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LOWER AND UPPER SOLUTIONS TO SEMILINEAR BOUNDARY VALUE PROBLEMS: AN ABSTRACT APPROACH

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Abstract. We provide an abstract setting for the theory of lower and upper solutions to some semilinear boundary value problems. In doing so, we need to introduce an abstract formulation of the Strong Maximum Principle. We thus obtain a general version of some existence results, both in the case where the lower and upper solutions are well-ordered, and in the case where they are not so. Applications are given, e.g. to boundary value problems associated to parabolic equations, as well as to elliptic equations.

1. Introduction

In this paper, we investigate the existence of solutions to boundary value problems of the type

$$\text{(P)}\qquad \begin{cases} \mathcal{L}u = F(x,t,u,\nabla_x u,\nabla_t u) & \text{in } Q,\\ \mathcal{B}u = 0 & \text{on } \partial Q,\end{cases}$$

where Q is a suitable bounded domain, $\mathcal{L}$ is a linear operator, $\mathcal{B}$ is a linear boundary operator, and F is a Carathéodory function. Typically, in the applications we have in mind, $\mathcal{L}$ will be a linear differential operator. Notice however that, according to the function space where $u = u(x,t)$ belongs, either of the gradients $\nabla_x u$ or $\nabla_t u$ might not appear in the equation.

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We provide an abstract setting to the method of lower and upper solutions, which differs from previously proposed ones, like, e.g. the one by H. Amann [1], [2], mainly by the fact that it is more directly related to the special structure of problem (P). In our construction, we introduce an abstract version of the Strong Maximum Principle, well fitting for our purposes, which, to our knowledge, has not been considered in the previous literature.

As for the applications of our abstract theorems, we have in mind some results of bifurcation type, on one hand, and some existence results in the case when the lower and upper solutions are not well-ordered, on the other hand. We will now explain how these two aims are pursued.

In order to simplify the exposition, in this section we will focus our attention on the particular case of a parabolic problem of the type

$$\tag{1.1} \begin{cases} \mathcal{L}u = F(x,t,u,\nabla_x u) & \text{in } \Omega \times]0,T[, \\ u = 0 & \text{on } \Gamma_1 \times]0,T[, \\ \displaystyle\sum_{i=1}^N b_i(x,t)\partial_{x_i} u + b_0(x,t)u = 0 & \text{on } \Gamma_2 \times]0,T[, \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases}$$

Here, $Q = \Omega \times]0,T[$, where Ω is a bounded regular domain in $\mathbb{R}^N$, and its boundary $\partial\Omega$ is the disjoint union of two closed sets Γ_1, Γ_2 (the cases $\Gamma_1 = \emptyset$ or $\Gamma_2 = \emptyset$ are admitted, of course). The parabolic differential operator is defined by

$$\mathcal{L}u = \partial_t u - \sum_{i,j=1}^N a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{i=1}^N a_i(x,t)\partial_{x_i} u + a_0(x,t)u,$$

and standard assumptions are made on the coefficients a_{ij}, a_i and b_i (see Section 7).

We denote by λ_1 the principal eigenvalue of $\mathcal{L}$, with the boundary conditions in (1.1), and by φ_1 the corresponding positive eigenfunction, with $\max \varphi_1 = 1$.

Let $\lambda_2 > \lambda_1$ be such that, for every function $q \in L^r(Q)$ satisfying $\lambda_1 \le q(x,t) \le \lambda_2$, each of the inequalities being strict on a subset of positive measure, the only solution of $\mathcal{L}u = q(x,t)u$, with the boundary conditions in (1.1), is the trivial one, $u = 0$.

Consider the class of functions

$$\mathcal{F}(\mathcal{I},\Lambda,K),$$

where $\mathcal{I} \subseteq \mathbb{R}$ is an interval, and Λ, K are some nonnegative constants. Its elements are the Carathéodory functions $f\colon Q \times \mathcal{I} \times \mathbb{R}^N \to \mathbb{R}$ which satisfy the following Bernstein–Nagumo growth condition:

$$|f(x,t,u,\xi)| \le h(x,t) + K\|\xi\|^2, \quad \text{for a.e. } (x,t) \in Q \text{ and every } (u,\xi) \in \mathcal{I} \times \mathbb{R}^N,$$

for some $h \in L^r(Q)$, with $r > N+2$ and $\|h\|_{L^r} \le \Lambda$.

For any open interval $I \subseteq \mathbb{R}$, we will use the notation

$$I\varphi_1 = \{u \in C^{1,0}(\overline{Q}) : (\inf I)\varphi_1 < u < (\sup I)\varphi_1\}.$$

Similarly if the interval is closed, with the strict inequalities replaced by non-strict ones. Moreover, we will denote by $I\varphi_1(\overline{Q})$ the interval obtained as the union of the images of the elements of $I\varphi_1$. Notice that, if $0 \in I$, then $I\varphi_1(\overline{Q}) = I$.

We will prove the following.

Theorem 1.1. *Let $I \subseteq \mathbb{R}$ be an open interval, and $\zeta \in L^r(Q)$ be a function such that*

$$\lambda_1 \le \zeta(x,t) \le \lambda_2 \quad \text{for a.e. } (x,t) \in Q,$$

the second inequality being strict on a subset of positive measure. Given a compact interval $[a,b]$, contained in I, let $\mathcal{I}$ be an open interval containing $[a,b]\varphi_1(\overline{Q})$. Let

$$F(x,t,u,\xi) = g(x,t,u,\xi)\,u + f(x,t,u,\xi).$$

There is a constant $\Lambda > 0$ such that, for every function g satisfying

$$\lambda_1 \le g(x,t,u,\xi) \le \zeta(x,t), \quad \text{for a.e. } (x,t) \in Q \text{ and every } (u,\xi) \in \mathcal{I} \times \mathbb{R}^N,$$

and every function $f \in \mathcal{F}(\mathcal{I},\Lambda,\Lambda)$, if there are a lower solution α and an upper solution β of (P) *verifying*

$$a\varphi_1 - \Lambda \le \alpha \le b\varphi_1, \quad a\varphi_1 \le \beta \le b\varphi_1 + \Lambda,$$

then problem (1.1) *has a solution $u \in I\varphi_1$. Moreover, if $\alpha \not\le \beta$, then*

$$u \in \overline{\{v \in C^{1,0}(\overline{Q}) : \alpha \not\le v \text{ and } v \not\le \beta\}}.$$

As a consequence of Theorem 1.1, concerning the Neumann-periodic problem

$$\text{(1.2)} \qquad \begin{cases} \mathcal{L}u = f(x,t,u,\nabla_x u) & \text{in } \Omega \times]0,T[, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times]0,T[, \\ u(x,0) = u(x,T) & \text{in } \Omega. \end{cases}$$

we have the following result, obtained in [8].

Corollary 1.2. *Let $I \subseteq \mathbb{R}$ be an open interval. Given a compact interval J, contained in I, there is a constant $\Lambda > 0$ such that, for every $f \in \mathcal{F}(I,\Lambda,\Lambda)$, if there are two constants α, β in J for which*

$$\text{(1.3)} \qquad f(x,t,\beta,0) \le 0 \le f(x,t,\alpha,0) \quad \text{a.e. in } Q,$$

then problem (1.2) *has a solution u, with $u(x,t) \in I$ for every $(x,t) \in \overline{Q}$.*

Indeed, in the setting of Corollary 1.2, we have that φ_1 is constantly equal to 1, so that $J\varphi_1(\overline{Q}) = J$, while α and β are constant lower and upper solutions.

Hence, choosing $\mathcal{I} = I$, it is readily seen that Corollary 1.2 is a consequence of Theorem 1.1.

Since α and β belong to the kernel of the differential operator, Corollary 1.2 is related to some co-bifurcation theorems, cf. [9], [10], although we showed in [8] that condition (1.3) cannot be replaced by the usual integral condition

$$\int_Q f(x,t,\beta,0)\,dx\,dt < 0 < \int_Q f(x,t,\alpha,0)\,dx\,dt.$$

This is due to the fact that the constant Λ appearing in the theorem is uniform with respect to a whole class of functions, while, in previous theorems available in the literature, the nonlinearity was usually fixed at the beginning, and then multiplied by a small parameter.

Besides from dealing with the more general problem (1.1), Theorem 1.1 generalizes the result proved in [8] in several directions. First of all, we are able to deal with lower and upper solutions which do not belong to the kernel of the differential operator. Also, more general nonlinear functions F are allowed. In addition, we have a more precise information on the location of the solution.

As a further consequence of Theorem 1.1, after a suitable change of variables, we obtain in Section 5 the following existence result, in the framework of non-well-ordered lower and upper solutions.

Corollary 1.3. *Let $\zeta \in L^r(Q)$ be a function such that*

$$\lambda_1 \le \zeta(x,t) \le \lambda_2 \quad \textit{for a.e. } (x,t) \in Q,$$

the second inequality being strict on a subset of positive measure. Assume that

$$F(x,t,u,\xi) = g(x,t,u,\xi)\,u + f(x,t,u,\xi),$$

where the function g verifies

$$\lambda_1 \le g(x,t,u,\xi) \le \zeta(x,t), \qquad \textit{for a.e. } (x,t) \in Q \textit{ and every } (u,\xi) \in \mathbb{R} \times \mathbb{R}^N,$$

and the function f is L^r-bounded: there is an $h \in L^r(Q)$ such that

$$|f(x,t,u,\xi)| \le h(x,t), \quad \textit{for a.e. } (x,t) \in Q \textit{ and every } (u,\xi) \in \mathbb{R} \times \mathbb{R}^N.$$

If (1.1) *has a lower solution α and an upper solution β, then* (1.1) *has a solution u. Moreover, if $\alpha \not\le \beta$, then*

$$u \in \overline{\{v \in C^{1,0}(\overline{Q}) : \alpha \not\le v \textit{ and } v \not\le \beta\}}.$$

Corollary 1.3 is due to C. De Coster and P. Omari [7], and extends to the parabolic problem a similar result proved for an elliptic problem by C. De Coster and M. Henrard in [5]. The main theorem in [5] is stated in a very general setting, including the case of asymmetric nonlinearities, as well. It generalizes a series of existence results in the presence of non-well-ordered lower and upper solutions, among which we mention those by H. Amann, A. Ambrosetti and G. Mancini [3], P. Omari [13], J.-P. Gossez and P. Omari [11], and P. Habets and P. Omari [12].

