\alpha}$*. Then, for every* $\beta \in\,]0,\infty[$,

$$\sup_{t\in[\beta,\infty[} \left| e^{-t\widetilde{A}_{\varepsilon_n}} u_n - e^{-tA_0} u_0 \right|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Proof. Suppose the theorem is not true. Then there are β, $\delta_0 \in\,]0,\infty[$ and there is a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sup_{t\in[\beta,\infty[} \left| e^{-t\widetilde{A}_{\varepsilon_{n_k}}} u_{n_k} - e^{-tA_0} u_0 \right|_{H_1^{\varepsilon_{n_k}}} \geq \delta_0 \quad \text{for all } k \in \mathbb{N}. \tag{3.13}$$

Taking a further subsequence, if necessary, and using condition (FSpec) we may also assume that there exists an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_{j\in[1..\ell]}$ corresponding to $(\lambda_{0,j})_{j\in[1..\ell]}$ and adapted to $(n_k)_k$. Let $\delta > 0$ be arbitrary. There is an $s_0 = s_0(\delta,\beta) > 0$ such that $s^{(\alpha+1)/2}e^{-st} < \delta$ for $s \geq s_0$ and $t \geq \beta$. Since $\lambda_{\varepsilon_{n_k},\ell+1} \to \infty$ as $k \to \infty$, there is a $k_0 = k_0(\delta,\beta) \in \mathbb{N}$ such that $\lambda_{\varepsilon_{n_k},\ell+1} > s_0$ for $k \geq k_0$. Since $\lambda_{\varepsilon_{n_k},j} \geq \lambda_{\varepsilon_{n_k},\ell+1}$ for all $k \in \mathbb{N}$ and $j \geq \ell+1$, we obtain

$$\lambda_{\varepsilon_{n_k},j} \geq s_0(\delta,\beta) \quad \text{for } k \geq k_0(\delta,\beta) \text{ and } j \geq \ell+1. \tag{3.14}$$

Let $t \geq \beta$ be arbitrary. Then

$$(3.15)\quad \big|e^{-t\widetilde{A}_{\varepsilon_{n_k}}}u_{n_k} - e^{-tA_0}u_0\big|_{H_1^{\varepsilon_{n_k}}} \leq \sum_{j=1}^{\ell}\big|e^{-t\lambda_{\varepsilon_{n_k},j}}u_{n_k}(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j} - e^{-t\lambda_{0,j}}\langle u_0, w_{0,j}\rangle_{H^0}w_{0,j}\big|_{H_1^{\varepsilon_{n_k}}} + \Big|e^{-t\widetilde{A}_{\varepsilon_{n_k}}}u_{n_k} - \sum_{j=1}^{\ell}e^{-t\lambda_{\varepsilon_{n_k},j}}u_{n_k}(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j}\Big|_{H_1^{\varepsilon_{n_k}}}.$$

Now (3.14) implies that, for all $k \geq k_0$,

$$(3.16)\quad \Big|e^{-t\widetilde{A}_{\varepsilon_{n_k}}}u_{n_k} - \sum_{j=1}^{\ell}e^{-t\lambda_{\varepsilon_{n_k},j}}u_{n_k}(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j}\Big|^2_{H_1^{\varepsilon_{n_k}}} = \sum_{j=\ell+1}^{\infty}(\lambda_{\varepsilon_{n_k},j}^{(\alpha+1)/2}e^{-t\lambda_{\varepsilon_{n_k},j}})^2\lambda_{\varepsilon_{n_k},j}^{-\alpha}|u_{n_k}(w_{\varepsilon_{n_k},j})|^2 \leq \delta^2\sum_{j=\ell+1}^{\infty}\lambda_{\varepsilon_{n_k},j}^{-\alpha}|u_{n_k}(w_{\varepsilon_{n_k},j})|^2 \leq \delta^2|u_{n_k}|^2_{H_{-\alpha}^{\varepsilon_{n_k}}} \leq \delta^2\widetilde{C},$$

where $\widetilde{C} := \sup_{k\in\mathbb{N}}|u_{n_k}|^2_{H_{-\alpha}^{\varepsilon_{n_k}}}$. Note that $\widetilde{C} < \infty$ by our assumptions (1) and (3). Let $j \in [1\,..\,\ell]$ be arbitrary. Then

$$(3.17)\quad |e^{-t\lambda_{\varepsilon_{n_k},j}}u_{n_k}(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j} - e^{-t\lambda_{0,j}}\langle u_0, w_{0,j}\rangle_{H^0}w_{0,j}|_{H_1^{\varepsilon_{n_k}}}$$
$$\leq |e^{-t\lambda_{\varepsilon_{n_k},j}}(u_{n_k} - v_{n_k})(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j}|_{H_1^{\varepsilon_{n_k}}} + |e^{-t\lambda_{\varepsilon_{n_k},j}}v_{n_k}(w_{\varepsilon_{n_k},j})(w_{\varepsilon_{n_k},j} - w_{0,j})|_{H_1^{\varepsilon_{n_k}}} + |e^{-t\lambda_{\varepsilon_{n_k},j}}(v_{n_k}(w_{\varepsilon_{n_k},j}) - \langle u_0, w_{0,j}\rangle_{H^0})w_{0,j}|_{H_1^{\varepsilon_{n_k}}} + |(e^{-t\lambda_{\varepsilon_{n_k},j}} - e^{-t\lambda_{0,j}})\langle u_0, w_{0,j}\rangle_{H^0}w_{0,j}|_{H_1^{\varepsilon_{n_k}}}$$
$$\leq |u_{n_k} - v_{n_k}|_{H_{-\alpha}^{\varepsilon_{n_k}}}|w_{\varepsilon_{n_k},j}|_{H_{\alpha}^{\varepsilon_{n_k}}}|w_{\varepsilon_{n_k},j}|_{H_1^{\varepsilon_{n_k}}} + |v_{n_k}|_{H_{-\alpha}^{\varepsilon_{n_k}}}|w_{\varepsilon_{n_k},j}|_{H_{\alpha}^{\varepsilon_{n_k}}}\cdot|w_{\varepsilon_{n_k},j} - w_{0,j}|_{H_1^{\varepsilon_{n_k}}} + |v_{n_k}(w_{\varepsilon_{n_k},j}) - \langle u_0, w_{0,j}\rangle_{H^0}|\cdot|w_{0,j}|_{H_1^{\varepsilon_{n_k}}} + |e^{-t\lambda_{\varepsilon_{n_k},j}} - e^{-t\lambda_{0,j}}|\cdot|\langle u_0, w_{0,j}\rangle_{H^0}|\cdot|w_{0,j}|_{H_1^{\varepsilon_{n_k}}}.$$

Note that, for every $\gamma \in [0,\infty[$, $|w_{\varepsilon_{n_k},j}|_{H_{\gamma}^{\varepsilon_{n_k}}} = \lambda_{\varepsilon_{n_k},j}^{\gamma/2}$. Moreover, $|w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \leq C|w_{0,j}|_{H_1^0}$ and

$$\sup_{t\in[\beta,\infty[}|e^{-t\lambda_{\varepsilon_{n_k},j}} - e^{-t\lambda_{0,j}}| \to 0 \quad \text{as } k \to \infty.$$

Hence, our assumptions and (3.17) show that

$$(3.18)\quad \sup_{t\in[\beta,\infty[}|e^{-t\lambda_{\varepsilon_{n_k},j}}u_{n_k}(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j} - e^{-t\lambda_{0,j}}\langle u_0, w_{0,j}\rangle_{H^0}w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \to 0$$

as $k \to \infty$. Thus formulas (3.15), (3.16), (3.18) and the fact that $\delta > 0$ is arbitrary imply that

$$\sup_{t\in[\beta,\infty[} \left|e^{-t\widetilde{A}_{\varepsilon_{n_k}}} u_{n_k} - e^{-tA_0} u_0\right|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty$$

which contradicts (3.13). The theorem is proved. □

Corollary 3.8. *Let* $(H^\varepsilon, \langle\cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *be a family which satisfies condition* (FSpec). *Suppose* $(\varepsilon_n)_n$ *is a sequence in* $]0,\varepsilon_0]$ *with* $\varepsilon_n \to 0$. *Let* $u_0 \in H^0$ *be arbitrary and let* $(u_n)_n$ *be a sequence such that* $u_n \in H^{\varepsilon_n}$ *for* $n \in \mathbb{N}$. *Suppose that*

$$|u_n - u_0|_{H^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Then, for every $\beta \in]0,\infty[$,

$$\sup_{t\in[\beta,\infty[} \left|e^{-tA_{\varepsilon_n}} u_n - e^{-tA_0} u_0\right|_{H_1^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Proof. Use Theorem 3.7 with $\alpha = 0$ and $v_n = u_0$ for all $n \in \mathbb{N}$. □

4. Nonlinear semiflows: convergence, compactness and index continuation

In this section we will introduce an abstract nonlinear convergence condition (Conv) which is similar to a corresponding condition from [5]. This will imply a number of singular convergence, compactness and Conley index continuation results. Most proofs in this section are omitted, since they are identical (mutatis mutandis) to the corresponding proofs in [5].

