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EXISTENCE RESULTS FOR GENERALIZED VARIATIONAL INEQUALITIES VIA TOPOLOGICAL METHODS

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Abstract. In this paper we find existence results for elliptic and parabolic nonlinear variational inequalities involving a multivalued map. Both cases of a lower semicontinuous multivalued map and an upper semicontinuous one are considered.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, we use topological methods to establish existence results for a class of nonlinear variational inequalities on convex closed sets. The inequalities considered involve a quasilinear operator of class S_+ and the nonlinear part is given by the sum of a Carathéodory map and a multivalued map (multimap). We take into account both the cases of elliptic variational inequalities and parabolic variational inequalities. We look for solutions in $W_0^{m,p}(\Omega) = W_0^{m,p}(\Omega, \mathbb{R})$ $(1 < p < \infty)$ and in $L^p([0,d], W_0^{m,p}(\Omega))$, $2 \leq p < \infty$, in the elliptic and parabolic case, respectively.

Problems of this kind have been studied by many authors and appear in many applications, such as the obstacle and bi-obstacle problem, or the elasto-plastic

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torsion problem, in which the set K is given by gradient conditions. We mention the works of S. Hu and N. Papageorgiou [9], S. Aizicovici, N. Papageorgiou and V. Staicu [1], M. Väth [16], K.Q. Lan [12], the monograph of S. Carl, Lee Vy Khoi and D. Motreanu [5], and the references therein. It is also worth to mention the work of B. Mordukhovich for the link between differential inclusions and variational inequalities (see [13], [14]) and of M. Kučera for the relation with partial differential equations (see [8]).

Observe that if the set on which the variational inequality is valid coincides with the whole space $W_0^{m,p}(\Omega)$ or $L^p([0,d],W_0^{m,p}(\Omega))$, the solutions of the variational inequalities are weak solutions of elliptic and parabolic partial differential inclusions involving a second order differential operator in divergence form. Different methods have been applied to solve these problems, the more used ones being the method of upper and lower solutions (see e.g. [5]) and the degree theory approach. The latter was first used for semilinear variational inequalities by A. Szulkin [15] and E. Miesermann [11]. In [12] the author proves existence results for variational inequalities involving a demicontinuous S-contractive, map A, i.e. $I-A$ is of S_+-type; he finds, as an application, weak solutions for semilinear second-order elliptic inequalities.

Concerning the multivalued case, in [16] a fixed point index is constructed for the studied partial differential inclusion. In [9] and in [1] degree theory methods based on the degree map for multivalued perturbation of a S_+ operator are applied: in [9] the authors prove existence results for a class of partial differential inclusions with an upper semicontinuous multivalued nonlinearity; in [1] multiplicity results are proved both for partial differential inclusions and variational inequalities with, as multimap involved, the generalized subdifferential of a locally Lipschitz function.

On the other hand, we consider both the cases of an upper semicontinuous and a lower semicontinuous general kind of multivalued nonlinearity. To solve the problem we use a linearization argument. More precisely, we define a suitable multivalued operator (multioperator) whose fixed points are the solutions of the variational inequalities considered.

We do not assume any regularity in terms of compactness, neither on the quasilinear operator nor on the nonlinearity part to apply the topological degree theory for completely continuous multimap (see [10]), to obtain the existence of at least a fixed point. Moreover, with this approach we do not require any restriction on the set K, as done in [5], see Example 5.1.

2. Preliminaries

A multimap $G: \mathbb{R}^k \multimap \mathbb{R}$ is said to be:

(a) (upper semicontinuous (u.s.c.)) if $G^{-1}(V) = \{x \in \mathbb{R}^k : G(x) \subset V\}$ is an open subset of $\mathbb{R}^k$ for every open $V \subseteq \mathbb{R}$;
(b) (lower semicontinuous (l.s.c.)) if $G^{-1}(Q) = \{x \in \mathbb{R}^k : G(x) \subset Q\}$ is a closed subset of $\mathbb{R}$ for every closed set $Q \subset \mathbb{R}$
(c) (closed) if its graph $G_F = \{(x, y) \in \mathbb{R}^k \times \mathbb{R} : y \in G(x)\}$ is a closed subset of $\mathbb{R}^k \times \mathbb{R}$;
(d) (compact) if it maps bounded sets into compact ones;
(e) (completely continuous) if it is u.s.c. and compact.

For u.s.c. multimaps the following relations hold.

Theorem 2.1 (see [10, Theorem 1.1.4]). *An u.s.c. multimap $G: \mathbb{R}^k \multimap \mathbb{R}$ with closed values is a closed multimap.*

Theorem 2.2 (see [10, Theorem 1.1.5]). *A closed multimap $G: \mathbb{R}^k \multimap \mathbb{R}$ with compact values, such that maps bounded sets into compact ones is u.s.c.*

A map $g: \mathbb{R}^k \to \mathbb{R}$ is said to be a *Carathéodory map* if it is measurable with respect to the first variable and continuous with respect to the other $k-1$ variables.

A map $g: \mathbb{R}^k \to \mathbb{R}$ is said to be a *selection* of the multivalued map G if $g(x) \in G(x)$ for any $x \in \mathbb{R}^k$.

Theorem 2.3 (see [7, Theorem 4.4.33]). *Let $\mathcal{D} \subset \mathbb{R}^k$ be a given domain and $G: \mathcal{D} \times \mathbb{R} \multimap \mathbb{R}$ be a multimap with closed convex values such that*

(a) *$(x, u) \multimap G(x, u)$ is measurable and*
(b) *$G(x, \cdot\,): \mathbb{R} \multimap \mathbb{R}$ is l.s.c. for all $x \in \mathcal{D}$,*

then G admits a Carathéodory selection, i.e. there exists a Carathéodory map $g: \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ such that $g(x, u) \in G(x, u)$ for all $(x, u) \in \mathcal{D} \times \mathbb{R}$.

A straightforward generalization of [10, Theorem 1.3.5] leads to the following result.

Theorem 2.4. *Let $\mathcal{D} \subset \mathbb{R}^k$ be a given domain and $G: \mathcal{D} \times \mathbb{R} \multimap \mathbb{R}$ be a multimap with compact values such that*

(a) *for every $u \in \mathbb{R}$ the multimap $G(\,\cdot\,, u): \mathcal{D} \multimap \mathbb{R}$ is measurable and*
(b) *for almost every $x \in \mathcal{D}$ the multimap $F(x, \cdot\,) \multimap \mathbb{R}$ is u.s.c.,*

then for every measurable map $q: \mathcal{D} \to \mathbb{R}$ there exists a measurable selection $\phi: \mathcal{D} \to \mathbb{R}$ of the multimap $\Phi: \mathcal{D} \multimap \mathbb{R}$,

$$\Phi(x) = G(x, q(x)),$$

i.e. ϕ is a measurable map and $\phi(x) \in \Phi(x, q(x))$ for almost every $x \in \mathcal{D}$.

Let E be a Banach space and E^* its dual space, an operator $A: E \to E^*$ is said *to satisfy the S_+ condition* if and only if the weak convergence of a sequence $\{u_n\} \subset E$ to $u \in E$ and the condition $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \le 0$ imply the strong convergence of $\{u_n\}$ to u in E.