Theorem 1.1 is indeed a particular case of a general existence result, which we state and prove in Section 3. It is based on an abstract setting, which we construct in Sections 2 and 3. There, we point out a set of assumptions needed for our purposes, which will be shown to be satisfied, in particular, by the parabolic problem (1.1) considered above (see Section 7), or by some elliptic problem, with Neumann, Dirichlet, or more general boundary conditions (see Section 8).

In using lower and upper solutions techniques, in their various forms, one of the main tools always needed is the Maximum Principle, in some of its formulations. Needless to say, we find it unavoidable in our abstract setting, too. So, in Section 2, we introduce an assumption, which can be interpreted as an abstract version of the Strong Maximum Principle, and could be of some independent interest by its own.

We will provide in Section 5 examples of applications of Theorem 1.1 to some equations with superlinear, or one-sided superlinear nonlinearities.

In Section 6, we will show how the case of asymmetric nonlinearities can be treated in our abstract setting, as well.

We conclude in Section 9 with some remarks on possible extensions of the theory and its applications.

2. The abstract setting

Let Ω be a bounded regular domain in $\mathbb{R}^N$, and Σ be a bounded regular domain in $\mathbb{R}^M$. Here, N and M are natural numbers, not both equal to zero. Set $Q = \Omega \times \Sigma$. The points in $\overline{Q}$ will be denoted by $z = (x, t)$, where $x \in \overline{\Omega}$ and $t \in \overline{\Sigma}$. If $M = 0$, we identify Q with Ω and write $z = x$. Symmetrically, if $N = 0$, we identify Q with Σ and write $z = t$.

We denote by $C^\sharp(\overline{Q})$ the space $C^{i,j}(\overline{Q})$, where i, j are two numbers in the set $\{0, 1\}$. Let us clarify this notation. If $i = j = 0$, then $C^\sharp(\overline{Q})$ is the space of continuous functions $u: \overline{Q} \to \mathbb{R}$. If $i = j = 1$, those functions are of class C^1. In the case $i = 1$, $j = 0$, the elements of $C^\sharp(\overline{Q})$ are the continuous functions $u(x, t)$ such that $\nabla_x u(x, t)$ is continuous on $\overline{Q}$, as well. Symmetrically, if $i = 0$, $j = 1$, the elements of $C^\sharp(\overline{Q})$ are those continuous functions such that $\nabla_t u(x, t)$ is continuous.

Notice that, if $M = 0$, the space $C^\sharp(\overline{Q})$ is identified with $C^i(\overline{\Omega})$. Similarly, if $N = 0$, we identify $C^\sharp(\overline{Q})$ with $C^j(\overline{\Sigma})$.

Moreover, in order to simplify the notation, we write $\mathbb{R}^\sharp = \mathbb{R}^{iN+jM}$ and set

$$\nabla_\sharp u = (\nabla_x u, \nabla_t u) \in \mathbb{R}^\sharp,$$

with the convention that, if $j = 0$ or $M = 0$, then $\nabla_\sharp u = \nabla_x u$, and if $i = 0$ or $N = 0$, then $\nabla_\sharp u = \nabla_t u$. Clearly enough, if both these cases occur, then $\nabla_\sharp u$ will simply not be considered.

Let $W(Q)$ be a Banach space of functions which is continuously and compactly imbedded in $C^\sharp(\overline{Q})$. Assume that $\mathcal{L}: W(Q) \to L^r(Q)$ is a linear operator, with $r > 1$, $F: Q \times \mathbb{R} \times \mathbb{R}^\sharp \to \mathbb{R}$ is a L^r-Carathéodory function, and $\mathcal{B}: C^\sharp(\overline{Q}) \to C(\partial Q)$ is a linear and continuous operator.

We are concerned with the boundary value problem (P), as stated at the beginning of the paper, which we can write, equivalently, as

$$\text{(P)} \qquad \begin{cases} \mathcal{L}u = F(z, u, \nabla_\sharp u) & \text{in } Q, \\ \mathcal{B}u = 0 & \text{on } \partial Q, \end{cases}$$

Recall that F is L^r-Carathéodory if:

(i) $F(\,\cdot\,, u, \xi)$ is measurable in Q, for every $(u, \xi) \in \mathbb{R} \times \mathbb{R}^\sharp$;
(ii) $F(z, \cdot\,, \cdot\,)$ is continuous in $\mathbb{R} \times \mathbb{R}^\sharp$, for almost every $z \in Q$;
(iii) for every $\rho > 0$ there is a $h_\rho \in L^r(Q)$ such that, if $|u| + \|\xi\| \le \rho$, then

$$|F(z, u, \xi)| \le h_\rho(z) \quad \text{for a.e. } z \in Q.$$

Here, and in the sequel, $\|\cdot\|$ denotes the euclidean norm in $\mathbb{R}^\sharp$.

The following assumption will be related to the Strong Maximum Principle.

Assumption A1. If $u \in W(Q)$ is such that

$$\min_{\overline{Q}} u < 0 \quad \text{and} \quad \mathcal{B}u \ge 0,$$

then there is a point $z_0 \in \overline{Q}$ with the following properties:

(a) $u(z_0) < 0$,
(b) there is no neighbourhood U of z_0 such that $\mathcal{L}u > 0$, almost everywhere on $U \cap Q$.

Let us introduce the subspaces

$$C^\sharp_{\mathcal{B}}(\overline{Q}) = \{u \in C^\sharp(\overline{Q}) : \mathcal{B}u = 0\}, \quad W_{\mathcal{B}}(Q) = \{u \in W(Q) : \mathcal{B}u = 0\},$$

endowed with the norms in $C^\sharp(\overline{Q})$ and $W(Q)$, respectively. These are Banach spaces, since the operator $\mathcal{B}$ is assumed to be linear and continuous. We will denote by $L: W_{\mathcal{B}}(Q) \to L^r(Q)$ the restriction of $\mathcal{L}$ to $W_{\mathcal{B}}(Q)$. The following is a standard invertibility assumption.

Assumption A2. There is a $\sigma < 0$ such that $L - \sigma I: W_{\mathcal{B}}(Q) \to L^r(Q)$ is invertible and the operator $(L - \sigma I)^{-1}: L^r(Q) \to W_{\mathcal{B}}(Q)$ is continuous. Here, I denotes the identity operator.

In order to have a control on the growth of $F(z,u,\xi)$ in the variable ξ, we need to introduce a suitable increasing function

$$\mathcal{G}\colon [0,\infty[\,\to [0,\infty[\,.$$

The following assumption will be related to the well-known Bernstein–Nagumo condition.

Assumption A3. Given two constants $M,\Lambda>0$, there is a constant $C>0$ such that, if $u\in W_{\mathcal{B}}(Q)$ verifies

$$\begin{cases} |Lu(z)|\le h(z)+\mathcal{G}(\|\nabla_\sharp u(z)\|) & \text{for a.e. } z\in Q,\\ |u(z)|\le M & \text{for every } z\in\overline{Q}, \end{cases}$$

for some $h\in L^r(Q)$ with $\|h\|_{L^r}\le\Lambda$, then $\|u\|_W\le C$.

Remark 2.1. A typical choice in the applications is the function $\mathcal{G}(y)=cy^2$, first proposed by Bernstein. Notice that, whenever $\mathcal{G}$ can be taken to be identically equal to 0, it is not difficult to prove that Assumption A3 is a consequence of Assumption A2.

Let us introduce the nonlinear operator $N\colon C^\sharp_{\mathcal{B}}(\overline{Q})\to L^r(Q)$ defined by

$$(Nu)(z)=F(z,u(z),\nabla_\sharp u(z)).$$

It is readily seen that N is continuous and maps bounded sets into bounded sets. Problem (P) can then be more rigorously stated as

$$Lu=Nu. \tag{2.1}$$

A *solution* of problem (P) will be a function $u\in W_{\mathcal{B}}(Q)$ which satisfies equation (2.1), almost everywhere in Q.

If σ is the number given by Assumption A2, equation (2.1) is equivalent to the fixed point problem

$$u=\mathcal{S}u,$$

where the operator $\mathcal{S}\colon C^\sharp_{\mathcal{B}}(\overline{Q})\to C^\sharp_{\mathcal{B}}(\overline{Q})$ is defined by

$$\mathcal{S}u=(L-\sigma I)^{-1}(Nu-\sigma u).$$

By Assumption A2, and the fact that $W(Q)$ is compactly imbedded in $C^\sharp(\overline{Q})$, we have that $\mathcal{S}$ is completely continuous, so that we can use Leray–Schauder degree theory.