Definition 4.1. Let $\varepsilon_0 > 0$ and $r \in \mathbb{N}$ be arbitrary. For each $i \in [1\,.\,.\,r]$ let $(H^\varepsilon_{(i)}, \langle\cdot\,,\cdot\rangle_{H^\varepsilon_{(i)}}, A_{i,\varepsilon})_{\varepsilon\in[0,\varepsilon_0]}$ be a family satisfying condition (FSpec).

For $\varepsilon \in [0,\varepsilon_0]$, $H^\varepsilon = \bigtimes_{i=1}^r H^\varepsilon_{(i)}$ is the product Hilbert space and $A_\varepsilon = \bigtimes_{i=1}^r A_{i,\varepsilon}$ is the product self-adjoint operator. Using the notation of Section 2, we set, for $\alpha \in \mathbb{R}$, $H^\varepsilon_\alpha = H_\alpha(A_\varepsilon)$. In particular, $H^\varepsilon_1 = \bigtimes_{i=1}^r H^\varepsilon_{(i),1}$. This space should not be confused with $H^\varepsilon_{(1)}$.

Let $\alpha \in [0,1[$ be given and for every $\varepsilon \in [0,\varepsilon_0]$ let $\widetilde{A}_\varepsilon = \widetilde{A}_{\varepsilon,-\alpha}\colon H^\varepsilon_{2-\alpha} \to H^\varepsilon_{-\alpha}$ be the extension of A_ε to $H^\varepsilon_{-\alpha}$. We say that the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ of maps *satisfies condition* (Conv) if the following properties are satisfied:

(1) $f_\varepsilon\colon H^\varepsilon_1 \to H^\varepsilon_{-\alpha}$ for every $\varepsilon \in [0,\varepsilon_0]$.

(2) $\lim_{\varepsilon\to 0^+} |e^{-t\widetilde{A}_\varepsilon} f_\varepsilon(u) - e^{-t\widetilde{A}_0} f_0(u)|_{H^\varepsilon_1} = 0$ for every $u \in H^0_1$ and every $t \in]0,\infty[$.

(3) For every $M \in [0,\infty[$ there is an $L = L_M \in [0,\infty[$ such that

$$|f_\varepsilon(u) - f_\varepsilon(v)|_{H^\varepsilon_{-\alpha}} \le L|u-v|_{H^\varepsilon_1}$$

for all $\varepsilon \in [0,\varepsilon_0]$ and u, $v \in H^\varepsilon_1$ satisfying $|u|_{H^\varepsilon_1}$, $|v|_{H^\varepsilon_1} \le M$.

(4) For every $u \in H_1^0$ there is an $\varepsilon_0' \in]0, \varepsilon_0]$ such that

$$\sup_{\varepsilon \in [0,\varepsilon_0']} |f_\varepsilon(u)|_{H^\varepsilon_{-\alpha}} < \infty.$$

The next result shows that the above condition (2) is valid uniformly for t bounded away from zero.

Proposition 4.2. *Assume condition* (Conv) *and let $\beta \in]0, \infty]$ be arbitrary. Then for every $u \in H_1^0$*

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [\beta,\infty[} |e^{-t\widetilde{A}_\varepsilon} f_\varepsilon(u) - e^{-t\widetilde{A}_0} f_0(u)|_{H_1^\varepsilon} = 0$$

Proof. Let $v = e^{-\beta\widetilde{A}_0} f_0(u) \in H_1^0$. For every $t \in [\beta, \infty[$ we have

$$\begin{aligned} |e^{-t\widetilde{A}_\varepsilon} f_\varepsilon(u) &- e^{-t\widetilde{A}_0} f_0(u)|_{H_1^\varepsilon} \\ &\le |e^{-(t-\beta)\widetilde{A}_\varepsilon}(e^{-\beta\widetilde{A}_\varepsilon} f_\varepsilon(u) - e^{-\beta\widetilde{A}_0} f_0(u))|_{H_1^\varepsilon} \\ &\quad + |e^{-(t-\beta)\widetilde{A}_\varepsilon} v - e^{-(t-\beta)\widetilde{A}_0} v|_{H_1^\varepsilon} \\ &\le |e^{-\beta\widetilde{A}_\varepsilon} f_\varepsilon(u) - e^{-\beta\widetilde{A}_0} f_0(u)|_{H_1^\varepsilon} + |e^{-(t-\beta)\widetilde{A}_\varepsilon} v - e^{-(t-\beta)\widetilde{A}_0} v|_{H_1^\varepsilon}. \end{aligned}$$

By a componentwise application of Theorem 3.6 we obtain

$$\lim_{\varepsilon \to 0} \sup_{s \in [0,\infty[} |e^{-s\widetilde{A}_\varepsilon} v - e^{-s\widetilde{A}_0} v|_{H_1^\varepsilon} = 0,$$

so the assertion follows from condition (Conv) part (2) (with $t = \beta$). □

Proposition 4.3. *Let $(H^\varepsilon_{(i)}, \langle \cdot\,, \cdot \rangle_{H^\varepsilon_{(i)}}, A_{i,\varepsilon})_{\varepsilon \in [0,\varepsilon_0]}$, $i \in [1\,.\,.\,r]$, be as in Definition* 4.1. *Then there exists a constant $C \in]1, \infty[$ such that*

$$|u|_{H_1^\varepsilon} \le C|u|_{H_1^0} \quad \textit{and} \quad |u|_{H_1^0} \le C|u|_{H_1^\varepsilon}$$

for all $u \in H_1^0$ and all $\varepsilon \in]0, \varepsilon_0]$. Moreover, for every $\varepsilon \in [0, \varepsilon_0]$ and every $u \in H^\varepsilon_{-\alpha}$

$$|e^{-\widetilde{A}_\varepsilon r} u|_{H_1^\varepsilon} \le C_0 r^{-(\alpha+1)/2} |u|_{H^\varepsilon_{-\alpha}}, \tag{4.1}$$

where $C_0 \in]0, \infty[$ is as in Remark 3.3.

Proof. This follows from the (FSpec) condition, formulas (2.12) and (2.15) and Remark 3.3. □

In the sequel, if $(H^\varepsilon_{(i)}, \langle \cdot\,, \cdot \rangle_{H^\varepsilon_{(i)}}, A_{i,\varepsilon})_{\varepsilon \in [0,\varepsilon_0]}$, $i \in [1\,.\,.\,r]$ and $(f_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ are as in Definition 4.1 then we will write, for every $\varepsilon \in [0, \varepsilon_0]$, $\pi_\varepsilon := \pi_{\widetilde{A}_\varepsilon, f_\varepsilon}$ to denote the local semiflow on H_1^ε generated by the abstract parabolic equation

$$\dot{u} = -\widetilde{A}_\varepsilon u + f_\varepsilon(u) \tag{4.2}$$

(cf. equation (2.16)) or, equivalently, the system

$$\dot{u}_i = -\widetilde{A}_{i,\varepsilon} u + f_{i,\varepsilon}(u), \quad i \in [1..r] \tag{4.3}$$

(cf. (2.19).)

For the rest of this section, unless otherwise specified, we assume that the families $(H^\varepsilon_{(i)}, \langle\cdot\,,\cdot\rangle_{H^\varepsilon_{(i)}}, A_{i,\varepsilon})_{\varepsilon\in[0,\varepsilon_0]}$, $i \in [1..r]$, and $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ are as in Definition 4.1.