The existence of solutions of the variational inequalities considered will be investigated by means of the well-known Ky Fan fixed point theorem.

Theorem 2.5 (Ky Fan). *Let X be a Hausdorff locally convex topological vector space, V be a closed convex subset of X and $G: V \multimap V$ be a compact u.s.c. multimap. Then G has a fixed point.*

Given a Banach space E we denote with $B_M(0)$ the ball with radius M and center 0. Moreover, given a domain $\mathcal{D} \subset \mathbb{R}^k$ we denote in the whole paper with $\|u\|_p$, $\|u\|_{m,p}$, $\|u\|_0$ the usual norm for $L^p(\mathcal{D}) = L^p(\mathcal{D}, \mathbb{R})$, $W^{m,p}(\mathcal{D}) = W^{m,p}(\mathcal{D}, \mathbb{R})$ and $W_0^{m,p}(\mathcal{D}) = W_0^{m,p}(\mathcal{D}, \mathbb{R})$, respectively.

3. Elliptic variational inequalities

We consider the following variational inequalities:

$$\begin{cases} \displaystyle\int_\Omega \sum_{|\alpha|\le m} A_\alpha(x, \eta(u(x)))(D^\alpha v(x) - D^\alpha u(x))\,dx \\ \displaystyle\qquad \ge \int_\Omega (g(x,\eta(u(x))) + f(x,u(x)) + h)(v(x) - u(x))\,dx & \text{for all } v \in K, \\ f(x,u(x)) \in F(x,u(x)) & \text{a.e. } x \in \Omega, \end{cases} \tag{3.1}$$

$$\begin{cases} \displaystyle\int_\Omega \sum_{|\alpha|\le m} A_\alpha(x, \eta(u(x)))(D^\alpha v(x) - D^\alpha u(x))\,dx \\ \displaystyle\qquad \ge \int_\Omega (g(x,\eta(u(x))) + f(x) + h)(v(x) - u(x))\,dx & \text{for all } v \in K, \\ f(x) \in F(x,u(x)) & \text{a.e. } x \in \Omega, \end{cases} \tag{3.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, K a closed convex subset of $W_0^{m,p}(\Omega)$ $(1 < p < \infty)$ with $0 \in K$, α a multiindex, $\eta(u) = \{D^\alpha u : |\alpha| \le m\}$, the function A_α maps $\Omega \times \mathbb{R}^{N_m}$ into $\mathbb{R}$ (with $N_m = (N+m)!/(N!m!)$), $g: \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ and $F: \Omega \times \mathbb{R} \multimap \mathbb{R}$ are a given map and multimap, respectively, finally $h \in (W_0^{m,p}(\Omega))^*$.

Let q be such that $1/p + 1/q = 1$, we assume the following hypotheses on the function $A_\alpha: \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$:

(A1) $x \to A_\alpha(x,\eta)$ is measurable in Ω for any $\eta \in \mathbb{R}^{N_m}$;
$\eta \to A_\alpha(x,\eta)$ is continuous for almost all (a.a.) $x \in \Omega$;

there exist a function $k_0 \in L^q(\Omega)$ and a constant ν such that

$$|A_\alpha(x,\eta)| \le k_0(x) + \nu(\|\eta\|^{p-1}), \quad \text{a.e. in } \Omega, \text{ for any } \eta \in \mathbb{R}^{N_m};$$

(A2) for all $x \in \Omega$ and η, η', $\eta \neq \eta'$,

$$\sum_{|\alpha|\le m} (A_\alpha(x,\eta) - A_\alpha(x,\eta'))(\eta_\alpha - \eta'_\alpha) > 0;$$

(A3) there exist a function $k_1 \in L^1(\Omega)$ and a constant μ such that

$$\sum_{|\alpha|\le m} A_\alpha(x,\eta)\eta_\alpha \ge \mu\|\eta\|^p - k_1(x), \quad \text{a.e. in } \Omega \text{ and for all } \eta \in \mathbb{R}^{N_m}.$$

As a consequence the function A_α generates an operator A from $W_0^{m,p}(\Omega)$ into its dual $(W_0^{m,p}(\Omega))^*$ defined by

$$\langle Au, \varphi\rangle = \int_\Omega \sum_{|\alpha|\le m} A_\alpha(x, \eta(u(x)))D^\alpha\varphi(x)\,dx.$$

A typical example that satisfies (A1)–(A3) is the p-Laplacian operator.

As it is well known, under previous hypotheses, the operator $A: W_0^{m,p}(\Omega) \to (W_0^{m,p}(\Omega))^*$ is continuous, bounded, monotone, and satisfies the S_+ condition (see e.g. [5 Theorem 2.109]).

Remark 3.1. Observe that if the set K coincides with the whole space $W_0^{m,p}(\Omega)$, the solutions of the variational inequalities (3.1) and (3.2) are weak solutions of the following partial differential inclusion:

$$\begin{cases} -h \in \displaystyle\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \eta(u(x))) + g(x, \eta(u(x))) + F(x, u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Given $q \in W_0^{m,p}(\Omega)$, consider the linearized variational inequality:

$$\langle A(u), v - u\rangle \ge \int_\Omega (g(x, \eta(q(x))) + f(x, q(x)) + h)(v(x) - u(x))\,dx, \tag{3.3}$$

for all $v \in K$, where $f: \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory selection of the multimap F.

Theorem 3.2. *Let $A_\alpha: \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ satisfy hypotheses* (A1)–(A3) *and $g: \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ be a Carathéodory map such that*

$$|g(x,\eta)| \le k_2(x) + c_1(\|\eta\|^\sigma) \quad \textit{a.e. in } \Omega, \textit{ for all } \eta \in \mathbb{R}^{N_m},$$

with $k_2 \in L^q(\Omega)$, $c_1 > 0$ and $1 \le \sigma < p - 1$. Let $F: \Omega \times \mathbb{R} \multimap \mathbb{R}$ be a measurable multimap with closed convex values such that:

(a) *$F(x, \cdot): \mathbb{R} \multimap \mathbb{R}$ is l.s.c. for all $x \in \Omega$;*
(b) *$\|F(x,u)\| \le a(x) + b|u|^\sigma$ almost everywhere in Ω, for all $u \in \mathbb{R}$ with $a \in L^q(\Omega)$, $b > 0$ and $1 \le \sigma < p - 1$.*

Then the problem (3.1) *has a nonempty and compact solutions set.*

Proof. Hypotheses (a)–(b) on the multimap F imply the existence of a Carathéodory selection (see Theorem 2.3) and hence the variational inequality (3.3) is well defined.

We can assume without loss of generality (w.l.o.g.) that $0 \in K$: if this is not the case, we can consider an element $u_0 \in K$ and solve the analogous problem:

$$\begin{cases} \langle \overline{A}(w), v' - w\rangle \geq \displaystyle\int_\Omega (\overline{g}(x, \eta(w(x))) + \overline{f}(x, w(x)) + h)(v'(x) - w(x))\,dx, \\ \hfill \text{for all } v' \in K_1, \\ \overline{f}(x, w(x)) \in \overline{F}(x, w(x)) \hfill \text{a.e. } x \in \Omega, \end{cases}$$

where $K_1 = K - u_0$, $w = u - u_0$, $\overline{A}\colon W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$, $\overline{g}\colon \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ and $\overline{F}\colon \Omega \times \mathbb{R} \multimap \mathbb{R}$ are defined by $\overline{A}(w) = A(w + u_0)$, $\overline{g}(x, \eta(w(x))) = g(x, \eta(w(x) + u_0(x)))$, $\overline{F}(x, w(x)) = F(x, w(x) + u_0(x))$, respectively.