Given two continuous functions $u,v\colon\overline{Q}\to\mathbb{R}$, we use the following notations:

$$u\le v \;\Leftrightarrow\; u(z)\le v(z) \quad \text{for every } z\in\overline{Q},$$
$$u< v \;\Leftrightarrow\; u(z)< v(z) \quad \text{for every } z\in\overline{Q}.$$

Let us consider the sets

$$C^\sharp_{\mathcal{B}^-}(\overline{Q}) = \{u \in C^\sharp(\overline{Q}) : \mathcal{B}u \le 0\}, \quad C^\sharp_{\mathcal{B}^+}(\overline{Q}) = \{u \in C^\sharp(\overline{Q}) : \mathcal{B}u \ge 0\},$$

endowed with the norm in $C^\sharp(\overline{Q})$. Notice that $C^\sharp_{\mathcal{B}} = C^\sharp_{\mathcal{B}^-} \cap C^\sharp_{\mathcal{B}^+}$.

We now introduce a further assumption.

Assumption A4. A relation $u \ll v$ is defined in $C^\sharp(\overline{Q})$, with the following properties:

$$\begin{aligned} u < v \ &\Rightarrow\ u \ll v, \\ u \ll v \ &\Rightarrow\ u \le v, \\ [u \le v \text{ and } v \ll w] \ &\Rightarrow\ u \ll w, \\ [u \ll v \text{ and } v \le w] \ &\Rightarrow\ u \ll w, \\ u \ll v \ &\Rightarrow\ u + w \ll v + w, \\ [c > 0 \text{ and } u \ll v] \ &\Rightarrow\ cu \ll cv, \end{aligned} \tag{2.2}$$

for every $u, v, w \in C^\sharp(\overline{Q})$ and every constant $c \in \mathbb{R}$. Sometimes, we will write $v \gg u$ instead of $u \ll v$. Moreover, the set

$$\{u \in C^\sharp_{\mathcal{B}^-}(\overline{Q}) : u \ll 0\}$$

is open in $C^\sharp_{\mathcal{B}^-}(\overline{Q})$. Equivalently, the set $\{u \in C^\sharp_{\mathcal{B}^+}(\overline{Q}) : u \gg 0\}$ is open in $C^\sharp_{\mathcal{B}^+}(\overline{Q})$.

Notice indeed that $u \gg 0$ if and only if $-u \ll 0$. Moreover, as a consequence of Assumption A4, one can easily see that the sets

$$\{u \in C^\sharp_{\mathcal{B}}(\overline{Q}) : u \ll 0\} \quad \text{and} \quad \{u \in C^\sharp_{\mathcal{B}}(\overline{Q}) : u \gg 0\}$$

are open in $C^\sharp_{\mathcal{B}}(\overline{Q})$. These facts will be frequently used in the sequel, by just mentioning Assumption A4.

Definition 2.2. A pair of functions $(\alpha, \beta) \in C^\sharp(\overline{Q}) \times C^\sharp(\overline{Q})$ is said to be *degree-admissible* for (P) if $\alpha \ll \beta$, and there is a constant $R_{\alpha,\beta} > 0$ with the following property: any solution u of (P) satisfying $\alpha \le u \le \beta$ is such that

$$\alpha \ll u \ll \beta \quad \text{and} \quad \|u\|_{C^\sharp} < R_{\alpha,\beta}.$$

In the following, it will be convenient to use the notation

$$[\alpha, \beta] = \{u \in C^\sharp(\overline{Q}) : \alpha \le u \le \beta\},$$

and correspondingly

$$[\alpha, \beta]_{\mathcal{B}} = \{u \in C^\sharp_{\mathcal{B}}(\overline{Q}) : \alpha \le u \le \beta\}.$$

If (α, β) is a degree-admissible pair, then the set

$$\mathcal{U}_{(\alpha,\beta)} = \{u \in C^{\sharp}_{\mathcal{B}}(\overline{Q}) : \alpha \ll u \ll \beta\}$$

is open in $C^{\sharp}_{\mathcal{B}}(\overline{Q})$. Indeed, by Assumption A4, the sets

$$A = \{u \in C^{\sharp}_{\mathcal{B}^+}(\overline{Q}) : u \gg \alpha\}, \quad B = \{u \in C^{\sharp}_{\mathcal{B}^-}(\overline{Q}) : u \ll \beta\},$$

are open in $C^{\sharp}_{\mathcal{B}^+}(\overline{Q})$ and $C^{\sharp}_{\mathcal{B}^-}(\overline{Q})$, respectively, and

$$\mathcal{U}_{(\alpha,\beta)} = (A \cap C^{\sharp}_{\mathcal{B}}(\overline{Q})) \cap (B \cap C^{\sharp}_{\mathcal{B}}(\overline{Q})).$$

Moreover, since the closure of $\mathcal{U}_{(\alpha,\beta)}$ satisfies

$$\overline{\mathcal{U}}_{(\alpha,\beta)} \subseteq [\alpha, \beta]_{\mathcal{B}},$$

the fixed points of $\mathcal{S}$ in $\overline{\mathcal{U}}_{(\alpha,\beta)}$ belong to $\mathcal{U}_{(\alpha,\beta)}$, and to the ball $B(0, R_{\alpha,\beta})$ in $C^{\sharp}_{\mathcal{B}}(\overline{Q})$. Hence, we can define

$$\deg(I - \mathcal{S}, \mathcal{U}_{(\alpha,\beta)}) = \mathrm{d}_{\mathrm{LS}}(I - \mathcal{S}, \mathcal{U}_{(\alpha,\beta)} \cap B(0, R_{\alpha,\beta})),$$

where d_{LS} denotes the Leray–Schauder degree.

The degree-admissible pairs considered in the sequel will often be made of lower and upper solutions, a concept that we precise now.

Definition 2.3. A function $\alpha \in W(Q)$ is a *lower solution* of (P) if

$$\begin{cases} \mathcal{L}\alpha \le F(z, \alpha, \nabla_{\sharp}\alpha) & \text{a.e. in } Q, \\ \mathcal{B}\alpha \le 0 & \text{on } \partial Q. \end{cases}$$

The function α is a *strict lower solution* if it is a lower solution and, for every solution u of (P) with $u \ge \alpha$, one has that $u \gg \alpha$.

Analogously, a function $\beta \in W(Q)$ is an *upper solution* of (P) if

$$\begin{cases} \mathcal{L}\beta \ge F(z, \beta, \nabla_{\sharp}\beta) & \text{a.e. in } Q, \\ \mathcal{B}\beta \ge 0 & \text{on } \partial Q. \end{cases}$$

The function β is a *strict upper solution* if it is an upper solution and, for every solution u of (P) with $u \le \beta$, one has that $u \ll \beta$.

3. Well-ordered lower and upper solutions

The following is a classical result on lower and upper solutions. It has been proved for different types of boundary value problems, with various kinds of differential operators. See [4] for a comprehensive review. We prove it here in our abstract setting.

Theorem 3.1. *Let Assumptions* A1–A3 *hold true, and let* α *be a lower solution and* β *be an upper solution of* (P) *satisfying* $\alpha \le \beta$. *Assume that there is a function* $\eta \in L^r(Q)$ *for which*

$$|F(z,u,\xi)| \le \eta(z) + \mathcal{G}(\|\xi\|), \quad \text{for a.e. } z \in Q, \quad \text{every } u \in [\alpha(z),\beta(z)] \text{ and every } \xi \in \mathbb{R}^\sharp.$$

Then, problem (P) *has a solution* u *such that* $\alpha \le u \le \beta$. *If, moreover, Assumption* A4 *holds and* (α,β) *is a degree-admissible pair, then*

$$\deg(I - \mathcal{S}, \mathcal{U}_{(\alpha,\beta)}) = 1.$$

Remark 3.2. Recall from Remark 2.1 that, in the case when the function $\mathcal{G}$ is identically equal to 0, Assumption A3 is not explicitely needed, since it is a consequence of Assumption A2. This is the case, e.g. when there is no dependence on $\nabla_\sharp u$ in problem (P), i.e. $F(z,u,\xi)$ does not depend on ξ.

Proof. We follow closely the proof in [5]. Set $M = \max\{\|\alpha\|_{L^\infty}, \|\beta\|_{L^\infty}\}$ and $\Lambda = \|h\|_{L^r}$, where

$$h(z) := \eta(z) + \mathcal{G}(\max\{\|\nabla_\sharp\alpha\|_{L^\infty}, \|\nabla_\sharp\beta\|_{L^\infty}\}).$$

Let $C > 0$ be the constant given by Assumption A3. As $W(Q)$ is continuously imbedded in $C^\sharp(\overline{Q})$, there exists a positive constant κ such that $\|u\|_{C^\sharp} \le \kappa\|u\|_W$ for every $u \in W(Q)$. Let $R = \kappa C$. We can assume $R > \max\{\|\nabla_\sharp\alpha\|_{L^\infty}, \|\nabla_\sharp\beta\|_{L^\infty}\}$. Let $\overline{F}\colon \overline{Q} \times \mathbb{R} \times \mathbb{R}^\sharp \to \mathbb{R}$ be defined by

$$\overline{F}(z,u,\xi) = \begin{cases} F(z,u,\xi) & \text{if } \alpha(z) \le u \le \beta(z) \text{ and } \|\xi\| \le R, \\ F\left(z,u,R\dfrac{\xi}{\|\xi\|}\right) & \text{if } \alpha(z) \le u \le \beta(z) \text{ and } \|\xi\| > R, \\ F(z,\alpha(z),\nabla_\sharp\alpha(z)) & \text{if } u < \alpha(z), \\ F(z,\beta(z),\nabla_\sharp\beta(z)) & \text{if } u > \beta(z). \end{cases}$$

Notice that $\overline{F}$ satisfies

$$|\overline{F}(z,u,\xi)| \le h(z) + \min\{\mathcal{G}(\|\xi\|), \mathcal{G}(R)\}, \quad \text{for a.e. } z \in Q, \text{ every } u \in \mathbb{R} \text{ and every } \xi \in \mathbb{R}^\sharp. \tag{3.1}$$

Let us define

$$\gamma(z,u) = \begin{cases} \alpha(z) & \text{if } u \le \alpha(z), \\ u & \text{if } \alpha(z) < u < \beta(z), \\ \beta(z) & \text{if } u \ge \beta(z), \end{cases}$$

and consider the modified problem

$$(\overline{\mathrm{P}}) \qquad \begin{cases} \mathcal{L}u - \sigma u = \overline{F}(z,u,\nabla_\sharp u) - \sigma\gamma(z,u) & \text{in } Q, \\ \mathcal{B}u = 0 & \text{on } \partial Q, \end{cases}$$

where $\sigma < 0$ is the number given by Assumption A2. The remaining part of the proof will be divided into four steps.