We will now state a number of convergence, compactness and continuation results. Using, in appropriate places, Proposition 4.2, Proposition 4.3 and applying componentwise Theorem 3.6, resp. Theorem 3.7, resp. Proposition 3.4, the proofs of these results are completely analogous to the proofs of the corresponding results from [5].

We begin by stating two singular convergence results for semiflows.

THEOREM 4.4. *Let $(\varepsilon_n)_n$ be a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$. Let $u_0 \in H^0_1$ and $(u_n)_n$ be a sequence with $u_n \in H^{\varepsilon_n}_1$ for every $n \in \mathbb{N}$ and*

$$|u_n - u_0|_{H^{\varepsilon_n}} \to 0 \quad \text{as } n \to \infty.$$

Let $b \in]0,\infty[$ and suppose that $u_n\pi_{\varepsilon_n}t$ and $u\pi_0 t$ are defined for all $n \in \mathbb{N}$ and $t \in [0,b]$. Moreover suppose there exists an $M' \in [0,\infty[$ such that $|u_n\pi_{\varepsilon_n}s|_{H^{\varepsilon_n}_1} \le M'$ for all $n \in \mathbb{N}$ and for all $s \in [0,b]$. Then for every $t \in]0,b]$ and every sequence $(t_n)_n$ in $]0,b]$ converging to t

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t_n|_{H^{\varepsilon_n}_1} \to 0 \quad \text{as } n \to \infty.$$

THEOREM 4.5. *Let $(\varepsilon_n)_n$ be a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and let $(t_n)_n$ be a sequence in $[0,\infty[$ with $t_n \to t_0$, for some $t_0 \in [0,\infty[$. Let $u_0 \in H^0_1$ and $(u_n)_n$ be a sequence with $u_n \in H^{\varepsilon_n}_1$ for every $n \in \mathbb{N}$ and*

$$|u_n - u_0|_{H^{\varepsilon_n}_1} \to 0 \quad \text{as } n \to \infty.$$

Assume $u_0\pi_0 t_0$ is defined. Then there exists an $n_0 \in \mathbb{N}$ such that $u_n\pi_{\varepsilon_n}t_n$ is defined for all $n \ge n_0$ and

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t_0|_{H^{\varepsilon_n}_1} \to 0 \quad \text{as } n \to \infty.$$

We also have the following admissibility (i.e. asymptotic compactness) results:

THEOREM 4.6. *Let $\varepsilon \in [0,\varepsilon_0]$ be arbitrary. Then every closed and bounded set in H^ε_1 is strongly π_ε-admissible.*

Theorem 4.7. *Suppose* $\kappa \in \,]0,\infty[$, $(\varepsilon_n)_n$ *is a sequence in* $]0,\varepsilon_0]$ *with* $\varepsilon_n \to 0$, $(t_n)_n$ *is a sequence in* $[0,\infty[$ *with* $t_n \ge \kappa$ *for every* $n \in \mathbb{N}$ *and* $(u_n)_n$ *is a sequence with* $u_n \in H_1^{\varepsilon_n}$ *for every* $n \in \mathbb{N}$. *Assume that there exists a* $C'' \in \,]0,\infty[$ *such that* $u_n\pi_{\varepsilon_n}t_n$ *is defined and*

$$|u_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}} \le C'' \quad \text{for all } n \in \mathbb{N} \text{ and for all } s \in [0,t_n].$$

Then there exist a $v \in H_1^0$ *and a sequence* $(n_k)_k$ *in* $\mathbb{N}$ *with* $n_k \to \infty$ *as* $k \to \infty$ *such that*

$$|u_{n_k}\pi_{\varepsilon_{n_k}}t_{n_k} - v|_{H_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty.$$

For $\varepsilon \in \,]0,\varepsilon_0]$ let $Q_\varepsilon\colon H_1^\varepsilon \to H_1^\varepsilon$ be the H_1^ε-orthogonal projection of H_1^ε onto H_0^ε.

We can now state the following Conley index continuation principle for singular families of abstract parabolic equations:

Theorem 4.8. *Let* N *be a closed and bounded isolating neighborhood of an invariant set* K_0 *relative to* π_0. *For* $\varepsilon \in \,]0,\varepsilon_0]$ *and for every* $\eta \in \,]0,\infty[$ *set*

$$N_{\varepsilon,\eta} := \{\, u \in H_1^\varepsilon \mid Q_\varepsilon u \in N \text{ and } |(I-Q_\varepsilon)u|_{H_1^\varepsilon} \le \eta \,\}$$

and $K_{\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(N_{\varepsilon,\eta})$ *i.e.* $K_{\varepsilon,\eta}$ *is the largest* π_ε*-invariant set in* $N_{\varepsilon,\eta}$. *Then for every* $\eta \in \,]0,\infty[$ *there exists an* $\varepsilon^{\mathrm{c}} = \varepsilon^{\mathrm{c}}(\eta) \in \,]0,\varepsilon_0]$ *such that for every* $\varepsilon \in \,]0,\varepsilon^{\mathrm{c}}]$ *the set* $N_{\varepsilon,\eta}$ *is a strongly admissible isolating neighborhood of* $K_{\varepsilon,\eta}$ *relative to* π_ε *and*

$$h(\pi_\varepsilon, K_{\varepsilon,\eta}) = h(\pi_0, K_0).$$

Furthermore, for every $\eta > 0$, *the family* $(K_{\varepsilon,\eta})_{\varepsilon\in[0,\varepsilon^{\mathrm{c}}(\eta)]}$ *of invariant sets, where* $K_{0,\eta} = K_0$, *is upper semicontinuous at* $\varepsilon = 0$ *with respect to the family* $|\cdot|_{H_1^\varepsilon}$ *of norms i.e.*

$$\lim_{\varepsilon\to0^+} \sup_{w\in K_{\varepsilon,\eta}} \inf_{u\in K_0} |w-u|_{H_1^\varepsilon} = 0.$$

Remark 4.9. The family $(K_{\varepsilon,\eta})_{\varepsilon\in]0,\varepsilon^{\mathrm{c}}(\eta)]}$ is asymptotically independent of η i.e. whenever η_1 and $\eta_2 \in \,]0,\infty[$ then there is an $\varepsilon' \in \,]0,\min(\varepsilon^{\mathrm{c}}(\eta_1),\varepsilon^{\mathrm{c}}(\eta_2))]$ such that $K_{\varepsilon,\eta_1} = K_{\varepsilon,\eta_2}$ for $\varepsilon \in \,]0,\varepsilon']$.

Finally, we have the following (co)homology index continuation principle:

Theorem 4.10. *Assume the hypotheses of Theorem* 4.8 *and for every* $\eta \in \,]0,\infty[$ *let* $\varepsilon^{\mathrm{c}}(\eta) \in \,]0,\varepsilon_0]$ *be as in that theorem. Let* $(P,\prec)$ *be a finite poset. Let* $(M_{p,0})_{p\in P}$ *be a* $\prec$*-ordered Morse decomposition of* K_0 *relative to* π_0. *For each* $p \in P$, *let* $V_p \subset N$ *be closed in* X_0 *and such that* $M_{p,0} = \mathrm{Inv}_{\pi_0}(V_p) \subset \mathrm{Int}_{H_1^0}(V_p)$. (*Such sets* V_p, $p \in P$, *exist.*) *For* $\varepsilon \in \,]0,\varepsilon_0]$, *for every* $\eta \in \,]0,\infty[$ *and* $p \in P$ *set* $M_{p,\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(V_{p,\varepsilon,\eta})$, *where*

$$V_{p,\varepsilon,\eta} := \{\, u \in H_1^\varepsilon \mid Q_\varepsilon u \in V_p \text{ and } |(I-Q_\varepsilon)u|_{H_1^\varepsilon} \le \eta \,\}.$$

Then for every $\eta \in \,]0,\infty[$ there is an $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\eta) \in \,]0,\varepsilon^{c}(\eta)]$ such that for every $\varepsilon \in \,]0,\widetilde{\varepsilon}]$ and $p \in P$, $M_{p,\varepsilon,\eta} \subset \mathrm{Int}_{H^{\varepsilon}_{1}}(V_{p,\varepsilon,\eta})$ and the family $(M_{p,\varepsilon,\eta})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $K_{\varepsilon,\eta}$ relative to π_ε and the (co)homology index braids of $(\pi_0, K_0, (M_{p,0})_{p\in P})$ and $(\pi_\varepsilon, K_{\varepsilon,\eta}, (M_{p,\varepsilon,\eta})_{p\in P}))$, $\varepsilon \in \,]0,\widetilde{\varepsilon}]$, are isomorphic and so they determine the same collection of C-connection matrices.