We split the proof in several steps and for sake of simplicity we assume $m = 1$.

Let U_f be the solution set of (3.3). Denote with T the multioperator

$$T\colon W_0^{1,p}(\Omega) \multimap W_0^{1,p}(\Omega), \quad q \mapsto \{U_f, f(x, q(x)) \in F(x, q(x))\}.$$

Step 1. The multioperator T has nonempty closed convex values.

Indeed, consider the functional $G\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as:

$$G(u) = \int_\Omega (g(x, q(x), Dq(x)) + f(x, q(x)) + h)u(x)\,dx.$$

We have:

$$\begin{aligned} |G(u)| &= \left| \int_\Omega (g(x, q(x), Dq(x)) + f(x, q(x)) + h)u(x)\,dx \right| \\ &\leq \int_\Omega (|g(x, q(x), Dq(x))| + |f(x, q(x))| + |h|)|u(x)|\,dx \\ &\leq \int_\Omega (k_2(x) + c_1(|q(x)|^\sigma + \|Dq(x)\|^\sigma) + a(x) + b|q(x)|^\sigma + |h|)|u(x)|\,dx \\ &\leq (\|k_2\|_q + \|a\ \|_q + \|h\|_q)\|u\|_p \\ &\quad + |\Omega|^{1-(\sigma+1)/p}(c_1(\|q\|_p^\sigma + \|Dq\|_p^\sigma) + b\|q\|_p^\sigma)\|u\|_p \leq C\|u\|_0, \end{aligned}$$

hence G is a linear and continuous operator, i.e. $G \in (W_0^{1,p}(\Omega))^*$.

Let $\chi(u)$ be the indicator function of K

$$\chi(u) = \begin{cases} 0 & \text{for } u \in K, \\ \infty & \text{for } u \in W_0^{1,p}(\Omega) \setminus K. \end{cases}$$

The problem (3.3) can be rewritten in the following equivalent form (see [17, pp. 874–875])

$$G \in \partial\chi(u) + A(u), \quad u \in K,$$

where

$$\partial\chi(u) = \begin{cases} u^* \in (W_0^{1,p}(\Omega))^*, \quad \langle u^*, u - v\rangle \geq 0 & \text{for all } v \in K,\ u \in K, \\ \emptyset & u \in W_0^{1,p}(\Omega) \setminus K. \end{cases}$$

The mapping $\partial\chi: W_0^{1,p}(\Omega) \multimap (W_0^{1,p}(\Omega))^*$ is maximal monotone, then, for the regularity properties of the operator A it follows that for any $b \in (W_0^{1,p}(\Omega))^*$ the inclusion

$$b \in \partial\chi(u) + A(u)$$

has at least a solution $u \in K$ (see [4]). In particular there exist solutions when $b = G$. Moreover, from the monotonicity and the continuity of the operator A we have that (3.3) is equivalent to the problem

$$\langle A(v), v - u\rangle \geq \int_\Omega (g(x, q(x), Dq(x)) + f(x, q(x)) + h)(v(x) - u(x))\,dx$$

for all $v \in K$. Hence, since F has convex values, the multioperator T has closed and convex values.

Step 2. The multioperator T is a closed operator.

Let $q_n \to q_0$ in $W_0^{1,p}(\Omega)$, $u_n \to u_0$ in $W_0^{1,p}(\Omega)$ where $u_n \in T(q_n)$, then, for all $v \in K$,

$$\langle A(u_n), v - u_n\rangle \geq \int_\Omega (g(x, q_n(x), Dq_n(x)) + f(x, q_n(x)) + h)(v(x) - u_n(x))\,dx$$

From the convergence of q_n in $W^{1,p}(\Omega)$, we can extract a subsequence $\{q_{n_k}\} \subset \{q_n\}$ such that:

$$\begin{aligned}\lim_{k\to\infty} &\int_\Omega (g(x, q_{n_k}(x), Dq_{n_k}(x)) + f(x, q_{n_k}(x)) + h)(v(x) - u_{n_k}(x))\,dx \\ &= \liminf_{n\to\infty} \int_\Omega (g(x, q_n(x), Dq_n(x)) + f(x, q_n(x)) + h)(v(x) - u_n(x))\,dx\end{aligned}$$

and $q_{n_k} \to q_0$, $Dq_{n_k} \to Dq_0$ almost everywhere in Ω. From the continuity of g with respect to the second and the third argument, the continuity of f with respect to the second argument, the Lebesgue convergence Theorem and Hölder inequality we have

$$\begin{aligned}\lim_{k\to\infty} &\int_\Omega (g(x, q_{n_k}(x), Dq_{n_k}(x)) + f(x, q_{n_k}(x)) + h)(v(x) - u_{n_k}(x))\,dx \\ &= \int_\Omega (g(x, q_0(x), Dq_0(x)) + f(x, q_0(x)) + h)(v(x) - u_0(x))\,dx.\end{aligned}$$

Moreover, K is closed, hence $u_0 \in K$ and from the continuity of A we have

$$\lim_{n\to\infty} \langle A(u_n), v - u_n\rangle = \langle A(u_0), v - u_0\rangle.$$

Then

$$\begin{aligned}\langle A(u_0), v-u_0\rangle &= \lim_{n\to\infty}\langle A(u_n), v-u_n\rangle\\ &\geq \liminf_{n\to\infty}\int_\Omega (g(x,q_n(x),Dq_n(x))+f(x,q_n(x))+h)(v(x)-u_n(x))\,dx\\ &= \int_\Omega (g(x,q_0(x),Dq_0(x))+f(x,q_0(x))+h)(v(x)-u_0(x))\,dx,\end{aligned}$$

then $u_0\in T(q_0)$ and T is closed.

Step 3. The multioperator T is a compact operator with compact and convex values.

To prove this, let $q_n\in W_0^{1,p}(\Omega)$ be such that $\|q_n\|_0<N$, for all n, with N a positive constant, and let $u_n\in T(q_n)$. Since, by hypothesis, $0\in K$ we may consider (3.3) with $v\equiv 0$, obtaining

$$\begin{aligned}\mu\|u_n\|_0^p-k_1(x) &\leq \langle A(u_n),u_n\rangle\\ &\leq \int_\Omega (g(x,q_n(x),Dq_n(x))+f(x,q_n(x))+h)u_n(x)\,dx\\ &\leq \int_\Omega (|g(x,q_n(x),Dq_n(x))|+|f(x,q_n(x))|+|h|)|u_n(x)|\,dx\\ &\leq (\|k_2\|_q+\|a\|_q+c_1|\Omega|^{1-(\sigma+1)/p}\|Dq_n\|_p^\sigma)\|u_n\|_p\\ &\quad+((c_1+b)|\Omega|^{1-(\sigma+1)/p}\|q_n\|_p^\sigma+\|h\|_q)\|u_n\|_p\leq C(\|u_n\|_0).\end{aligned}$$