Step 1. Every solution u of $(\overline{\mathrm{P}})$ is such that $\alpha \le u \le \beta$.

Let us prove that $\alpha \le u$. Set $v = u - \alpha$, and assume by contradiction that $\min v < 0$. Since $\mathcal{B}v = \mathcal{B}u - \mathcal{B}\alpha = -\mathcal{B}\alpha \ge 0$, by Assumption A1 there is a point $z_0 \in \overline{Q}$ such that $v(z_0) < 0$, and there is no neighbourhood U of z_0 such that $\mathcal{L}v > 0$, almost everywhere on $U \cap Q$. On the other hand, as $v(z_0) < 0$, there is a neighbourhood V of z_0 such that $v < 0$ on $V \cap Q$, i.e., $u < \alpha$ on $V \cap Q$. Hence

$$\begin{aligned}\mathcal{L}v = \mathcal{L}u - \mathcal{L}\alpha &= \overline{F}(z, u, \nabla_\sharp u) - \sigma(\gamma(z, u) - u) - \mathcal{L}\alpha \\ &= F(z, \alpha, \nabla_\sharp \alpha) - \sigma(\alpha - u) - \mathcal{L}\alpha \ge \sigma v > 0,\end{aligned}$$

almost everywhere on $V \cap Q$, a contradiction. In a similar way it can be shown that $u \le \beta$.

Step 2. Every solution u of $(\overline{\mathrm{P}})$ solves (P).

Let u be a solution of $(\overline{\mathrm{P}})$. As, by Step 1, $\alpha \le u \le \beta$, we have $\gamma(z, u) = u$ and, by (3.1),

$$|(Lu)(z)| = |\overline{F}(z, u(z), \nabla_\sharp u(z))| \le h(z) + \mathcal{G}(\|\nabla_\sharp u(z)\|).$$

By Assumption A3,

$$\|u\|_{C^\sharp} \le \kappa \|u\|_W \le \kappa C = R,$$

and therefore $\overline{F}(z, u, \nabla_\sharp u) = F(z, u, \nabla_\sharp u)$.

Step 3. Problem $(\overline{\mathrm{P}})$ admits a solution.

Let us introduce the operators Γ, $\overline{N}\colon C^\sharp_{\mathcal{B}}(\overline{Q}) \to L^r(Q)$ defined by

$$(\Gamma u)(z) = \gamma(z, u(z)), \quad (\overline{N}u)(z) = \overline{F}(z, u(z), \nabla_\sharp u(z)).$$

Clearly, Γ is continuous, and, despite the fact that the function $\overline{F}$ might not be continuous, one can see that the operator $\overline{N}$ is continuous. Moreover, by (3.1), for every $u \in C^\sharp_{\mathcal{B}}(\overline{Q})$,

$$|\overline{F}(z, u, \nabla_\sharp u)| \le h(z) + \mathcal{G}(R) \quad \text{for a.e. } z \in Q,$$

so that, for some positive constant C_1,

$$\|\overline{N}u - \sigma\Gamma u\|_{L^r} \le C_1 \quad \text{for every } u \in C^\sharp_{\mathcal{B}}(\overline{Q}). \tag{3.2}$$

Problem $(\overline{\mathrm{P}})$ is equivalent to the fixed point problem

$$u = \overline{\mathcal{S}}u,$$

where the operator $\overline{\mathcal{S}}\colon C^\sharp_{\mathcal{B}}(\overline{Q}) \to C^\sharp_{\mathcal{B}}(\overline{Q})$ is defined by

$$\overline{\mathcal{S}}u = (L - \sigma I)^{-1}(\overline{N}u - \sigma\Gamma u).$$

By Assumption A2, the fact that $W(Q)$ is compactly imbedded in $C^\sharp(\overline{Q})$, and (3.2), we have that $\overline{\mathcal{S}}$ is completely continuous and its image is bounded, hence contained in an open ball $B(0,K)$ in $C^\sharp_{\mathcal{B}}(\overline{Q})$. Therefore, by classical degree theory,

$$\mathrm{d}_{\mathrm{LS}}(I-\overline{\mathcal{S}},\, B(0,K)) = 1,$$

and problem $(\overline{\mathrm{P}})$ has a solution $u \in B(0,K)$.

Step 4. Computation of the degree.

If (α,β) is a degree-admissible pair, all fixed points of $\mathcal{S}$ belonging to $[\alpha,\beta]_{\mathcal{B}}$ are in $\mathcal{U}_{(\alpha,\beta)} \cap B(0,R_{\alpha,\beta})$. Taking $R > R_{\alpha,\beta}$ from the beginning of the proof, we have that all fixed points of $\overline{\mathcal{S}}$ are in $[\alpha,\beta]_{\mathcal{B}}$ (by Step 1), and they are fixed points of $\mathcal{S}$ (by Step 2). Hence, all fixed points of $\overline{\mathcal{S}}$ belong to $\mathcal{U}_{(\alpha,\beta)}\cap B(0,R_{\alpha,\beta})$, and since $\mathcal{S}$ and $\overline{\mathcal{S}}$ coincide on $\mathcal{U}_{(\alpha,\beta)} \cap B(0,R_{\alpha,\beta})$, we have

$$\deg(I-\mathcal{S},\mathcal{U}_{(\alpha,\beta)}) = \deg(I-\overline{\mathcal{S}},\mathcal{U}_{(\alpha,\beta)}).$$

Taking $K \geq R_{\alpha,\beta}$ in Step 3, by the excision property of the degree we have

$$\mathrm{d}_{\mathrm{LS}}(I-\overline{\mathcal{S}},\mathcal{U}_{(\alpha,\beta)} \cap B(0,R_{\alpha,\beta})) = \mathrm{d}_{\mathrm{LS}}(I-\overline{\mathcal{S}},B(0,K)) = 1,$$

thus ending the proof. □

4. Non-well-ordered lower and upper solutions

In this section, the problem we consider will be written in the form

$$\text{(P)} \qquad \begin{cases} \mathcal{L}u = g(z,u,\nabla_\sharp u)\,u + f(z,u,\nabla_\sharp u) & \text{in } Q, \\ \mathcal{B}u = 0 & \text{on } \partial Q, \end{cases}$$

where $f,g\colon Q\times\mathbb{R}\times\mathbb{R}^\sharp \to \mathbb{R}$ are L^r-Carathéodory functions. We introduce the following assumption on the existence of a "first" eigenvalue.

Assumption A5. There is a number $\lambda_1 \geq 0$ and a function $\varphi_1 \in W_{\mathcal{B}}(Q)$, with $\varphi_1 \gg 0$, such that $\ker(L-\lambda_1 I) = \{c\varphi_1 : c \in \mathbb{R}\}$. We will assume that $\max\varphi_1 = 1$.

Lemma 4.1. *Let Assumptions* A4 *and* A5 *hold. Given a bounded set $\mathcal{A}$ in $W(Q)$, there is a constant $C_{\mathcal{A}} \geq 0$ such that, if $w \in \mathcal{A}$ satisfies $\mathcal{B}w \leq 0$, then $w \leq C_{\mathcal{A}}\varphi_1$, and if $w \in \mathcal{A}$ satisfies $\mathcal{B}w \geq 0$, then $w \geq -C_{\mathcal{A}}\varphi_1$.*

Proof. We prove the first inequality, the second one being analogous. By contradiction, assume that, for every $n \in \mathbb{N}$, there is a $w_n \in \mathcal{A}$ with $\mathcal{B}w_n \leq 0$, and $w_n \not\leq n\varphi_1$. Since $\mathcal{A}$ is bounded and $W(Q)$ is compactly imbedded in $C^\sharp(\overline{Q})$, there is a subsequence, still denoted $(w_n)_n$, and a function $\overline{w} \in C^\sharp(\overline{Q})$, such that $w_n \to \overline{w}$ in $C^\sharp(\overline{Q})$. Since $\mathcal{B}$ is continuous, $\mathcal{B}\overline{w} \leq 0$. Hence, w_n and $\overline{w}$ are in $C^\sharp_{\mathcal{B}-}(\overline{Q})$. By Assumptions A4 and A5, there is a $\varepsilon > 0$ such that $\varphi_1 - \varepsilon\overline{w} \gg 0$. By Assumption A4, it has to be $w_n \ll \varphi_1/\varepsilon$ for n large enough, a contradiction. □

The following concept has been introduced in [12], in the framework of an elliptic problem.