Remark 4.11. Again, for each $p \in P$, the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in[0,\widetilde{\varepsilon}(\eta)]}$, where $M_{p,0,\eta} = M_{p,0}$ is upper semicontinuous at $\varepsilon = 0$ with respect to the family $|\cdot|_{H^{\varepsilon}_{1}}$ of norms and the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in]0,\widetilde{\varepsilon}(\eta)]}$ is asymptotically independent of η.

5. An application to systems of parabolic equations with large diffusion

In this section we verify the abstract conditions introduced in the previous sections for the family (E_ε) of equations introduced in section 1. We thus obtain singular convergence and singular compactness results with the ensuing Conley index and index braid continuation principles for the corresponding family π_ε of semiflows.

5.1. Let N be a positive integer and $\widetilde{\varepsilon}_0$ be a positive real number. Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $\Gamma = \partial\Omega$.

For each $\varepsilon \in \,]0,\widetilde{\varepsilon}_0]$, let $d_\varepsilon\colon \mathbb{R}^N \to \mathbb{R}$ be a positive smooth function. For each $\varepsilon \in \,]0,\widetilde{\varepsilon}_0]$ define

$$\sigma_1(\varepsilon) := \min\{d_\varepsilon(x) \mid x \in \mathrm{Cl}\,\Omega\} \text{ and } \sigma_2(\varepsilon) := \max\{d_\varepsilon(x) \mid x \in \mathrm{Cl}\,\Omega\}. \tag{5.1}$$

Assume that

$$\sigma_1(\varepsilon) \to \infty \quad \text{as } \varepsilon \to 0 \tag{5.2}$$

and

$$\sup\left\{\frac{\sigma_2(\varepsilon)}{\sigma_1(\varepsilon)} \;\middle|\; \varepsilon \in \,]0,\widetilde{\varepsilon}_0]\right\} < \infty. \tag{5.3}$$

For each $\varepsilon \in \,]0,\widetilde{\varepsilon}_0]$, let $V_\varepsilon \in L^{p_0}(\Omega)$ and $b_\varepsilon \in L^{q_0}(\Gamma)$ with

$$p_0 \begin{cases} \geq 1, & \text{for } N = 1; \\ > 1, & \text{for } N = 2; \\ \geq N/2, & \text{for } N \geq 3 \end{cases}$$

and

$$q_0 \begin{cases} \geq 1, & \text{for } N = 1; \\ > 1, & \text{for } N = 2; \\ \geq N-1, & \text{for } N \geq 3 \end{cases}$$

and assume that

$$\frac{1}{|\Omega|}\int_\Omega V_\varepsilon \, dx \to V_0 \quad \text{and} \quad \frac{1}{|\Gamma|}\int_\Gamma b_\varepsilon \, d\sigma \to b_0 \quad \text{as } \varepsilon \to 0. \tag{5.4}$$

Here, dx is the N-Lebesgue measure and $d\sigma$ is the surface measure on Γ.

Let $\gamma\colon H^1(\Omega)\to H^{1/2}(\Gamma)$ be the trace operator. For $\lambda\in\mathbb{R}$ and $\varepsilon\in\,]0,\widetilde{\varepsilon}_0]$ define the bilinear form $\tau_\varepsilon\colon H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ by

$$\tau_\varepsilon(u,v)=\int_\Omega d_\varepsilon\nabla u\nabla v\,dx+\int_\Omega(\lambda+V_\varepsilon)uv\,dx+\int_\Gamma b_\varepsilon\gamma(u)\gamma(v)\,d\sigma, \tag{5.5}$$

for u, $v\in H^1(\Omega)$. It follows from [13] (cf. formula (29)) that for each $\varepsilon\in\,]0,\widetilde{\varepsilon}_0]$ the bilinear form τ_ε is defined and continuous on $H^1(\Omega)\times H^1(\Omega)$. Moreover, [13, Theorem 3.1] implies that there exist a $\lambda_0\in\,]0,\infty[$ and an $\varepsilon_0\in\,]0,\widetilde{\varepsilon}_0]$ such that for all $\lambda>\lambda_0$ and for all $\varepsilon\in\,]0,\varepsilon_0]$, $\sigma_1(\varepsilon)-\lambda_0>0$ and

$$\tau_\varepsilon(u,u)\geq(\sigma_1(\varepsilon)-\lambda_0)\int_\Omega|\nabla u|^2\,dx+(\lambda-\lambda_0)\int_\Omega|u|^2\,dx,\quad u\in H^1(\Omega).$$

This implies that there exist λ_0, $\widetilde{\mu}\in\,]0,\infty[$ and an $\varepsilon_0'\in\,]0,\widetilde{\varepsilon}_0]$ such that for all $\lambda>\lambda_0$,

$$\tau_\varepsilon(u,u)\geq\widetilde{\mu}|u|^2_{H^1(\Omega)},\quad u\in H^1(\Omega),\ \varepsilon\in\,]0,\varepsilon_0']. \tag{5.6}$$

For the rest of this paper, we will assume that $\lambda>\lambda_0$. For each $\varepsilon\in\,]0,\varepsilon_0']$ the pair $(\tau_\varepsilon,\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Omega)})$ defines an operator $\mathbf{A}_\varepsilon\colon D(\mathbf{A}_\varepsilon)\to\mathbf{H}^\varepsilon:=L^2(\Omega)$. Specifically, let $D(\mathbf{A}_\varepsilon)$ be the set of all $u\in H^1(\Omega)$ such that there is a $w=w_u\in L^2(\Omega)$ with the property that

$$\tau_\varepsilon(u,v)=\langle w,v\rangle_{L^2(\Omega)}$$

for all $v\in H^1(\Omega)$. Then w_u is uniquely determined by u, the set $D(\mathbf{A}_\varepsilon)$ is a dense linear subspace both of $H^1(\Omega)$ and of $L^2(\Omega)$, and the map

$$\mathbf{A}_\varepsilon\colon D(\mathbf{A}_\varepsilon)\to L^2(\Omega),\quad u\mapsto w_u \tag{5.7}$$

is a linear positive self-adjoint operator in $(L^2(\Omega),\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Omega)})$ with $\mathbf{A}_\varepsilon^{-1}$ compact.

Remark 5.1. Let $\varepsilon\in\,]0,\varepsilon_0']$ and $\lambda>\lambda_0$. It is proved in [13] that $D(\mathbf{A}_\varepsilon)$ is the set of all $u\in H^1(\Omega)$ such that $-\operatorname{Div}(d_\varepsilon\nabla u)+V_\varepsilon u\in L^2(\Omega)$ and $d_\varepsilon\partial_\nu u+b_\varepsilon u=0$ in Γ. Here, ν is the exterior normal vector field on $\partial\Omega$ and $d_\varepsilon\partial_\nu u$ is the conormal derivative of u in some generalized sense. The linear operator $\mathbf{A}_\varepsilon$ is then given by

$$\mathbf{A}_\varepsilon u=-\operatorname{Div}(d_\varepsilon\nabla u)+(\lambda+V_\varepsilon)u\quad\text{for } u\in D(\mathbf{A}_\varepsilon).$$

Define

$$\mu:=V_0+\frac{|\Gamma|}{|\Omega|}\,b_0+\lambda. \tag{5.8}$$

It follows from [13, Theorem 3.4] that $\mu>0$. Let $\mathbf{H}^0$ be the set of (equivalence classes of) constant real functions on Ω and define $\mathbf{A}_0\colon\mathbf{H}^0\to\mathbf{H}^0$ by

$$\mathbf{A}_0u=\mu u,\quad u\in\mathbf{H}^0. \tag{5.9}$$

For $\varepsilon \in]0, \varepsilon_0']$ set $\mathbf{H}^\varepsilon = L^2(\Omega)$ and let $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{H}^\varepsilon} = \langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Omega)}$. Moreover, let $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{H}^0}$ be the restriction of $\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Omega)}$ to $\mathbf{H}^0 \times \mathbf{H}^0$. Notice that $\mathbf{H}^\varepsilon_0 = \mathbf{H}^\varepsilon$ for all $\varepsilon \in [0, \varepsilon_0']$.