Since $p>1$, by the Young inequality u_n is uniformly bounded, i.e. there exists a subsequence, that weakly converges in $W_0^{1,p}(\Omega)$ to $u_0\in W_0^{1,p}(\Omega)$. Moreover, from the convexity and the closure of K, we have $u_0\in K$. It follows

$$\begin{aligned}0 &\leq \lim_{n\to\infty}\bigg|\int_\Omega (g(x,q_n(x),Dq_n(x))+f(x,q_n(x))+h)(u_0(x)-u_n(x))\,dx\bigg|\\ &\leq \lim_{n\to\infty}(\|k_2\|_q+\|a\|_q+c_1|\Omega|^{1-(\sigma+1)/p}\|Dq_n\|_p^\sigma\\ &\quad+(c_1+b)|\Omega|^{1-(\sigma+1)/p}\|q_n\|_p^\sigma+\|h\|_q)\|u_0-u_n\|_p=0.\end{aligned}$$

Substituting $v=u_0$ in (3.1), we have

$$\begin{aligned}&\limsup_{n\to\infty}\langle A(u_n),u_n-u_0\rangle\\ &\qquad\leq \lim_{n\to\infty}\int_\Omega (g(x,q_n(x),Dq_n(x))+f(x,q_n(x))+h)(u_n(x)-u_0(x))=0.\end{aligned}$$

Since A satisfies the S^+ condition, $u_n\to u_0$ in $W^{1,p}(\Omega)$, hence T is a compact operator. Finally, by Step 2, T has closed values, hence has compact values.

Step 4. There exists a ball $B_M(0)$ such that $T(B_M(0))\subset B_M(0)$.

In fact, let $u \in T(q)$, as before we have:

$$\begin{aligned}\mu\|u\|_0^p - \|k_1\|_1 \le (&\|k_2\|_q + \|a\|_q + c_1|\Omega|^{1-(\sigma+1)/p}\|Dq\|_p^\sigma \\ &+ (c_1+b)|\Omega|^{1-(\sigma+1)/p}\|q\|_p^\sigma + \|h\|_q)\|u\|_p \\ \le\, & C(\|q\|_0^\sigma + \|q\|_0)\|u\|_0.\end{aligned}$$

Since $\sigma < p-1$ and $p > 1$, there exists a constant $M > 0$ such that $\|u\|_0 < M$ for any $u \in T(q)$ with $q \in B_M(0)$.

Then, by the Ky-Fan fixed point theorem, there exists a fixed point $u \in T(u)$, i.e. a solution of (3.1). Moreover, since by Step 3, T is a compact operator and the fixed point set is a bounded set we have that it is compact. $\square$

Now, to solve (3.2), given $q \in W_0^{m,p}(\Omega)$, consider the linearized variational inequality:

$$\langle A(u), v-u\rangle \ge \int_\Omega (g(x,\eta(q(x))) + f(x) + h)(v(x)-u(x))\,dx, \tag{3.4}$$

for all $v \in K$, where $f\colon \Omega \to \mathbb{R}$ is a measurable selection of the multimap $F(\,\cdot\,,q(\,\cdot\,))$.

Theorem 3.3. *Let $A_\alpha\colon \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ satisfy hypotheses* (A1)–(A3) *and $g\colon \Omega\times\mathbb{R}^{N_m} \to \mathbb{R}$ be a Carathéodory map such that*

$$|g(x,\eta)| \le k_2(x) + c_1(\|\eta\|^\sigma) \quad \text{a.e. in } \Omega, \text{ for all } \eta \in \mathbb{R}^{N_m}$$

with $k_2 \in L^q(\Omega)$, $c_1 > 0$ and $1 \le \sigma < p-1$. Let $F\colon \Omega\times\mathbb{R} \multimap \mathbb{R}$ be a multimap with compact and convex values such that

(a) *$F(\,\cdot\,,u)$ is measurable for all $u \in \mathbb{R}$;*
(b) *$F(x,\,\cdot\,)\colon \mathbb{R} \multimap \mathbb{R}$ is u.s.c. for all $x \in \Omega$;*
(c) *$\|F(x,u)\| \le a(x) + b|u|^\sigma$ almost everywhere in Ω, for all $u \in \mathbb{R}$ with $a \in L^q(\Omega)$, $b > 0$ and $1 \le \sigma < p-1$.*

Then the problem (3.2) *has a nonempty and compact solution set.*

Proof. Under hypotheses (a)–(b) the multimap $F((\,\cdot\,),q(\,\cdot\,))$ admits a measurable selection $f\colon\Omega\to\mathbb{R}$ (see Theorem 2.4). So (3.4) is well defined.

As before we assume $m = 1$ and it is possible to prove the existence of at least a solution of (3.4). The proof scheme is similar to Theorem 3.2 but we need to prove the closeness of the multimap T in a different way.

Denoting with U_f the solution set of (3.4), we introduce the solution multioperator

$$T\colon W_0^{1,p}(\Omega) \multimap W_0^{1,p}(\Omega), \quad q \to \{U_f, f(x) \in F(x,q(x))\}.$$

Let $q_n \to q_0$ in $W_0^{1,p}(\Omega)$, $u_n \to u_0$ in $W_0^{1,p}(\Omega)$ where $u_n \in T(q_n)$, we want to prove that $u_0 \in T(q_0)$. From $u_n \in T(q_n)$ we have

$$\langle A(u_n), v - u_n\rangle \geq \int_\Omega (g(x, q_n(x), Dq_n(x)) + f_n(x) + h)(v(x) - u_n(x))\, dx$$

for all $v \in K$, where $f_n(x) \in F(x, q_n(x))$ for almost all $x \in \Omega$. Since the sequence $\{q_n\}$ converges in $W^{1,p}(\Omega)$, there exists a subsequence $\{q_{n_k}\}$ in $L^p(\Omega)$ converging to q_0 almost everywhere in Ω. From the Egoroff's Theorem the sequence q_{n_k} converges almost uniformly to q_0, i.e. there exists a zero-measure set O such that $q_{n_k}(x)$ converges uniformly to $q_0(x)$ for all $x \in \Omega \setminus O$. Moreover, from the hypothesis (c) on F, $|f_{n_k}(x)| \leq \|F(x, q_{n_k}(x))\| \leq a(x) + b|q_{n_k}|^\sigma$. Hence there exists a constant $L > 0$ such that $\|f_{n_k}(x)\|_q \leq L$, and there exists a subsequence $\{f_{n_k}\}$, denoted as the sequence, that weakly converges in $L^q(\Omega)$ to a function f_0. From Mazur's lemma a convex combination $\{\widetilde{f}_{n_k}\}$ of $\{f_{n_k}\}$, converges to f_0 with respect to the norm of $L^1(\Omega)$. Passing to a subsequence we can assume that $\{\widetilde{f}_{n_k}\}$ converges almost everywhere to f_0. We show that $f_0(x) \in F(x, q_0(x))$ for almost all $x \in \Omega$.