Definition 4.2. A pair of functions $(\psi_1,\psi_2)\in L^r(Q)\times L^r(Q)$ is said to be *admissible* if it satisfies $\psi_1\le\lambda_1\le\psi_2$ almost everywhere in Q and, for every $q\in L^r(Q)$, with $\psi_1\le q\le\psi_2$ almost everywhere in Q, if u is a solution of

$$\begin{cases}\mathcal{L}u=q(z)\,u & \text{in } Q,\\ \mathcal{B}u=0 & \text{on } \partial Q,\end{cases}\tag{4.1}$$

then, either $u=0$, or $u\ll 0$, or $u\gg 0$.

The set of admissible pairs is not empty, since it contains the pair of constant functions (λ_1,λ_1). In the applications, one usually relates the admissibility of the pair (ψ_1,ψ_2) to the non-interaction of ψ_2 with the higher eigenvalues of the operator L.

Lemma 4.3. *Let Assumptions* A2, A4 *and* A5 *hold. Given an admissible pair of functions (ψ_1,ψ_2), there are two positive constants c_{ψ_1,ψ_2} and C_{ψ_1,ψ_2} such that, for every $q\in L^r(Q)$, with $\psi_1\le q\le\psi_2$ almost everywhere in Q, if u is a solution of* (4.1)*, then*

$$c_{\psi_1,\psi_2}\|u\|_{L^\infty}\varphi_1\le|u|\le C_{\psi_1,\psi_2}\|u\|_{L^\infty}\varphi_1.$$

Proof. The inequalities clearly hold when $u=0$. Assume $u\ne0$, and let $v=u/\|u\|_{L^\infty}$. Then, v solves (4.1), and since the pair (ψ_1,ψ_2) is admissible, it has to be either $v\gg0$, or $v\ll0$. Assume $v\gg0$, the other case being treated similarly. We want to prove that

$$c_{\psi_1,\psi_2}\varphi_1\le v\le C_{\psi_1,\psi_2}\varphi_1.$$

By contradiction, assume that for every $n\in\mathbb{N}$ there is a function $v_n\in W_{\mathcal{B}}(Q)$ satisfying the following properties: $\|v_n\|_{L^\infty}=1$, $v_n\gg0$, $\mathcal{L}v_n=q_n\,v_n$ for some function $q_n\in L^r(Q)$ with $\psi_1\le q_n\le\psi_2$ almost everywhere in Q, $\mathcal{B}v_n=0$ and either $v_n\not\ge\varphi_1/n$, or $v_n\not\le n\varphi_1$.

Let $N_n\colon L^\infty(Q)\to L^r(Q)$ be defined as

$$(N_nu)(z)=q_n(z)u(z).$$

It is a continuous operator, which transforms bounded subsets of $L^\infty(Q)$ into bounded subsets of $L^r(Q)$. Let $\sigma\in\mathbb{R}$ be the number given by Assumption A2. Then,

$$v_n=(L-\sigma I)^{-1}(N_nv_n-\sigma v_n).$$

By Assumption A2, as $\|v_n\|_{L^\infty}=1$, there is a constant $C>0$ such that $\|v_n\|_W\le C$. By Lemma 4.1, $v_n\le n\varphi_1$, for n large enough. So, it has to be $v_n\not\ge\varphi_1/n$. Since $W(Q)$ is compactly imbedded in $C^\sharp(\overline{Q})$, there are a subsequence, still denoted by $(v_n)_n$, and a function $\overline{v} \in C^\sharp(\overline{Q})$ such that $v_n \to \overline{v}$ in $C^\sharp(\overline{Q})$. Moreover, passing to a further subsequence, we may assume that there is a $\overline{q} \in L^r(Q)$ for which $q_n \rightharpoonup \overline{q}$ (weakly) in $L^r(Q)$. Since the set of functions

$$\{p \in L^r(Q) : \psi_1(z) \leq p(z) \leq \psi_2(z), \text{ for a.e. } z \in Q\}$$

is closed and convex, it is weakly closed, so that

$$\psi_1(z) \leq \overline{q}(z) \leq \psi_2(z), \quad \text{for a.e. } z \in Q.$$

Then, by compactness, $\overline{v} = (L - \sigma I)^{-1}(\overline{q}(\,\cdot\,)\overline{v} - \sigma\overline{v})$, i.e. $\overline{v} \in W_{\mathcal{B}}(Q)$ and

$$L\overline{v} = \overline{q}(\,\cdot\,)\overline{v}.$$

Since the pair (ψ_1, ψ_2) is admissible, either $\overline{v} = 0$, or $\overline{v} \gg 0$, or $\overline{v} \ll 0$. As $v_n \gg 0$, by Assumption A4 the third possibility is excluded. Since $\|v_n\|_\infty = 1$, also $\|\overline{v}\|_\infty = 1$, so that $\overline{v} \neq 0$. Therefore it has to be $\overline{v} \gg 0$. By Assumption A4, there is $\varepsilon > 0$ such that $\overline{v} - \varepsilon\varphi_1 \gg 0$. Using Assumption A4 again, it has to be $v_n \gg \varepsilon\varphi_1$ for n large enough, a contradiction. □

Let c_{ψ_1,ψ_2} and C_{ψ_1,ψ_2} be the two positive constants given by Lemma 4.3. Given two numbers a, b, set

$$\iota_a = \begin{cases} \dfrac{C_{\psi_1,\psi_2}}{c_{\psi_1,\psi_2}}\,a & \text{if } a \leq 0, \\[2mm] \dfrac{c_{\psi_1,\psi_2}}{C_{\psi_1,\psi_2}}\,a & \text{if } a \geq 0, \end{cases} \qquad \kappa_b = \begin{cases} \dfrac{c_{\psi_1,\psi_2}}{C_{\psi_1,\psi_2}}\,b & \text{if } b \leq 0, \\[2mm] \dfrac{C_{\psi_1,\psi_2}}{c_{\psi_1,\psi_2}}\,b & \text{if } b \geq 0. \end{cases} \tag{4.2}$$

Notice that

$$\iota_a \leq a \quad \text{and} \quad \kappa_b \geq b. \tag{4.3}$$

We will also need the following assumption, which in the applications will be satisfied by the positive constant functions.

Assumption A6. There is a function $\varphi_0 \in W(Q)$ such that

$$\varphi_0 > 0, \quad \mathcal{L}\varphi_0 \geq 0 \quad \text{and} \quad \mathcal{B}\varphi_0 \geq 0.$$

We will assume that $\max \varphi_0 = 1$.

We define the set

$$\mathcal{F}(\mathcal{I}, \Lambda, K),$$

where $\mathcal{I} \subseteq \mathbb{R}$ is an interval, and Λ, K are some nonnegative constants. Its elements are the Carathéodory functions $f\colon Q \times \mathcal{I} \times \mathbb{R}^\sharp \to \mathbb{R}$ which satisfy the following Bernstein–Nagumo growth condition:

$$|f(z, u, \xi)| \leq h(z) + K\mathcal{G}(\|\xi\|), \quad \text{for a.e. } z \in Q \text{ and every } (u, \xi) \in \mathcal{I} \times \mathbb{R}^\sharp,$$

for some $h \in L^r(Q)$, with $\|h\|_{L^r} \leq \Lambda$.

For an open interval $I \subseteq \mathbb{R}$, we use the notation

$$I\varphi_1 = \{u \in C^\sharp(\overline{Q}) : (\inf I)\varphi_1 < u < (\sup I)\varphi_1\}.$$

Similarly if the interval is closed, with the strict inequalities replaced by non-strict ones. Moreover, we will denote by $I\varphi_1(\overline{Q})$ the interval obtained as the union of the images of the elements of $I\varphi_1$.

We are now in a position to state our main result.

THEOREM 4.4. *Let Assumptions* A1–A6 *hold true. Let* (ψ_1, ψ_2) *be an admissible pair of functions, and* a, b *be two real numbers, with* $a \le b$. *Let* $I \subseteq \mathbb{R}$ *be an open interval containing* $[\iota_a, \kappa_b]$, *and* $\mathcal{I} \subseteq \mathbb{R}$ *be an open interval containing* $[\iota_a, \kappa_b]\varphi_1(\overline{Q})$, *where* ι_a *and* κ_b *are given by* (4.2). *There is a constant* $\Lambda > 0$ *such that, for every function* g *satisfying*

$$\psi_1(z) \le g(z, u, \xi) \le \psi_2(z), \quad \text{for a.e. } z \in Q \text{ and every } (u, \xi) \in \mathcal{I} \times \mathbb{R}^\sharp,$$

and every function $f \in \mathcal{F}(\mathcal{I}, \Lambda, \Lambda)$, *if there are a lower solution* α *and an upper solution* β *of* (P) *verifying*

$$a\varphi_1 - \Lambda \le \alpha \le b\varphi_1, \quad a\varphi_1 \le \beta \le b\varphi_1 + \Lambda,$$

then problem (P) *has a solution* $u \in I\varphi_1$. *Moreover, if* $\alpha \not\le \beta$, *then*

$$u \in \overline{\{v \in C^\sharp_{\mathcal{B}}(\overline{Q}) : \alpha \not\le v \text{ and } v \not\le \beta\}}. \tag{4.4}$$

REMARK 4.5. It can be seen that (4.4) implies

$$\alpha \not\ll u \quad \text{and} \quad u \not\ll \beta.$$

Indeed, let u satisfy (4.4) and assume by contradiction that $\alpha \ll u$. Let $(v_n)_n$ be a sequence in $C^\sharp_{\mathcal{B}}(\overline{Q})$ such that $\alpha \not\le v_n$, $v_n \not\le \beta$, and $v_n \to u$ in $C^\sharp(\overline{Q})$. By Assumption A4, $\alpha \ll v_n$, for n sufficiently large, contradicting $\alpha \not\le v_n$. Hence, (4.4) implies $\alpha \not\ll u$. In the same way one can see that (4.4) implies $u \not\ll \beta$, as well.