Recall that $\mathbf{H}^\varepsilon_\alpha := H_\alpha(\mathbf{A}_\varepsilon)$ for $\varepsilon \in [0, \varepsilon_0']$ and $\alpha \in \mathbb{R}$. Then, if $\varepsilon \in]0, \varepsilon_0']$, it follows that $\mathbf{H}^\varepsilon_1 = H_1(\mathbf{A}_\varepsilon) = H^1(\Omega)$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{H}^\varepsilon_1} = \tau_\varepsilon(\,\cdot\,,\,\cdot\,)$. Furthermore, $\mathbf{H}^0_1 = \mathbf{H}^0$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{H}^0_1}$ is the restriction of $\mu\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\Omega)}$ to $\mathbf{H}^0 \times \mathbf{H}^0$.

PROPOSITION 5.2. *With the notation introduced above, there exists an* $\varepsilon_0 \in]0, \varepsilon_0']$ *such that the family* $(\mathbf{H}^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{H}^\varepsilon}, \mathbf{A}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies condition* (FSpec).

PROOF. It is clear that (1), (2) and (3) of condition (FSpec) are satisfied for all $\varepsilon \in]0, \varepsilon_0']$.

An application of (5.4), the definition of τ_ε and estimate (5.6) implies that there exist an $\varepsilon_0 \in]0, \varepsilon_0']$ and a constant $C \in]1, \infty[$ such that

$$|u|_{\mathbf{H}^\varepsilon_1} \leq C|u|_{\mathbf{H}^0_1} \quad \text{and} \quad |u|_{\mathbf{H}^0_1} \leq C|u|_{\mathbf{H}^\varepsilon_1}$$

for all $u \in \mathbf{H}^0_1$ and all $\varepsilon \in]0, \varepsilon_0]$. This proves that $(\mathbf{H}^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{H}^\varepsilon}, \mathbf{A}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies part (4) of condition (FSpec).

For every $\varepsilon \in]0, \varepsilon_0]$ let $(\lambda_{\varepsilon,j})_j$ be the repeated sequence of eigenvalues of $\mathbf{A}_\varepsilon$ and $(w_{\varepsilon,j})_j$ be a corresponding $\mathbf{H}^\varepsilon$-orthonormal sequence of eigenfunctions. By [13, Corollary 3.5] we may choose the eigenfunctions $w_{\varepsilon,1}$ to be nonnegative. Notice that μ is the eigenvalue of $\mathbf{A}_0$.

Let $(\varepsilon_n)_n$ be a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$. It follows from formulas (41) and (42) in [13, Theorem 3.4] that

$$\lambda_{\varepsilon_n,1} \to \mu \quad \text{as } n \to \infty, \quad \text{and} \quad \lambda_{\varepsilon_n,j} \to \infty \quad \text{as } n \to \infty \text{ for all } j \geq 2.$$

Let $\mathbf{1}_\Omega$ be (the equivalence class) of the constant function on Ω equal 1 and $\mathbf{1}_\Gamma$ be (the equivalence class) of the constant function on Γ equal 1 (the former equivalence class is taken with respect to the N-dimensional Lebesgue measure on Ω, while the latter is taken with respect to the surface measure on Γ). It follows that $\gamma(\mathbf{1}_\Omega) = \mathbf{1}_\Gamma$. Define $w_{0,1} := |\Omega|^{-1/2}\mathbf{1}_\Omega$. It follows that that $w_{0,1}$ is an eigenfunction of $\mathbf{A}_0$ corresponding to the eigenvalue μ and $|w_{0,1}|_{\mathbf{H}^0} = 1$. Moreover, for any sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ it follows from [13, Corollary 3.5] that

$$|w_{\varepsilon_{n_k},1} - w_{0,1}|_{H^1(\Omega)} \to 0 \quad \text{as } k \to \infty.$$

In particular,

$$|w_{\varepsilon_{n_k},1} - w_{0,1}|_{L^2(\Omega)} \to 0 \quad \text{as } k \to \infty$$

Thus, by Hölder inequality, for every $u \in L^2(\Omega)$,

$$\langle u, w_{\varepsilon_{n_k},1}\rangle_{L^2(\Omega)} \to \langle u, w_{0,1}\rangle_{L^2(\Omega)} \quad \text{as } k \to \infty.$$

This implies that for every $u \in \mathbf{H}^0$

$$\langle u, w_{\varepsilon_{n_k},1}\rangle_{\mathbf{H}^{\varepsilon_{n_k}}} \to \langle u, w_{0,1}\rangle_{\mathbf{H}^0} \quad \text{as } k \to \infty.$$

Now we only need to prove that

$$|w_{\varepsilon_{n_k},1} - w_{0,1}|_{\mathbf{H}_1^{\varepsilon_{n_k}}} \to 0 \quad \text{as } k \to \infty.$$

For each $k \in \mathbb{N}$ we have, by a simple calculation,

$$\begin{aligned}
|w_{\varepsilon_{n_k},1} &- w_{0,1}|^2_{\mathbf{H}_1^{\varepsilon_{n_k}}} = \tau_{\varepsilon_{n_k}}(w_{\varepsilon_{n_k},1} - w_{0,1}, w_{\varepsilon_{n_k},1} - w_{0,1}) \\
&= \tau_{\varepsilon_{n_k}}(w_{\varepsilon_{n_k},1}, w_{\varepsilon_{n_k},1}) - 2\tau_{\varepsilon_{n_k}}(w_{\varepsilon_{n_k},1}, w_{0,1}) + \tau_{\varepsilon_{n_k}}(w_{0,1}, w_{0,1}) \\
&= \lambda_{\varepsilon_{n_k},1}\langle w_{\varepsilon_{n_k},1}, w_{\varepsilon_{n_k},1}\rangle_{L^2(\Omega)} - 2\lambda_{\varepsilon_{n_k},1}\langle w_{\varepsilon_{n_k},1}, w_{0,1}\rangle_{L^2(\Omega)} \\
&\quad + |\Omega|^{-1}\bigg(\int_\Omega (\lambda + V_{\varepsilon_{n_k}})\,dx + \int_\Gamma b_{\varepsilon_{n_k}}\,d\sigma\bigg) \\
&\quad \to \mu - 2\mu + \mu = 0, \quad \text{as } k \to \infty.
\end{aligned}$$

Hence part (5) of condition (FSpec) is satisfied. The proof is complete. □

5.2. Let N, $\widetilde{\varepsilon}_0$, Ω, p_0 and q_0 be as in subsection 5.1. Let $r \in \mathbb{N}$ be arbitrary and for each $i \in [1\,.\,.\,r]$ and each $\varepsilon \in\,]0, \widetilde{\varepsilon}_0]$, let $d_{i,\varepsilon}$, $V_{i,\varepsilon}$ and $b_{i,\varepsilon}$ be functions and $V_{i,0}$, $b_{i,0}$ be constants such that all conditions of subsection 5.1 are satisfied. Define the bilinear form $\tau_{i,\varepsilon}$ as in (5.5). Now choose λ_0, $\widetilde{\mu} \in\,]0, \infty[$ and an $\varepsilon_0' \in\,]0, \widetilde{\varepsilon}_0]$ such that for all $\lambda > \lambda_0$ and all $i \in [1\,.\,.\,r]$, the estimate (5.6) is satisfied by $\tau_{i,\varepsilon}$.

Let $\lambda > \lambda_0$. Let $i \in [1\,.\,.\,r]$ be arbitrary. In the notation of subsection 5.1, for $\varepsilon \in [0, \varepsilon_0']$ let $H^\varepsilon_{(i)} = \mathbf{H}^\varepsilon$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon_{(i)}} = \langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{H}^\varepsilon}$. For $\varepsilon \in\,]0, \varepsilon_0']$, define the operator $A_{i,\varepsilon}$, as $\mathbf{A}_\varepsilon$ where τ_ε in formula (5.7) is replaced by $\tau_{i,\varepsilon}$. Set

$$\mu_i := V_{i,0} + \frac{|\Gamma|}{|\Omega|}\, b_{i,0} + \lambda$$

and define the operator $A_{i,0}$ as $\mathbf{A}_0$ in formula (5.9) (with μ replaced by μ_i).