From the upper semicontinuity of the multimap F there exists an index k_0 such that $F(x, q_{n_k}(x)) \subset W_\varepsilon(F(x, q_0(x)))$ for all $x \in \Omega \setminus O$ and $k \geq k_0$. Then $f_{n_k}(x) \in W_\varepsilon(F(x, q_0(x)))$ for almost all $x \in \Omega$. From the convexity of the values of F

$$\widetilde{f}_{n_k}(x) \in W_\varepsilon(F(x, q_0(x))), \quad k \geq k_0,$$

for almost all $x \in \Omega$. It follows $f_0(x) \in F(x, q_0(x))$ for almost all $x \in \Omega$. Moreover, by the continuity of the operator A,

$$\lim_{n\to\infty} \langle A(u_n), v - u_n\rangle = \langle A(u_0), v - u_0\rangle$$

and hence

$$\begin{aligned}
\langle A(u_0), v - u_0\rangle &= \lim_{n\to\infty} \langle A(u_n), v - u_n\rangle \\
&\geq \liminf_{n\to\infty} \int_\Omega (g(x, q_n(x), Dq_n(x)) + f_n(x) + h)(v(x) - u_n(x))\, dx \\
&= \int_\Omega (g(x, q_0(x), Dq_0(x)) + f_0(x) + h)(v(x) - u_0(x))\, dx.
\end{aligned}$$

The last equality follows from the continuity of the functions q_n, the weak convergence up to subsequence of f_n and the strong convergence of the sequence u_n. Finally, K is closed, $u_0 \in K$ and hence $u_0 \in T(q_0)$. □

4. Evolution variational inequalities

We consider now the parabolic case. To this aim let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $J = [0, d]$, and K a closed convex subset of $X_0 = L^p(J, W_0^{m,p}(\Omega))$ $(2 \leq p < \infty)$ we look for functions $u \in Y_0 \cap K$,

$u(0,\cdot)=\overline{u}(\cdot)$, $Y_0=\{u\in X_0, u_t\in X_0^*\}$, solutions of the following variational inequalities:

$$(4.1)\quad \begin{cases} \displaystyle\int_J\int_\Omega \frac{\partial u}{\partial t}(v(t,x)-u(t,x))\,dx\,dt \\ \qquad \displaystyle + \int_J\int_\Omega \sum_{|\alpha|\le m} A_\alpha(t,x,\eta(u(t,x)))(D^\alpha v(t,x)-D^\alpha u(t,x))\,dx\,dt \\ \quad \displaystyle \ge \int_J\int_\Omega (g(t,x,\eta(u(t,x)))+f(t,x,u(t,x))+h)(v(t,x)-u(t,x))\,dx\,dt \\ \hfill \text{for all } v\in K, \\ f(t,x,u(t,x))\in F(t,x,u(t,x)) \hfill \text{a.e. in } J\times\Omega, \end{cases}$$

$$(4.2)\quad \begin{cases} \displaystyle\int_J\int_\Omega \frac{\partial u}{\partial t}(v(t,x)-u(t,x))\,dx\,dt \\ \qquad \displaystyle + \int_J\int_\Omega \sum_{|\alpha|\le m} A_\alpha(t,x,\eta(u(t,x)))(D^\alpha v(t,x)-D^\alpha u(t,x))\,dx\,dt \\ \quad \displaystyle \ge \int_J\int_\Omega (g(t,x,\eta(u(t,x)))+f(t,x)+h)(v(t,x)-u(t,x))\,dx\,dt \\ \hfill \text{for all } v\in K, \\ f(t,x)\in F(t,x,u(t,x)) \hfill \text{a.e. in } J\times\Omega, \end{cases}$$

where, as before, α is a multiindex, $\eta(u)=\{D^\alpha u : |\alpha|\le m\}$, the function A_α maps $J\times\Omega\times\mathbb{R}^{N_m}$ into $\mathbb{R}$ (with $N_m=(N+m)!/(N!m!)$), and where $g\colon J\times\Omega\times\mathbb{R}^{N_m}\to\mathbb{R}$ and $F\colon J\times\Omega\times\mathbb{R}\multimap\mathbb{R}$ are a given map and multimap respectively, finally $h\in X_0^*$.

Let q be such that $1/p+1/q=1$, we assume the following hypotheses on the function $A_\alpha\colon J\times\Omega\times\mathbb{R}^{N_m}\to\mathbb{R}$:

(A4) $(t,x)\to A_\alpha(t,x,\eta)$ is measurable in $J\times\Omega$ for any $\eta\in\mathbb{R}^{N_m}$;
$\eta\to A_\alpha(t,x,\eta)$ is continuous for almost all $x\in\Omega$;
there exist a function $k_0\in L^q(J\times\Omega)$ and a constant ν such that

$$|A_\alpha(t,x,\eta)|\le k_0(t,x)+\nu(\|\eta\|^{p-1}),\quad \text{a.e. in } J\times\Omega,\ \text{for any } \eta\in\mathbb{R}^{N_m};$$

(A5) for all $(t,x)\in J\times\Omega$, and for all η, η', $\eta\ne\eta'$,

$$\sum_{|\alpha|\le m}(A_\alpha(t,x,\eta)-A_\alpha(t,x,\eta'))(\eta_\alpha-\eta'_\alpha)>0;$$

(A6) there exist a function $k_1\in L^1(J\times\Omega)$ and a constant $\mu>0$ such that

$$\sum_{|\alpha|\le m}A_\alpha(t,x,\eta)\eta_\alpha\ge\mu\|\eta\|^p-k_1(t,x),\quad \text{a.e. in } J\times\Omega \text{ and for all } \eta\in\mathbb{R}^{N_m}.$$

Remark 4.1. Observe that if the set K coincides with the whole space X_0, the solutions of the variational inequalities (4.1) and (4.2) are weak solutions of the following parabolic partial differential inclusion:

$$\tag{4.3} \begin{cases} -h \in -\dfrac{\partial u}{\partial t} + \displaystyle\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \eta(u(t,x))) \\ \qquad\qquad + \; g(t,x,\eta(u(t,x))) + F(t,x,u(t,x)) & \text{in } J \times \Omega, \\ u(t,x) = 0 & \text{on } J \times \partial\Omega, \\ u(0,x) = \overline{u}(x) & \text{a.e. in } \Omega. \end{cases}$$

Let A from $J \times W_0^{m,p}(\Omega)$ into the dual $(W_0^{m,p}(\Omega))^*$ be defined by

$$\langle A(t,u), \varphi \rangle = \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(t,x,\eta(u(t,x))) D^\alpha \varphi(t,x)\, dx.$$

Defining $\widetilde{A}: Y_0 \to Y_0^*$ as $\widetilde{A}(u)(t) = A(t,u(t))$, by hypotheses (A4)–(A6), we have that the operator $\widetilde{A}$ is continuous, bounded and satisfies the S_+ condition (see [9]). Moreover, we define the operator $L: Y_0 \subseteq X_0 \to X_0^*$ as $L(u) = u_t$. It is known that the operator $L: Y_0 \subseteq X_0 \to X_0^*$ is a closed maximal monotone operator (see [9]). Given $q \in X_0$, we can consider the linearized variational inequality:

$$\tag{4.4} \begin{aligned} &\langle L(u) + \widetilde{A}(u), v - u \rangle \\ &\quad \geq \int_J \int_\Omega (g(t,x,\eta(q(t,x))) + f(t,x,q(t,x)) + h)(v(t,x) - u(t,x))\, dx\, dt, \end{aligned}$$

for all $v \in K$, where $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory selection of the multimap F.