PROOF. If the lower solution α and the upper solution β are well-ordered, the result follows from Theorem 3.1, independently of the choice of Λ. So, we will focus our attention on the case where $\alpha \not\le \beta$.

Let $\mu := \min \varphi_0$. Recall that, by Assumption A6, it is $\mu > 0$. Fix $\varepsilon > 0$ and a positive integer m_0 such that

$$\left[\iota_a - \varepsilon - \frac{2}{m_0\mu}, \kappa_b + \varepsilon + \frac{2}{m_0\mu}\right] \subseteq I$$

and

$$[\iota_a - \varepsilon, \kappa_b + \varepsilon]\varphi_1(\overline{Q}) + \left[-\frac{2}{m_0\mu}, \frac{2}{m_0\mu}\right] \subseteq \mathcal{I}.$$

We will prove the statement by taking $\Lambda = 1/m$, with $m \geq m_0$ a sufficiently large integer. By contradiction, assume that for every $m \geq m_0$ there are functions f_m and g_m, with

$$\psi_1(z) \leq g_m(z,u,\xi) \leq \psi_2(z), \quad \text{for a.e. } z \in Q \text{ and every } (u,\xi) \in \mathcal{I} \times \mathbb{R}^\sharp,$$

$f_m \in \mathcal{F}(\mathcal{I}, 1/m, 1/m)$, and there are a lower solution α_m and an upper solution β_m of the problem

$$(\mathrm{P}_m) \qquad \begin{cases} \mathcal{L}u = g_m(z,u,\nabla_\sharp u)\, u + f_m(z,u,\nabla_\sharp u) & \text{in } Q, \\ \mathcal{B}u = 0 & \text{on } \partial Q, \end{cases}$$

with

$$a\varphi_1 - \frac{1}{m} \leq \alpha_m \leq b\varphi_1, \quad a\varphi_1 \leq \beta_m \leq b\varphi_1 + \frac{1}{m}, \tag{4.5}$$

and $\alpha_m \not\leq \beta_m$, for which (P_m) has no solution u with $u \in I\varphi_1$, or (P_m) has no solution u in the closure of the set $\{v \in C^\sharp_{\mathcal{B}}(\overline{Q}) : \alpha_m \not\leq v \text{ and } v \not\leq \beta_m\}$. In particular, we are assuming $a < b$, since otherwise $\alpha_m \leq \beta_m$.

The strategy of the proof is to modify the functions f_m and g_m in order to create a new pair of lower and upper solutions, which are well-ordered with respect to α_m and β_m. By the use of degree arguments, we will then be able to prove the existence of a solution u_m of the modified problem which remains in the region where the functions have not been modified, so that u_m will indeed be a solution of (P_m), and we will see how this leads to a contradiction with the above.

Define the modified function $\widetilde{f}_m : Q \times \mathbb{R} \times \mathbb{R}^\sharp \to \mathbb{R}$ as

$$\widetilde{f}_m(z,u,\xi) = \begin{cases} \dfrac{3\lambda_1+1}{m\mu} & \text{if } u \leq (\iota_a - \varepsilon)\varphi_1(z) - \dfrac{2}{m\mu}\varphi_0(z), \\ \dfrac{3\lambda_1+1}{m\mu} + \dfrac{m\mu f_m(z,u,\xi) - 3\lambda_1 - 1}{\varphi_0(z)}\Big(u - (\iota_a-\varepsilon)\varphi_1(z) + \dfrac{2}{m\mu}\varphi_0(z)\Big) & \\ \qquad \text{if } (\iota_a - \varepsilon)\varphi_1(z) - \dfrac{2}{m\mu}\varphi_0(z) \leq u \leq (\iota_a - \varepsilon)\varphi_1(z) - \dfrac{1}{m\mu}\varphi_0(z), & \\ f_m(z,u,\xi) & \text{if } (\iota_a - \varepsilon)\varphi_1(z) - \dfrac{1}{m\mu}\varphi_0(z) \leq u \leq (\kappa_b + \varepsilon)\varphi_1(z) + \dfrac{1}{m\mu}\varphi_0(z), \\ -\dfrac{3\lambda_1+1}{m\mu} + \dfrac{m\mu f_m(z,u,\xi) + 3\lambda_1 + 1}{\varphi_0(z)}\Big((\kappa_b+\varepsilon)\varphi_1(z) + \dfrac{2}{m\mu}\varphi_0(z) - u\Big) & \\ \qquad \text{if } (\kappa_b + \varepsilon)\varphi_1(z) + \dfrac{1}{m\mu}\varphi_0(z) \leq u \leq (\kappa_b + \varepsilon)\varphi_1(z) + \dfrac{2}{m\mu}\varphi_0(z), & \\ -\dfrac{3\lambda_1+1}{m\mu} & \text{if } u \geq (\kappa_b + \varepsilon)\varphi_1(z) + \dfrac{2}{m\mu}\varphi_0(z), \end{cases}$$

define the modified function $\widetilde{g}_m\colon Q\times\mathbb{R}\times\mathbb{R}^\sharp\to\mathbb{R}$ as

$$\widetilde{g}_m(z,u,\xi)=\begin{cases}\lambda_1 & \text{if } u\le(\iota_a-\varepsilon)\varphi_1(z)-\dfrac{2}{m\mu}\varphi_0(z),\\ \lambda_1+m\mu\,\dfrac{g_m(z,u,\xi)-\lambda_1}{\varphi_0(z)}\Big(u-(\iota_a-\varepsilon)\varphi_1(z)+\dfrac{2}{m\mu}\varphi_0(z)\Big) & \\ \qquad\text{if } (\iota_a-\varepsilon)\varphi_1(z)-\dfrac{2}{m\mu}\varphi_0(z)\le u\le(\iota_a-\varepsilon)\varphi_1(z)-\dfrac{1}{m\mu}\varphi_0(z), & \\ g_m(z,u,\xi) & \text{if } (\iota_a-\varepsilon)\varphi_1(z)-\dfrac{1}{m\mu}\varphi_0(z)\le u\le(\kappa_b+\varepsilon)\varphi_1(z)+\dfrac{1}{m\mu}\varphi_0(z),\\ \lambda_1+m\mu\,\dfrac{g_m(z,u,\xi)-\lambda_1}{\varphi_0(z)}\Big((\kappa_b+\varepsilon)\varphi_1(z)+\dfrac{2}{m\mu}\varphi_0(z)-u\Big) & \\ \qquad\text{if } (\kappa_b+\varepsilon)\varphi_1(z)+\dfrac{1}{m\mu}\varphi_0(z)\le u\le(\kappa_b+\varepsilon)\varphi_1(z)+\dfrac{2}{m\mu}\varphi_0(z), & \\ \lambda_1 & \text{if } u\ge(\kappa_b+\varepsilon)\varphi_1(z)+\dfrac{2}{m\mu}\varphi_0(z),\end{cases}$$

and consider the modified problem

$$(\widetilde{\mathrm{P}}_m)\qquad\begin{cases}\mathcal{L}u=\widetilde{g}_m(z,u,\nabla_\sharp u)\,u+\widetilde{f}_m(z,u,\nabla_\sharp u) & \text{in } Q,\\ \mathcal{B}u=0 & \text{on } \partial Q.\end{cases}$$

Notice that $\widetilde{g}_m$ is a L^r-Carathéodory function,

$$\psi_1(z)\le\widetilde{g}_m(z,u,\xi)\le\psi_2(z)\quad\text{for a.e. } z\in Q \text{ and every } (u,\xi)\in\mathbb{R}\times\mathbb{R}^\sharp,\tag{4.6}$$

and, for m sufficiently large,

$$\widetilde{f}_m\in\mathcal{F}\Big(\mathbb{R},\frac{3\lambda_1+1}{m\mu}(1+|Q|^{1/r}),\frac{1}{m}\Big).\tag{4.7}$$

Let $\widetilde{N}_m\colon C^\sharp_{\mathcal{B}}(\overline{Q})\to L^r(Q)$, be defined as

$$(\widetilde{N}_m u)(z)=\widetilde{g}_m(z,u(z),\nabla_\sharp u(z))\,u(z)+\widetilde{f}_m(z,u(z),\nabla_\sharp u(z)).$$

Let $\sigma\in\mathbb{R}$ be the number given by Assumption A2, and let $\widetilde{\mathcal{S}}_m\colon C^\sharp_{\mathcal{B}}(\overline{Q})\to C^\sharp_{\mathcal{B}}(\overline{Q})$ be defined as

$$\widetilde{\mathcal{S}}_m u=(L-\sigma I)^{-1}(\widetilde{N}_m u-\sigma u).$$

Recall that solving problem $(\widetilde{\mathrm{P}}_m)$ is equivalent to finding a fixed point of $\widetilde{\mathcal{S}}_m$.