It follows from Proposition 5.2 that there is an $\varepsilon_0 \in\,]0, \varepsilon_0']$ such that the family $(H^\varepsilon_{(i)}, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon_{(i)}}, A_{i,\varepsilon})_{\varepsilon\in[0,\varepsilon_0]}$, $i \in [1\,.\,.\,r]$, is as in Definition 4.1.

In what follows let

$$2^*_\Omega = \begin{cases} \dfrac{2N}{N-2}, & \text{if } N \geq 3; \\ \text{an arbitrary } p^* \in\,]0, \infty[, & \text{if } N = 2; \\ \infty, & \text{if } N = 1 \end{cases}$$

and

$$2^*_\Gamma = \begin{cases} \dfrac{2(N-1)}{N-2}, & \text{if } N \geq 3; \\ \text{an arbitrary } p^{**} \in\,]0, \infty[, & \text{if } N = 2; \\ \infty, & \text{if } N = 1. \end{cases}$$

By interpolation theory (cf. [16]) for every $i \in [1..r]$, every $\theta \in [0,1]$ and every $\varepsilon \in [0, \varepsilon_0]$ there is a continuous imbedding from $H^\varepsilon_{(i),\theta}$ to $H^\theta(\Omega)$ with imbedding constant $C_{1,\theta} \in \,]0,\infty[$ independent of $\varepsilon \in [0,\varepsilon_0]$ and $i \in [1..r]$. Furthermore, there is a continuous imbedding from $H^\theta(\Omega)$ into $L^{p_{\theta,\Omega}}(\Omega)$ with imbedding constant $C_{2,\theta} \in \,]0,\infty[$. Here,

$$p_{\theta,\Omega} = \left(\theta \frac{1}{2^*_\Omega} + (1-\theta)\frac{1}{2}\right)^{-1}.$$

Moreover, for every $\rho \in [0,1]$ there is a continuous imbedding from $H^{\rho/2}(\Gamma)$ into $L^{p_{\rho,\Gamma}}(\Gamma)$ with imbedding constant $C_{3,\rho} \in \,]0,\infty[$. Here,

$$p_{\rho,\Gamma} = \left(\rho \frac{1}{2^*_\Gamma} + (1-\rho)\frac{1}{2}\right)^{-1}.$$

Finally, by [11], for every $\theta \in \,]1/2, 1]$ there is a bounded linear trace operator $\gamma = \gamma_\theta \colon H^\theta(\Omega) \to H^{\theta-(1/2)}(\Gamma)$ with a bound $C_{4,\theta} \in \,]0,\infty[$. Now the continuity of the functions $\theta \mapsto p_{\theta,\Omega}$ and $\theta \mapsto p_{2\theta-1,\Gamma}$ at $\theta = 1$ implies the following result.

Lemma 5.3. *Let $q_2 \in \,](1-(1/2^*_\Omega))^{-1}, \infty[$ and $q_3 \in \,](1-(1/2^*_\Gamma))^{-1}, \infty[$ be arbitrary. Then there is a $\theta \in \,]1/2, 1]$ such that*

$$p_2 = \frac{q_2}{q_2 - 1} < p_{\theta,\Omega} \quad \textit{and} \quad p_3 = \frac{q_3}{q_3 - 1} < p_{2\theta-1,\Gamma}.$$

Set $\alpha = \theta$ and let $C_5 \in \,]0,\infty[$ (resp. $C_6 \in \,]0,\infty[$) be a bound of the imbedding $L^{p_{\alpha,\Omega}}(\Omega) \to L^{p_2}(\Omega)$ (resp. $L^{p_{2\alpha-1,\Gamma}}(\Gamma) \to L^{p_3}(\Gamma)$). Then, whenever $i \in [1..r]$, $\Phi_i \in L^{q_2}(\Omega)$, $\Psi_i \in L^{q_3}(\Gamma)$, $\varepsilon \in \,]0, \varepsilon_0]$ and $h_i \in H^\varepsilon_{(i),\alpha}$, then $\Phi_i h_i \in L^1(\Omega)$, $\Psi_i \gamma(h_i) \in L^1(\Gamma)$,

$$\int_\Omega |\Phi_i h_i| \, dx \le C_{1,\alpha} C_{2,\alpha} C_5 |\Phi|_{L^{q_2}(\Omega)} |h_i|_{H^\varepsilon_{(i),\alpha}},$$

$$\textit{and} \quad \int_\Gamma |\Psi_i \gamma(h_i)| \, d\sigma \le C_{1,\alpha} C_{4,\alpha} C_{3,2\alpha-1} C_6 |\Psi|_{L^{q_3}(\Gamma)} |h_i|_{H^\varepsilon_{(i),\alpha}}.$$

In particular, there is a unique $f_{i,\varepsilon} \in H^\varepsilon_{(i),-\alpha}$ such that

$$f_{i,\varepsilon}(h_i) = \int_\Omega \Phi_i h_i \, dx + \int_\Gamma \Psi_i \gamma(h_i) \, d\sigma, \quad h_i \in H^\varepsilon_{(i),\alpha}.$$

Moreover,

$$|f_{i,\varepsilon}|_{H^\varepsilon_{(i),-\alpha}} \le C_{7,\alpha} (|\Phi_i|_{L^{q_2}(\Omega)} + |\Psi_i|_{L^{q_3}(\Gamma)})$$

where $C_{7,\alpha} = \max(C_{1,\alpha} C_{2,\alpha} C_5, C_{1,\alpha} C_{4,\alpha} C_{3,2\alpha-1} C_6)$.

We define the map $f_\varepsilon \colon H^\varepsilon_\alpha \to \mathbb{R}$ by

$$f_\varepsilon(h) = \sum_{i=1}^r f_{i,\varepsilon}(h_i), \quad h = (h_1, \ldots, h_r) \in H^\varepsilon_\alpha.$$

Then $f_\varepsilon \in H^\varepsilon_{-\alpha}$ and in the notation of Section 2, $f_{i,\varepsilon} = \Lambda_{(i),\alpha}(f_\varepsilon)$ for $i \in [1..r]$.

Theorem 5.4. *For each $i \in [1\,.\,.\,r]$ and each $\varepsilon \in [0,\varepsilon_0]$, let $\Phi_{i,\varepsilon}\colon H^1(\Omega,\mathbb{R}^r) \to L^{q_2}(\Omega)$ and $\Psi_{i,\varepsilon}\colon H^{1/2}(\Gamma,\mathbb{R}^r) \to L^{q_3}(\Gamma)$ be maps satisfying the following assumptions:*

(1) *For all $M \in [0,\infty[$ there is an $L = L_M \in [0,\infty[$ such that*
(a) *for all $\varepsilon \in [0,\varepsilon_0]$ and all u, $v \in H^1(\Omega,\mathbb{R}^r)$ such that $|u|_{H^1(\Omega,\mathbb{R}^r)}$, $|v|_{H^1(\Omega,\mathbb{R}^r)} \le M$,*

$$|\Phi_{i,\varepsilon}(u) - \Phi_{i,\varepsilon}(v)|_{L^{q_2}(\Omega)} \le L|u-v|_{H^1(\Omega,\mathbb{R}^r)}$$

(b) *for all $\varepsilon \in [0,\varepsilon_0]$ and all u, $v \in H^{1/2}(\Gamma,\mathbb{R}^r)$ with $|u|_{H^{1/2}(\Gamma,\mathbb{R}^r)}$, $|v|_{H^{1/2}(\Gamma,\mathbb{R}^r)} \le M$,*

$$|\Psi_{i,\varepsilon}(u) - \Psi_{i,\varepsilon}(v)|_{L^{q_3}(\Gamma)} \le L|u-v|_{H^{1/2}(\Gamma,\mathbb{R}^r)}.$$

(2) *For every $u \in H^0$,*

$$|\Phi_{i,\varepsilon}(u) - \Phi_{i,0}(u)|_{L^{q_2}(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

(3) *For every $u \in H^{1/2}(\Gamma,\mathbb{R}^r)$,*

$$|\Psi_{i,\varepsilon}(u) - \Psi_{i,0}(u)|_{L^{q_3}(\Gamma)} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Let $\alpha \in \,]1/2,1]$ be as in Lemma 5.3. For $i \in [1\,.\,.\,r]$, $\varepsilon \in [0,\varepsilon_0]$ and $u \in H_1^\varepsilon$ define, for $h_i \in H^\varepsilon_{(i),\alpha}$,

$$f_{i,\varepsilon}(u)(h_i) = \int_\Omega \Phi_{i,\varepsilon}(u)h_i\,dx + \int_\Gamma \Psi_{i,\varepsilon}(\gamma(u))\gamma(h_i)\,d\sigma.$$

Moreover, we define the map $f_\varepsilon(u)\colon H^\varepsilon_\alpha \to \mathbb{R}$ by

$$f_\varepsilon(u)(h) = \sum_{i=1}^r f_{i,\varepsilon}(u)(h_i), \quad h = (h_1,\ldots,h_r) \in H^\varepsilon_\alpha.$$

Then $f_\varepsilon(u) \in H^\varepsilon_{-\alpha}$ and in the notation of section 2, $f_{i,\varepsilon}(u) = \Lambda_{(i),\alpha}(f_\varepsilon(u))$ for $i \in [1\,.\,.\,r]$. Finally, the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ of maps satisfies condition (Conv).