We have three possible cases for the set K. It has non empty interior, denoted by $\mathrm{int}(K)$; eather it has non empty relatively interior, in this case we solve the problem on X_0', the smallest subspace of X_0 containing K or it is reduced to a single point $K = \{0\}$. We solve the problem in the case $0 \in \mathrm{int}(K)$ and the other two cases follow easily.

Theorem 4.2. *Let $Y_0 \cap \mathrm{int}(K) \neq \emptyset$, $A_\alpha: J \times \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ satisfy hypotheses* (A4)–(A6)*, and $g: J \times \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ be a Carathéodory map such that*

$$|g(t,x,\eta)| \leq k_2(t,x) + c_1(\|\eta\|^\sigma) \quad \text{a.e. in } J \times \Omega, \ \text{for all } \eta \in \mathbb{R}^{N_m}$$

with $k_2 \in L^q(J \times \Omega)$, $c_1 > 0$ and $1 \leq \sigma < p - 1$.

Let $F: J \times \Omega \times \mathbb{R} \multimap \mathbb{R}$ be a measurable multimap with closed convex values such that

(a) *$F(t,x,\cdot): \mathbb{R} \multimap \mathbb{R}$ is l.s.c. for all $(t,x) \in J \times \Omega$;*
(b) *$\|F(t,x,u)\| \leq a(t,x) + b|u|^\sigma$ almost everywhere in $J \times \Omega$, for all $u \in \mathbb{R}$ with $a \in L^q(J \times \Omega)$, $b > 0$ and $1 \leq \sigma < p - 1$.*

Then the problem (4.1) *has a nonempty and compact solution set.*

Proof. Hypotheses (a)–(b) on the multimap F imply the existence of a Carathéodory selection and hence the variational inequality (3.3) is well defined (see again Theorem 2.3).

As before, for sake of simplicity we assume $m = 1$ and, denoting with U_f the solution set of (4.4), we define the multioperator T as

$$T\colon X_0 \multimap X_0, \quad q \mapsto \{U_f, f(t,x,q(t,x)) \in F(t,x,q(t,x))\}.$$

As for the elliptic variational inequalities we can assume without losss of generality that $0 \in K$.

The multioperator T is a compact multioperator with nonempty closed convex values. Indeed, as before we consider the linear and continuous functional $G\colon X_0 \to \mathbb{R}$ defined as:

$$G(u) = \int_J \int_\Omega (g(t,x,q(t,x),Dq(t,x)) + f(t,x,q(t,x)) + h)u(t,x)\,dx\,dt.$$

So, denoting with $\chi(u)$ the indicator function of K, problem (4.4) can be rewritten in the following equivalent form (see [17, pp. 893–894])

$$G \in \partial\chi + L(u) + \widetilde{A}(u), \quad u \in K.$$

Since by hypothesis $Y_0 \cap \operatorname{int}(K) \neq \emptyset$, the operator $\partial\chi(u) + L(u)$ is a maximal monotone operator as sum of two maximal monotone operators, then, for the regularity properties of the operator $\widetilde{A}$ it follows that for any $b \in X_0^*$ the inclusion

$$b \in \partial\chi(u) + L(u) + \widetilde{A}(u)$$

has at least a solution $u \in K$ (see [4]). In particular there exist solutions when $b = G$. Moreover, from the monotonicity and continuity of the operator $\widetilde{A}$ and from the monotonicity and linearity of the operator L we have that (4.4) is equivalent to the problem

$$\begin{aligned}&\langle L(v) + \widetilde{A}(v), v - u\rangle \\ &\geq \int_J \int_\Omega (g(t,x,q(t,x),Dq(t,x)) + f(t,x,q(t,x)) + h)(v(t,x) - u(t,x))\,dx\,dt\end{aligned}$$

for all $v \in K$. Therefore, recalling that F has convex values, the multioperator T has closed and convex values.

T is a closed operator. To this aim let $q_n \to q_0$ in X_0 and $u_n \to u_0$ in X_0 with $u_n \in T(q_n)$, we claim that $u_0 \in T(q_0)$.

We find an estimate for $\|Lu_n\|$. Notice that since K is closed convex and $0 \in \operatorname{int}(K)$ we have that for any $v \in X_0$ there exists $\gamma \in \mathbb{R}$ and a $v_k \in K$ such that $v = \gamma v_k$. So

$$\|Lu_n\| = \sup_{v\in X_0:\|v\|_{X_0}\le 1} |\langle Lu_n, v\rangle| = \sup_{\|\gamma v_k\|\le 1} |\langle Lu_n, \gamma v_k\rangle|,$$

we have

$$\begin{aligned}|\langle Lu_n, \gamma v_k\rangle| &= |\gamma||\langle Lu_n, v_k\rangle| = |\gamma||\langle Lu_n, -v_k\rangle| \\ &\le |\gamma||\langle Lu_n, u_n - v_k\rangle| + |\gamma||\langle Lu_n, u_n\rangle|.\end{aligned}$$

Since $v_k \in K$ we obtain

$$\begin{aligned}|\langle Lu_n, u_n - v_k\rangle| \le \bigg| \int_J\int_\Omega &(g(t,x,q_n(t,x), Dq_n(t,x))+ \\ &+ f(t,x,q_n(t,x)) + h)(u_n(t,x) - v_k(t,x))\,dx\,dt\bigg| + |\langle \widetilde{A}u_n, u_n - v_k\rangle|.\end{aligned}$$

From growth conditions on maps g and F and on the operator A_α we have:

$$\begin{aligned}|\langle Lu_n, u_n - v_k\rangle| \le &(\|a\|_q + \|k_2\|_q + \|k_0\|_q + \|h\|_{X_0^*})(\|u_n\|_{X_0} + \|v_k\|_{X_0}) \\ &+ c_1(d|\Omega|)^{(1-(\sigma+1)/p)}\|Dq_n\|^\sigma_{X_0}(\|u_n\|_{X_0} + \|v_k\|_{X_0}) \\ &+ (c_1+b)(d|\Omega|)^{(1-(\sigma+1)/p)}\|q_n\|^\sigma_{X_0}(\|u_n\|_{X_0} + \|v_k\|_{X_0}) \\ &+ \nu(\|Du_n\|^{p-1}_{X_0} + \|u_n\|^{p-1}_{X_0})(\|u_n\|_{X_0} + \|v_k\|_{X_0}).\end{aligned}$$

Since, from the convergence of the sequences $\{q_n\}$ and $\{u_n\}$ we have the existence of two constants $M_1 > 0$ and $M_2 > 0$ such that $\|q_n\|_{X_0} \le M_1$ and $\|u_n\|_{X_0} \le M_2$, we obtain the existence of a constant N_1 such that

$$|\langle Lu_n, u_n - v_k\rangle| \le N_1.$$

Choosing $v \equiv 0$ in (4.4), as before we have the existence of a constant N_2 such that

$$|\langle Lu_n, u_n\rangle| \le N_2,$$

therefore the norm of Lu_n is uniformly bounded. Then, up to subsequence, there exists $v_0 \in X_0^*$ such that $Lu_n \rightharpoonup v_0$. By the definition of the operator L we have that $v_0 = Lu_0$, i.e. $u_n \rightharpoonup u_0$ in Y_0 up to subsequence. From the compact embedding $Y_0 \subset L^p(J, L^p(\Omega))$ and the continuous embedding $Y_0 \subset C(J, L^p(\Omega))$ it follows $u_n(0) \to u_0(0)$ and $u_n(d) \to u_0(d)$ and

$$\begin{aligned}&\lim_{n\to\infty} \langle Lu_n, u_n - u_0\rangle \\ &\quad = \lim_{n\to\infty}\bigg(\frac{1}{2}\|u_n(d) - u_0(d)\|_2^2 - \frac{1}{2}\|u_n(0) - u_0(0)\|_2^2 - \langle Lu_0, u_n - u_0\rangle\bigg) = 0.\end{aligned}$$