Define the functions $\widetilde{\alpha}_m$ and $\widetilde{\beta}_m$ as follows:

$$\widetilde{\alpha}_m(z)=(\iota_a-\varepsilon)\varphi_1(z)-\frac{3}{m\mu}\varphi_0(z),\qquad\widetilde{\beta}_m(z)=(\kappa_b+\varepsilon)\varphi_1(z)+\frac{3}{m\mu}\varphi_0(z).$$

Notice that, by (4.3), (4.5), and Assumptions A5 and A6,

$$\widetilde{\alpha}_m\ll\alpha_m\ll\widetilde{\beta}_m,\qquad\widetilde{\alpha}_m\ll\beta_m\ll\widetilde{\beta}_m.\tag{4.8}$$

Let us prove that, for the modified problem $(\widetilde{\mathrm{P}}_m)$,

$$\widetilde{\alpha}_m \text{ is a strict lower solution.}$$

Indeed, we have

$$\widetilde{f}_m(z, \widetilde{\alpha}_m, \nabla_\sharp \widetilde{\alpha}_m) = \frac{3\lambda_1 + 1}{m\mu} \quad \text{and} \quad \widetilde{g}_m(z, \widetilde{\alpha}_m, \nabla_\sharp \widetilde{\alpha}_m) = \lambda_1,$$

so that, by Assumption A6,

$$\begin{aligned} \mathcal{L}\widetilde{\alpha}_m &= \lambda_1 \widetilde{\alpha}_m + \frac{3\lambda_1}{m\mu}\varphi_0 - \frac{3}{m\mu}\mathcal{L}(\varphi_0) \leq \lambda_1 \widetilde{\alpha}_m + \frac{3\lambda_1 + 1}{m\mu} \\ &= \widetilde{g}_m(z, \widetilde{\alpha}_m, \nabla_\sharp \widetilde{\alpha}_m)\widetilde{\alpha}_m + \widetilde{f}_m(z, \widetilde{\alpha}_m, \nabla_\sharp \widetilde{\alpha}_m). \end{aligned}$$

Moreover, by Assumption A6,

$$\mathcal{B}\widetilde{\alpha}_m = -\frac{3}{m\mu}\mathcal{B}(\varphi_0) \leq 0,$$

so that $\widetilde{\alpha}_m$ is a lower solution. In order to show that it is a strict lower solution, let u be a solution of $(\widetilde{\mathrm{P}}_m)$ with $u \geq \widetilde{\alpha}_m$. We will prove that $u > \widetilde{\alpha}_m$. By contradiction, assume that $\min(u - \widetilde{\alpha}_m) = 0$. Set $v = u - \widetilde{\alpha}_m - (1/(m\mu))\varphi_0$. Then, $\min v < 0$ and, by Assumption A6, $\mathcal{B}v = (2/(m\mu))\mathcal{B}(\varphi_0) \geq 0$. By Assumption A1, there is a point $z_0 \in \overline{Q}$ such that $v(z_0) < 0$, and there is no neighbourhood U of z_0 such that $\mathcal{L}v > 0$, almost everywhere on $U \cap Q$.

Being $v(z_0) < 0$, there is a neighbourhood V of z_0 such that $u < (\iota_a - \varepsilon)\varphi_1 - (2/(m\mu))\varphi_0$ in $V \cap Q$. By Assumption A6,

$$\begin{aligned} \mathcal{L}v &= \mathcal{L}u - \mathcal{L}\Big(\widetilde{\alpha}_m + \frac{1}{m\mu}\varphi_0\Big) = \lambda_1 u + \frac{3\lambda_1 + 1}{m\mu} - \mathcal{L}\Big(\widetilde{\alpha}_m + \frac{1}{m\mu}\varphi_0\Big) \\ &\geq \lambda_1 \widetilde{\alpha}_m + \frac{3\lambda_1 + 1}{m\mu} - \mathcal{L}\Big(\widetilde{\alpha}_m + \frac{1}{m\mu}\varphi_0\Big) = \frac{3\lambda_1 + 1}{m\mu} - \frac{3\lambda_1}{m\mu}\varphi_0 + \frac{2}{m\mu}\mathcal{L}(\varphi_0) > 0, \end{aligned}$$

almost everywhere in $V \cap Q$, a contradiction.

In the same way we can show that

$$\widetilde{\beta}_m \text{ is a strict upper solution.}$$

The pair $(\widetilde{\alpha}_m, \widetilde{\beta}_m)$ is degree-admissible for $(\widetilde{\mathrm{P}}_m)$: by (4.6), (4.7), and Assumption A3 (or A2), since $W(Q)$ is continuously imbedded in $C^\sharp(\overline{Q})$, there is a constant $R > 0$ such that $\|u\|_{C^\sharp} < R$ for every solution u of $(\widetilde{\mathrm{P}}_m)$ with $\widetilde{\alpha}_m \leq u \leq \widetilde{\beta}_m$. Since $\widetilde{\alpha}_m$ and $\widetilde{\beta}_m$ are strict, for such a solution we also have that $\widetilde{\alpha}_m \ll u \ll \widetilde{\beta}_m$.

The following Claim will give us a solution u_m of $(\widetilde{\mathrm{P}}_m)$ with some localization properties, which will permit us to show, passing to subsequences, that u_m lies in the region where f_m and g_m have not been modified.

Claim. There is a solution u_m of $(\widetilde{\mathrm{P}}_m)$ such that

$$\widetilde{\alpha}_m \ll u_m \ll \widetilde{\beta}_m, \tag{4.9}$$

and

$$u_m \in \overline{\{v \in C^\sharp_{\mathcal{B}}(\overline{Q}) : \alpha_m \not\leq v \text{ and } v \not\leq \beta_m\}}. \tag{4.10}$$

For the proof, we distinguish three cases.

Case 1. The pairs $(\widetilde{\alpha}_m, \beta_m)$ and $(\alpha_m, \widetilde{\beta}_m)$ are degree-admissible for $(\widetilde{\mathrm{P}}_m)$. By Theorem 3.1, we have

$$\begin{aligned} \deg(I - \widetilde{\mathcal{S}}_m, \mathcal{U}_{(\widetilde{\alpha}_m,\beta_m)}) &= 1, \\ \deg(I - \widetilde{\mathcal{S}}_m, \mathcal{U}_{(\alpha_m,\widetilde{\beta}_m)}) &= 1, \\ \deg(I - \widetilde{\mathcal{S}}_m, \mathcal{U}_{(\widetilde{\alpha}_m,\widetilde{\beta}_m)}) &= 1. \end{aligned}$$

Since $\alpha_m \not\leq \beta_m$, the sets $\mathcal{U}_{(\widetilde{\alpha}_m,\beta_m)}$ and $\mathcal{U}_{(\alpha_m,\widetilde{\beta}_m)}$ are disjoint. Moreover, they are both contained in $\mathcal{U}_{(\widetilde{\alpha}_m,\widetilde{\beta}_m)}$ and

$$\deg(I - \widetilde{\mathcal{S}}_m,\mathcal{U}_{(\widetilde{\alpha}_m,\widetilde{\beta}_m)}) \neq \deg(I - \widetilde{\mathcal{S}}_m, \mathcal{U}_{(\widetilde{\alpha}_m,\beta_m)}) + \deg(I - \widetilde{\mathcal{S}}_m, \mathcal{U}_{(\alpha_m,\widetilde{\beta}_m)}).$$

By the additivity property of the degree, there is a solution u_m of $(\widetilde{\mathrm{P}}_m)$ such that $u_m \in \mathcal{U}_{(\widetilde{\alpha}_m,\widetilde{\beta}_m)}$ and

$$u_m \notin \overline{\mathcal{U}}_{(\widetilde{\alpha}_m,\beta_m)} \cup \overline{\mathcal{U}}_{(\alpha_m,\widetilde{\beta}_m)}.$$

Since $\widetilde{\alpha}_m$ and $\widetilde{\beta}_m$ are strict, we have (4.9). Let us see that, in this case, $\alpha_m \not\leq u_m$ and $u_m \not\leq \beta_m$. By contradiction, assume $\alpha_m \leq u_m$. Let $(v_{m,k})_k$ be the sequence in $C^\sharp_{\mathcal{B}}(\overline{Q})$ defined by

$$v_{m,k} = u_m + \frac{1}{k}\varphi_1.$$

By Assumption A5, $v_{m,k} \gg \alpha_m$. Since $u_m \ll \widetilde{\beta}_m$ and, by Assumption A4 the set $\{v \in C^\sharp_{\mathcal{B}}(\overline{Q}) : v \ll \widetilde{\beta}_m\}$ is open in $C^\sharp_{\mathcal{B}}(\overline{Q})$, for k large enough we have that $v_{m,k} \ll \widetilde{\beta}_m$. Therefore, $v_{m,k} \in \mathcal{U}_{(\alpha_m,\widetilde{\beta}_m)}$. Since $v_{m,k} \to u_m$ in $C^\sharp(\overline{Q})$, it has to be $u_m \in \overline{\mathcal{U}}_{(\alpha_m,\widetilde{\beta}_m)}$, a contradiction. In the same way one proves that $u_m \not\leq \beta_m$.

Case 2. The pair $(\widetilde{\alpha}_m, \beta_m)$ is not degree-admissible for $(\widetilde{\mathrm{P}}_m)$. Then, for every $R > 0$ there is a solution u_m of $(\widetilde{\mathrm{P}}_m)$ with

$$\widetilde{\alpha}_m \leq u_m \leq \beta_m, \tag{4.11}$$

such that, either $\widetilde{\alpha}_m \not\ll u_m$, or $u_m \not\ll \beta_m$, or $\|u_m\|_{C^\sharp} \geq R$. By (4.6), (4.7) and Assumption A3 (or A2), we see that, for R large enough, the third possibility is excluded. Recalling that $\widetilde{\alpha}_m$ is strict, we have that $\widetilde{\alpha}_m \ll u_m$, and we deduce that it has to be $u_m \not\ll \beta_m$. On the other hand, it cannot be $\alpha_m \leq u_m$, because this would imply that $\alpha_m \leq \beta_m$. Moreover, by (4.8) and the fact that $\widetilde{\alpha}_m$ is strict, (4.11) implies that (4.9) holds.