Remark. By the definition of 2^*_Ω and 2^*_Γ we may, for $N = 1$, 2, take q_2 and q_3 arbitrary in $]1,\infty[$.

Proof of Theorem 5.4. Lemma 5.3 implies that the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies (1) of condition (Conv). Let $M \in [0,\infty[$ be arbitrary and $L = L_M$ be as in assumption (1). If $i \in [1\,.\,.\,r]$, $\varepsilon \in [0,\varepsilon_0]$ and u, $v \in H^\varepsilon_1$ with $|u|_{H^\varepsilon_1}$, $|v|_{H^\varepsilon_1} \le \min(M/C_{1,1}, M/(C_{1,1}C_{4,1}))$ then u, $v \in H^1(\Omega,\mathbb{R}^r)$ with $|u|_{H^1(\Omega,\mathbb{R}^r)}$, $|v|_{H^1(\Omega,\mathbb{R}^r)} \le M$ and $|\gamma(u)|_{H^{1/2}(\Gamma,\mathbb{R}^r)}$, $|\gamma(v)|_{H^{1/2}(\Gamma,\mathbb{R}^r)} \le M$ so

$$\begin{aligned}|f_{i,\varepsilon}(u) &- f_{i,\varepsilon}(v)|_{H^\varepsilon_{-\alpha}} \le C_{7,\alpha}|\Phi_{i,\varepsilon}(u) - \Phi_{i,\varepsilon}(v)|_{L^{q_2}(\Omega)} \\ &+ C_{7,\alpha}|\Psi_{i,\varepsilon}(\gamma(u)) - \Psi_{i,\varepsilon}(\gamma(v))|_{L^{q_3}(\Gamma)} \\ &\le C_{7,\alpha}(L + LC_{4,1})|u-v|_{H^1(\Omega,\mathbb{R}^r)} \le C_{7,\alpha}(L+LC_{4,1})C_{1,1}|u-v|_{H^\varepsilon_1}.\end{aligned}$$

This together with assumption (1) implies part (3) of condition (Conv). If $i \in [1..r]$ and $u \in H_1^0$ then

$$|f_{i,\varepsilon}(u)|_{H^\varepsilon_{-\alpha}} \leq C_{7,\alpha}(|\Phi_{i,\varepsilon}(u)|_{L^{q_2}(\Omega,\mathbb{R}^r)} + |\Psi_{i,\varepsilon}(\gamma(u))|_{L^{q_3}(\Gamma,\mathbb{R}^r)}).$$

This together with assumptions (2) and (3) easily implies part (4) of condition (Conv).

Now let $w \in H_1^0$ be arbitrary and $(\varepsilon_n)_n$ be a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$. Let $t \in]0,\infty[$ be arbitrary. We will show that

$$\lim_{n\to\infty} |e^{-t\widetilde{A}_{\varepsilon_n}} f_{\varepsilon_n}(w) - e^{-t\widetilde{A}_0} f_0(w)|_{H_1^{\varepsilon_n}} = 0, \tag{5.10}$$

proving (2) of condition (Conv).

By the considerations in section 2 we only have to show that, for every $i \in [1..r]$, every $w \in H_1^0$, every sequence $(\varepsilon_n)_n$ in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and every $t \in]0,\infty[$

$$\lim_{n\to\infty} |e^{-t\widetilde{A}_{i,\varepsilon_n}} f_{i,\varepsilon_n}(w) - e^{-t\widetilde{A}_{i,0}} f_{i,0}(w)|_{H^{\varepsilon_n}_{(i),1}} = 0. \tag{5.11}$$

It follows from Proposition 5.2 that the family $(H^\varepsilon_{(i)}, \langle\cdot,\cdot\rangle_{H^\varepsilon_{(i)}}, A_{i,\varepsilon})_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition (FSpec). For $n \in \mathbb{N}$ set $u_n = f_{i,\varepsilon_n}(w)$ and define $v_n \in H^{\varepsilon_n}_{i,-\alpha}$ by

$$v_n(h_i) = \int_\Omega \Phi_{i,0}(w) h_i\, dx + \int_\Gamma \Psi_{i,0}(\gamma(w))\gamma(h_i)\, d\sigma, \quad h_i \in H^{\varepsilon_n}_{i,\alpha}.$$

Finally, set $u = f_{i,0}(w)$. Then

$$\begin{aligned} |u_n - v_n|_{H^{\varepsilon_n}_{i,-\alpha}} \leq{}& C_{7,\alpha}(|\Phi_{i,\varepsilon_n}(w) - \Phi_{i,0}(w)|_{L^{q_2}(\Omega)} \\ &+ |\Psi_{i,\varepsilon_n}(\gamma(w)) - \Psi_{i,0}(\gamma(w))|_{L^{q_3}(\Gamma)}). \end{aligned} \tag{5.12}$$

Notice that the right hand side of this estimate goes to zero as $n \to \infty$.

Let $C_8 \in]0,\infty[$ be a bound for the imbedding $H^1(\Omega) \to H^\alpha(\Omega)$. Then, with obvious notation, we obtain, for every $j \in \mathbb{N}$,

$$\begin{aligned} |v_n(w_{i,\varepsilon_n,j}) - u(w_{i,0,j})| \leq{}& |\Phi_{i,0}(w)|_{L^{q_2}(\Omega)} |w_{i,\varepsilon_n,j} - w_{i,0,j}|_{L^{p_2}(\Omega)} \\ &+ |\Psi_{i,0}(\gamma(w))|_{L^{q_3}(\Gamma)} |\gamma(w_{i,\varepsilon_n,j} - w_{i,0,j})|_{L^{p_3}(\Gamma)} \leq \widetilde{C} |w_{i,\varepsilon_n,j} - w_{i,0,j}|_{H^{\varepsilon_n}_{i,1}}, \end{aligned}$$

where

$$\widetilde{C} := C_5 C_{2,\alpha} C_8 C_{1,1} |\Phi_0(w)|_{L^{q_2}(\Omega)} + C_6 C_{3,2\alpha-1} C_{4,\alpha} C_8 C_{1,1} |\Psi_0(\gamma(w))|_{L^{q_3}(\Gamma,\mathbb{R}^r)}.$$

Hence

$$|v_n(w_{i,\varepsilon_n,j}) - u(w_{i,0,j})| \to 0 \quad \text{as } n \to \infty. \tag{5.13}$$

Now, for all $n \in \mathbb{N}$,

$$|v_n|_{H^{\varepsilon_n}_{-\alpha}} \leq C_{7,\alpha}(|\Phi_{i,0}(w)|_{L^{p_2}(\Omega,\mathbb{R}^r)} + |\Psi_{i,0}(\gamma(w))|_{L^{p_3}(\Gamma)}). \tag{5.14}$$

Formulas (5.12)–(5.14) imply that the assumptions of Theorem 3.7 are satisfied. An application of Theorem 3.7 implies (5.10). The proof is complete. □

Now assume the following

Hypothesis 5.5. *For $i \in [1\,.\,.\,r]$ and $\varepsilon \in [0,\varepsilon_0]$, $\varphi_{i,\varepsilon}\colon \Omega \times \mathbb{R}^r \to \mathbb{R}$ and $\psi_{i,\varepsilon}\colon \Gamma \times \mathbb{R}^r \to \mathbb{R}$, $(x,s) \mapsto \varphi_{i,\varepsilon}(x,s)$, $(x,s) \mapsto \psi_{i,\varepsilon}(x,s)$, are functions such that*