Hence, recalling the convergence $u_n \to u_0$ in X_0 and that K is closed ($u_0 \in K$) we have

$$\begin{aligned}(4.5)\quad &\limsup_{n\to\infty}(\langle \widetilde{A}(u_n), u_n - u_0\rangle + \langle Lu_n, u_n - u_0\rangle)\\ &\quad\le \limsup_{n\to\infty}\int_J\int_\Omega (g(t,x,q(t,x),Dq(t,x)) + f(t,x,q(t,x)) + h)\\ &\qquad\qquad\cdot (u_n(t,x) - u_0(t,x))\,dx\,dt = 0.\end{aligned}$$

The operator $\widetilde{A}$ satisfies the S_+ condition; from the previous inequality we have that $u_n \to u_0$ in Y_0, in particular $Lu_n \to Lu_0$ in X_0^*. Finally,

$$\begin{aligned}&\langle Lu_0, v - u_0\rangle + \langle \widetilde{A}(u_0), v - u_0\rangle = \lim_{k\to\infty}\langle Lu_{n_k}, v - u_{n_k}\rangle + \langle \widetilde{A}(u_{n_k}), v - u_{n_k}\rangle\\ &\ge \int_J\int_\Omega (g(t,x,q_0(t,x),Dq_0(t,x)) + f(t,x,q_0(t,x)) + h)(v(t,x) - u_0(t,x))\,dx\,dt,\end{aligned}$$

where u_{n_k} and q_{n_k} are the sequences that verify the inferior limit. Then $u_0 \in T(q_0)$.

To prove the compactness, let $q_n \in X_0$ be such that $\|q_n\|_0 < N$, for all n, with N a positive constant, and let $u_n \in T(q_n)$. Observe that

$$\langle L(u), u\rangle = \frac{1}{2}\|u(d)\|_2^2 - \frac{1}{2}\|u(0)\|_2^2.$$

Moreover, from (A6) we have that

$$\begin{aligned}\langle \widetilde{A}(u), u\rangle &\ge \int_J\int_\Omega (\mu\|\eta(u(t,x))\|^p - k_1(t,x))\,dx\,dt\\ &= \int_J \mu\|u(t)\|_0^p\,dt - \int_J\int_\Omega k_1(t,x)\,dx\,dt = \mu\|u\|_{X_0}^p - \widetilde{k}_1.\end{aligned}$$

By hypothesis $0 \in K$; we may consider (4.4) with $v \equiv 0$, obtaining

$$\begin{aligned}&\mu\|u_n\|_{X_0}^p - \widetilde{k}_1 - \frac{1}{2}\|u_n(0)\|_2^2\\ &\quad\le \mu\|u_n\|_{X_0}^p - \widetilde{k}_1 - \frac{1}{2}\|u_n(0)\|_2^2 + \frac{1}{2}\|u_n(d)\|_2^2 \le \langle L(u_n) + \widetilde{A}(u_n), u_n\rangle\\ &\quad\le \left|\int_J\int_\Omega (g(t,x,q_n(t,x),Dq_n(t,x)) + f(t,x,q_n(t,x)) + h)u_n(t,x)\,dx\,dt\right|.\end{aligned}$$

From the growth conditions on maps g and F, we obtain

$$\mu\|u_n\|_{X_0}^p - \widetilde{k}_1 - \frac{1}{2}\|u_n(0)\|_2^2 \le C\|u_n\|_{X_0}.$$

Since $p \ge 2$, by the Young inequality u_n is uniformly bounded, i.e. there exists a subsequence, that weakly converges in X_0 to $u_0 \in X$. Moreover, from the convexity and the closure of K, we have $u_0 \in K$. As before it is possible to prove the uniform boundedness of $\|Lu_n\|$, hence to show that $u_n \rightharpoonup u_0$ in Y_0 up to subsequence. Since Y_0 is compactly embedded in $L^p(J, L^p(\Omega))$, then $u_n \to u_0$

in $L^p(J, L^p(\Omega))$. So inequality (4.5) holds and we have the strong convergence $u_n \to u_0$ in Y_0, i.e. the compactness of the operator T.

As for the elliptic case it is possible to prove the existence of a constant $M > 0$ such that $\|u\|_{X_0} < M$ for any $u \in T(q)$, with $\|q\|_{X_0} < M$. Therefore we have that there exists a ball $B_M(0)$ such that $T(B_M(0)) \subset B_M(0)$, then by the Ky-Fan fixed point theorem we obtain a solution of (4.1). Moreover by the compactness of the solution operator T we have the compactness of its fixed point set. $\square$

For parabolic variational inequalities the existence theorem for u.s.c. multimap F is still valid.

Theorem 4.3. *Let $Y_0 \cap \operatorname{int}(K) \neq \emptyset$, $A_\alpha: J \times \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ satisfy hypotheses* (A4)–(A6)*, and $g: J \times \Omega \times \mathbb{R}^{N_m} \to \mathbb{R}$ be a Carathéodory map such that*

$$|g(t,x,\eta)| \le k_2(t,x) + c_1(\|\eta\|^\sigma) \quad \text{a.e. in } J \times \Omega, \text{ for all } \eta \in \mathbb{R}^{N_m}$$

with $k_2 \in L^q(J \times \Omega)$, $c_1 > 0$ and $1 \le \sigma < p-1$. Let $F: J \times \Omega \times \mathbb{R} \multimap \mathbb{R}$ be a multimap with compact and convex values such that

(a) *$F(\,\cdot\,, \,\cdot\,, u)$ is measurable for all $u \in \mathbb{R}$;*
(b) *$F(t,x,\,\cdot\,): \mathbb{R} \multimap \mathbb{R}$ is u.s.c. for all $(t,x) \in J \times \Omega$;*
(c) *$\|F(t,x,u)\| \le a(t,x) + b|u|^\sigma$ almost everywhere in $J \times \Omega$, for all $u \in \mathbb{R}$ with $a \in L^q(J \times \Omega)$, $b > 0$ and $1 \le \sigma < p-1$.*

Then the problem (4.2) *has a nonempty and compact solution set.*

Proof. As before given $q \in X_0$ we consider the linearized problem

$$\begin{aligned}(4.6)\quad &\langle L(u) + \widetilde{A}(u), v - u\rangle \\ &\qquad \ge \int_J \int_\Omega (g(t,x,\eta(q(t,x))) + f(t,x) + h)(v(t,x) - u(t,x))\,dx\,dt,\end{aligned}$$

for all $v \in K$, where $f: J \times \Omega \to \mathbb{R}$ is a measurable selection of the multimap $F(\,\cdot\,, q(\,\cdot\,))$, which exists again by Theorem 2.4. Moreover, denoting with U_f the solution set of (4.6), we introduce the solution multioperator

$$T: X_0 \multimap X_0, \quad q \to \{U_f, f(t,x) \in F(t,x,q(t,x))\}.$$

As for the elliptic variational inequalities the proof scheme is similar to Theorem 4.1 but we need to prove the closeness of the multimap T in a different way.

Given $q_n \to q_0$ and $u_n \to u_0$ in X_0, with $u_n \in T(q_n)$, as in Theorem 3.3 it is possible to find a sequence of selections $\{f_n\} \subset L^q(J, L^q(\Omega))$, $f_n(t,x) \in F(t,x,q_n(t,x))$ almost everywhere in $J \times \Omega$, such that $\{f_n\}$ weakly converges to f_0 with $f_0(t,x) \in F(t,x,q_0(t,x))$. Now, as in the proof of Theorem 4.1 we have

$$\lim_{n\to\infty} \langle L(u_n) + \widetilde{A}(u_n), v - u_n\rangle = \langle L(u_0) + \widetilde{A}(u_0), v - u_0\rangle.$$

Hence

$$
\begin{aligned}
&\langle L(u_0) + \widetilde{A}(u_0), v - u_0\rangle = \lim_{n\to\infty} \langle L(u_n) + \widetilde{A}(u_n), v - u_n\rangle \\
&\geq \liminf_{n\to\infty} \int_J \int_\Omega (g(t,x,q_n(t,x), Dq_n(t,x)) + f_n(t,x) + h)(v(t,x) - u_n(t,x))\,dx\,dt \\
&= \int_J \int_\Omega (g(t,x,q_0(t,x), Dq_0(t,x)) + f_0(t,x) + h)(v(t,x) - u_0(t,x))\,dx\,dt
\end{aligned}
$$

and we have the conclusion. □

5. Examples

We stress that to solve problems (3.1), (3.2), (4.1), (4.2) we do not require any additional conditions on the set K beside closeness and convexity unlike similar results in the literature. In Chapter 5 of [5] some existence results for the following variational inequality

$$
\begin{cases}
\displaystyle\int_\Omega A_0(x, Du(x))(Dv(x) - Du(x)) \\
\qquad \displaystyle\geq \int_\Omega F(x,u(x))(v(x) - u(x))\,dx \quad \text{for all } v \in K, \\
u \in K
\end{cases}
\tag{5.1}
$$

are collected, where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with Lipschitz boundary, K is a closed and convex subset of $W_0^{1,p}(\Omega)$, $1 < p < \infty$, $A_0\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory monotone map satisfying analogous hypotheses as (A2)–(A3) and $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory map satisfying an analogous hypothesis as (b) of Theorem 3.2.

In Theorem 5.5. of [5] the existence of k subsolutions $\underline{u}_1, \ldots, \underline{u}_k$ of (5.1) as well as the following condition on the set K are required

$$
\underline{u}_j \vee K \subset K, \quad 1 \leq j \leq k. \tag{5.2}
$$

We point out that variational inequality (5.1) is in the same class of variational inequalities as (3.1). In order to solve it we do not need condition (5.2) and to assume the existence of subsolutions. The following example shows as, even in very simple cases, condition (5.2) may not be satisfied.

Example 5.1. We consider the following problem

$$
\min_{u\in K} \left\{ \int_C |Du|^2\,dx\,dy,\ u \in u_0 + W_0^{1,2}(C, \mathbb{R}) \right\}, \tag{5.3}
$$

where $C \subset \mathbb{R}^2$ is the unit ball centered in zero, u_0 is an harmonic function and

$$
K = \{v \in u_0 + W_0^{1,2} : v = u_0 + w,\ w \text{ is subharmonic}\}. \tag{5.4}
$$

Finding minimizers of problem (5.3) is equivalent to solve the variational inequality:

$$\langle Du, Dv - Du\rangle \geq 0 \quad \text{for all } v \in K, \tag{5.5}$$

i.e. we obtain a variational inequality of type (3.1). We can apply Theorem 3.2 obtaining the existence of a solution. We point out that the set K defined in (5.4) does not satisfy condition (5.2). Indeed it is well known that any subsolution (supersolution) of (5.5) is a superharmonic (subharmonic) function and vice-versa. Moreover given a subharmonic function $\underline{u} \in W_0^{1,2}(C, \mathbb{R})$ and a superharmonic function $\overline{u} \in W_0^{1,2}(C, \mathbb{R})$, we have

$$\underline{u}(x) \leq \overline{u}(x) \quad \text{a.e. } x \in C.$$

Then, for all $\underline{u}_i$, $j = 1, \ldots, k$ subsolutions:

$$\underline{u}_j \vee K = \underline{u}_j, \quad 1 \leq j \leq k.$$

If $\underline{u}_j \in K$, $\underline{u}_j$ is either a subsolution and a supersolution, and hence a solution, and there is nothing to prove. Then in general $\underline{u}_j \notin K$.

We give a physical example from which partial differential inclusions of the type (4.3) arise. We generalize the one dimensional heating problem in [3] to a multidimensional one.

Example 5.2. We consider a problem of heat dissipation in an isotropic homogeneous bounded body $B \subset \mathbb{R}^3$, which has to be maintained at a constant temperature u_0. The problem is expressed by the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Delta u - D(t,x) = \displaystyle\sum_{i=1}^{N} q_i(t,u) f_i(x) & t \in [0,T],\ x \in B, \\ u(t,x) = u_0(t,x) & t \in [0,T],\ x \in \partial B. \end{cases}$$

The function $u(t,x) \in W^{1,p}([0,T] \times B, \mathbb{R})$ describes the change of temperature at point x and time t due to the dispersion $D(t,x)$. Heat is supplied by N sources $f_i \in L^\infty(B_i, \mathbb{R})$, $i = 1, \ldots, N$ of bounded heating output $q_i(t,u)$, $i = 1, \ldots, N$, where $B_i \subset B$, in order to keep the body B at a constant temperature for any $t > 0$. The heating output is represented by N measurable functions $q_i\colon [0,T] \times \mathbb{R} \mapsto \mathbb{R}$, continuous with respect to the second variable.

Now given a constant $c \in \mathbb{R}$ we consider all the possible amounts of heat subject to the constraint

$$\sum_{i=1}^{N} q_i(t,u) = c, \tag{5.6}$$

for all $t \in [0,T]$ and $x \in B$. Defining the multimap $F\colon [0,T] \times \Omega \times \mathbb{R} \multimap \mathbb{R}$ as

$$F(t,x,u) = \bigg\{ \sum_{i=1}^{N} q_i(t,u) f_i(x) : q_i \text{ satisfying (5.6)} \bigg\},$$

we obtain an analogous problem as (4.3), i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \Delta u - D(t,x) \in F(t,x,u) & t \in [0,T],\ x \in B, \\ u(t,x) = u_0(t,x) & t \in [0,T],\ x \in \partial B. \end{cases}$$

In this way we can obtain a solution that is optimal with respect to the controls $q_i(t,u)$ satisfying (5.6).

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