1. *there is a null set N_Ω in Ω with $\varphi_{i,\varepsilon}(x,\,\cdot\,) \in C^1(\mathbb{R}^r,\mathbb{R})$ for all $x \in \Omega \setminus N_\Omega$;*
2. *there is a null set N_Γ in Γ (rel. to the surface measure on Γ) with $\psi_{i,\varepsilon}(x,\,\cdot\,) \in C^1(\mathbb{R}^r,\mathbb{R})$ for all $x \in \Gamma \setminus N_\Gamma$;*
3. *for all $s \in \mathbb{R}^r$, $\varphi_{i,\varepsilon}(\,\cdot\,,s)$ and $D_s\varphi_{i,\varepsilon}(\,\cdot\,,s)$ is measurable on Ω;*
4. *for all $s \in \mathbb{R}^r$, $\psi_{i,\varepsilon}(\,\cdot\,,s)$ and $D_s\psi_{i,\varepsilon}(\,\cdot\,,s)$ is measurable on Γ.*

Moreover, $q_2 \in \,](1-(1/2^*_\Omega))^{-1}, 2^*_\Omega[$ and $q_3 \in \,](1-(1/2^*_\Gamma))^{-1}, 2^*_\Gamma[$ and

$$r_2 = \frac{2^*_\Omega q_2}{2^*_\Omega - q_2}, \quad r_3 = \frac{2^*_\Gamma q_3}{2^*_\Gamma - q_3}, \quad \beta_2 = \frac{2^*_\Omega}{q_2} - 1, \quad \beta_3 = \frac{2^*_\Gamma}{q_3} - 1.$$

There is a constant $\widetilde{C} \in \,]0,\infty[$ and functions $a_2 \in L^{r_2}(\Omega)$, $b_2 \in L^{q_2}(\Omega)$, $a_3 \in L^{r_3}(\Gamma)$, $b_3 \in L^{q_3}(\Gamma)$ such that, for all $\varepsilon \in [0,\varepsilon_0]$,

$$\begin{aligned}
\|D_s\varphi_{i,\varepsilon}(x,s)\| &\le \widetilde{C}(a_2(x) + \|s\|^{\beta_2}), && \text{for } (x,s) \in (\Omega \setminus N_\Omega) \times \mathbb{R}^r,\\
|\varphi_{i,\varepsilon}(x,0)| &\le b_2(x), && \text{for } x \in \Omega \setminus N_\Omega,\\
\|D_s\psi_{i,\varepsilon}(x,s)\| &\le \widetilde{C}(a_3(x) + \|s\|^{\beta_3}), && \text{for } (x,s) \in (\Gamma \setminus N_\Gamma) \times \mathbb{R}^r,\\
|\psi_{i,\varepsilon}(x,0)| &\le b_3(x), && \text{for } x \in \Gamma \setminus N_\Gamma.
\end{aligned}$$

Finally, as $\varepsilon \to 0^+$,

$$\begin{aligned}
|\varphi_{i,\varepsilon}(x,s) - \varphi_0(x,s)| &\to 0, && \text{for } (x,s) \in (\Omega \setminus N_\Omega) \times \mathbb{R}^r,\\
|\psi_{i,\varepsilon}(x,s) - \psi_0(x,s)| &\to 0, && \text{for } (x,s) \in (\Gamma \setminus N_\Gamma) \times \mathbb{R}^r.
\end{aligned}$$

Theorem 5.6. *Assume Hypothesis 5.5. For $i \in [1\,.\,.\,r]$ and $\varepsilon \in [0,\varepsilon_0]$ and $u \in H^1(\Omega,\mathbb{R}^r)$ (resp. $u \in H^{1/2}(\Gamma,\mathbb{R}^r)$) define $\Phi_{i,\varepsilon}(u)(x) = \varphi_{i,\varepsilon}(x,u(x))$ (resp. $\Psi_{i,\varepsilon}(u)(x) = \psi_{i,\varepsilon}(x,u(x))$) for $x \in \Omega$ (resp. $x \in \Gamma$).*

Then $\Phi_{i,\varepsilon}\colon H^1(\Omega,\mathbb{R}^r) \to L^{q_2}(\Omega,\mathbb{R})$ and $\Psi_{i,\varepsilon}\colon H^{1/2}(\Gamma,\mathbb{R}^r) \to L^{q_3}(\Gamma,\mathbb{R})$ are defined and satisfy the assumptions of Theorem 5.4.

Proof. Use results and arguments in [7, Chapter 2]. □

Finally we obtain the following

Corollary 5.7. *For $i \in [1\,.\,.\,r]$ and $\varepsilon \in [0,\varepsilon_0]$ let $\varphi_{i,\varepsilon}\colon \Omega \times \mathbb{R}^r \to \mathbb{R}$ and $\psi_{i,\varepsilon}\colon \Gamma \times \mathbb{R}^r \to \mathbb{R}$, $(x,s) \mapsto \varphi_{i,\varepsilon}(x,s)$, $(x,s) \mapsto \psi_{i,\varepsilon}(x,s)$, be functions as in Theorem 5.6. For $\varepsilon \in [0,\varepsilon_0]$ and $u \in H^1(\Omega,\mathbb{R}^r)$ (resp. $u \in H^{1/2}(\Gamma,\mathbb{R}^r)$) define $\Phi_{i,\varepsilon}(u)(x) = \varphi_{i,\varepsilon}(x,u(x))$ (resp. $\Psi_{i,\varepsilon}(u)(x) = \psi_{i,\varepsilon}(x,u(x))$) for $x \in \Omega$ (resp.*

$x \in \Gamma$) and let $\alpha \in]1/2, 1]$ be as in Lemma 5.3. *For $i \in [1\,.\,.\,r]$, $\varepsilon \in [0, \varepsilon_0]$ and $u \in H_1^\varepsilon$ define,*

$$f_{i,\varepsilon}(u)(h) = \int_\Omega \Phi_{i,\varepsilon}(u) h \, dx + \int_\Gamma \Psi_{i,\varepsilon}(\gamma(u))\gamma(h) \, d\sigma, \quad h_i \in H^\varepsilon_{(i),\alpha}.$$

Moreover, we define the map $f_\varepsilon(u) \colon H^\varepsilon_\alpha \to \mathbb{R}$ by

$$f_\varepsilon(u)(h) = \sum_{i=1}^r f_{i,\varepsilon}(u)(h_i), \quad h = (h_1, \ldots, h_r) \in H^\varepsilon_\alpha.$$

Then $f_\varepsilon(u) \in H^\varepsilon_{-\alpha}$ and in the notation of Section 2, *$f_{i,\varepsilon}(u) = \Lambda_{(i),\alpha}(f_\varepsilon(u))$ for $i \in [1\,.\,.\,r]$. Finally, the family $(f_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ of maps satisfies condition* (Conv).

Proof. This follows from Theorems 5.6 and 5.4. □

With the family $(H^\varepsilon_{(i)}, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon_{(i)}}, A_{i,\varepsilon})_{\varepsilon \in [0,\varepsilon_0]}$, $i \in [1\,.\,.\,r]$ as in this subsection and the family $(f_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ as in Corollary 5.7 consider, for every $\varepsilon \in [0, \varepsilon_0]$, the corresponding abstract parabolic equation (4.2) (or, equivalently, the system (4.3)) and the corresponding local semiflow π_ε on H^ε_1.

If $\varepsilon > 0$ then Remark 5.1 shows that system (4.3) can be regarded as the abstract formulation of the system (E_ε) of boundary value problems introduced in Section 1.

If $\varepsilon = 0$, then using the notation from the proof of Proposition 5.2 we obtain from Corollary 5.7 and formula (2.22) with $\ell_i = 1$ and $v_{i,1} = |\Omega|^{-1/2}\mathbf{1}_\Omega$, $i \in [1\,.\,.\,r]$, that system (4.3) is just the system (E_0) from section 1 of ordinary differential equations on the r-dimensional linear subspace $H^1_c(\Omega, \mathbb{R}^r)$ of $H^1(\Omega, \mathbb{R}^r)$ consisting of (equivalence classes) of constant functions.

We conclude that all convergence, compactness and index continuation results of section 4 hold in the present case.

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