Considering again the sequence $v_{m,k} = u_m + (1/k)\varphi_1$, since $\alpha_m \not\leq u_m$, for k large enough we have that $\alpha_m \not\leq v_{m,k}$. Let us see that $v_{m,k} \not\leq \beta_m$. By contradiction, assume $v_{m,k} \leq \beta_m$. Then, $\beta_m - u_m \geq (1/k)\varphi_1 \gg 0$, so that $u_m \ll \beta_m$, a contradiction. Hence, for k large enough, $v_{m,k} \in \{v \in C^\sharp_{\mathcal{B}}(\overline{Q}) : \alpha_m \not\leq v \text{ and } v \not\leq \beta_m\}$, so that, since $v_{m,k} \to u_m$ in $C^\sharp(\overline{Q})$, we have (4.10).

Case 3. The pair $(\alpha_m, \widetilde{\beta}_m)$ is not degree-admissible for $(\widetilde{\mathrm{P}}_m)$. This case is analogous to Case 2. The proof of the Claim is thus completed.

Let $(u_m)_m$ be the sequence provided by the above Claim. Recall that, as seen in Remark 4.5, (4.10) implies

$$\alpha_m \not\ll u_m \quad \text{and} \quad u_m \not\ll \beta_m. \tag{4.12}$$

By a compactness argument, we will now show that, for a subsequence, u_m is indeed a solution of (P_m) and belongs to $I\varphi_1$. Having proved (4.10), this will give us the required contradiction.

By (4.6), (4.7), (4.9), and Assumption A3, there is a constant $C > 0$ such that

$$\|u_m\|_W \leq C, \quad \text{for every } m.$$

Being $W(Q)$ compactly imbedded in $C^\sharp(\overline{Q})$, there are a subsequence, still denoted by $(u_m)_m$, and a function $\overline{u} \in C^\sharp_{\mathcal{B}}(\overline{Q})$ such that $u_m \to \overline{u}$ in $C^\sharp(\overline{Q})$. In particular, for every $z \in \overline{Q}$ we have $\|\nabla_\sharp u_m(z)\| \leq \overline{c}$, for some constant $\overline{c} > 0$. Hence, by (4.7),

$$|\widetilde{f}_m(z, u_m(z), \nabla_\sharp u_m(z))| \leq h_m(z) + \frac{1}{m}\mathcal{G}(\overline{c}),$$

for some $h_m \in L^r(Q)$ such that

$$\|h_m\|_{L^r} \leq \frac{3\lambda_1 + 1}{m\mu}(1 + |Q|^{1/r}),$$

so that

$$\widetilde{f}_m(\,\cdot\,, u_m(\,\cdot\,), \nabla_\sharp u_m(\,\cdot\,)) \to 0 \quad \text{in } L^r(Q).$$

Moreover, by (4.6), there is a $\overline{q} \in L^r(Q)$ such that, for a subsequence,

$$\widetilde{g}_m(\,\cdot\,, u_m(\,\cdot\,), \nabla_\sharp u_m(\,\cdot\,)) \rightharpoonup \overline{q}(\,\cdot\,) \quad \text{(weakly) in } L^r(Q). \tag{4.13}$$

Then, by compactness,

$$\widetilde{\mathcal{S}}_m u_m = (L - \sigma I)^{-1}(\widetilde{N}_m u_m - \sigma u_m) \to (L - \sigma I)^{-1}(\overline{q}(\,\cdot\,)\overline{u} - \sigma\overline{u}) \quad \text{in } C^\sharp(\overline{Q}).$$

As $u_m = \widetilde{\mathcal{S}}_m u_m$, we get $\overline{u} = (L - \sigma I)^{-1}(\overline{q}(\,\cdot\,)\overline{u} - \sigma\overline{u})$, i.e. $\overline{u} \in W_{\mathcal{B}}(Q)$ and $L\overline{u} = \overline{q}(\,\cdot\,)\overline{u}$. Since the set of functions

$$\{p \in L^r(Q) : \psi_1(z) \leq p(z) \leq \psi_2(z), \quad \text{for a.e. } z \in Q\}$$

is closed and convex, it is weakly closed, so that, by (4.6) and (4.13),

$$\psi_1(z) \leq \overline{q}(z) \leq \psi_2(z), \quad \text{for a.e. } z \in Q.$$

Recalling that the pair (ψ_1, ψ_2) is admissible, it has to be either $\overline{u} = 0$, or $\overline{u} \gg 0$, or $\overline{u} \ll 0$. We want to show that

$$\iota_a \varphi_1 \leq \overline{u} \leq \kappa_b \varphi_1. \tag{4.14}$$

We consider three different cases.

Case 1. $0 \leq a < b$.

Assume by contradiction that $\overline{u} \not\leq \kappa_b \varphi_1$. Then, $\overline{u} \gg 0$ and, by Lemma 4.3, we have that $\overline{u} \leq C_{\psi_1,\psi_2} \|\overline{u}\|_{L^\infty} \varphi_1$. Being $\overline{u} \not\leq \kappa_b \varphi_1$, it has to be $C_{\psi_1,\psi_2} \|\overline{u}\|_{L^\infty} > \kappa_b$. Again by the same lemma and Assumption A5,

$$\overline{u} \geq c_{\psi_1,\psi_2} \|\overline{u}\|_{L^\infty} \varphi_1 \gg \frac{c_{\psi_1,\psi_2}}{C_{\psi_1,\psi_2}} \kappa_b \varphi_1 = b\varphi_1.$$

Hence, $\overline{u} \gg b\varphi_1$ so that, since $u_m \to \overline{u}$ in $C^\sharp(\overline{Q})$, by Assumption A4, $u_m \gg b\varphi_1$, for m sufficiently large. Since $\alpha_m \leq b\varphi_1$, we get $\alpha_m \ll u_m$, in contradiction with (4.12).

Assume now by contradiction that $\iota_a \varphi_1 \not\leq \overline{u}$. If $a = 0$, then $\iota_a = 0$ and $\overline{u} \not\geq 0$, so that $\overline{u} \ll 0$. By Assumption A4, $u_m \ll 0$ for m large enough, and since $\beta_m \geq 0$, we have $u_m \ll \beta_m$, in contradiction with (4.12). If $a > 0$, we get a similar contradiction if $\overline{u} \leq 0$. Hence, it has to be $\overline{u} \gg 0$. By Lemma 4.3, since $\iota_a \varphi_1 \not\leq \overline{u}$, we have that $c_{\psi_1,\psi_2} \|\overline{u}\|_{L^\infty} < \iota_a$. Again by the same lemma and Assumption A5,

$$\overline{u} \leq C_{\psi_1,\psi_2} \|\overline{u}\|_{L^\infty} \varphi_1 \ll \frac{C_{\psi_1,\psi_2}}{c_{\psi_1,\psi_2}} \iota_a \varphi_1 = a\varphi_1.$$

Hence, $\overline{u} \ll a\varphi_1$ so that, by Assumption A4, $u_m \ll a\varphi_1$, for m sufficiently large. Since $\beta_m \geq a\varphi_1$, we get $u_m \ll \beta_m$, in contradiction with (4.12).

Case 2. $a < 0 < b$.

In this case, $\iota_a \leq a < 0 < b \leq \kappa_b$. The inequality $\overline{u} \leq \kappa_b \varphi_1$ is proved as in Case 1. Assume by contradiction that $\iota_a \varphi_1 \not\leq \overline{u}$. Then, it has to be $\overline{u} \ll 0$. By Lemma 4.3, since $\iota_a \varphi_1 \not\leq \overline{u}$, we have that $-C_{\psi_1,\psi_2} \|\overline{u}\|_{L^\infty} < \iota_a$. Again by the same lemma and Assumption A5,

$$\overline{u} \leq -c_{\psi_1,\psi_2} \|\overline{u}\|_{L^\infty} \varphi_1 \ll \frac{c_{\psi_1,\psi_2}}{C_{\psi_1,\psi_2}} \iota_a \varphi_1 = a\varphi_1.$$

Hence, $\overline{u} \ll a\varphi_1$ so that, by Assumption A4, $u_m \ll a\varphi_1$, for m sufficiently large. Since $\beta_m \geq a\varphi_1$, we get $u_m \ll \beta_m$, in contradiction with (4.12).

Case 3. $a < b \leq 0$.

This case is the symmetrical of Case 1, and its proof is completely analogous.

By (4.14) and Assumption A5, we have

$$(\iota_a - \varepsilon)\varphi_1 \ll \overline{u} \ll (\kappa_b + \varepsilon)\varphi_1.$$

Since $u_m \to \overline{u}$ in $C^\sharp(\overline{Q})$, by Assumption A4 we have that, for m sufficiently large,

$$(\iota_a - \varepsilon)\varphi_1 \ll u_m \ll (\kappa_b + \varepsilon)\varphi_1.$$

So, u_m is a solution to problem (P_m). By the choice of ε, we have that $u_m \in I\varphi_1$, and since (4.10) holds, we get a contradiction, which ends the proof. □

Remark 4.6. We emphasize the fact that, in Theorem 4.4, the choice of the constant Λ is made uniformly for a whole class of functions and lower and upper solutions. This point has been investigated more carefully in [8].

5. Some consequences, in the abstract setting

In this section, we consider again the problem

$$\text{(P)} \qquad \begin{cases} \mathcal{L}u = g(z,u,\nabla_\sharp u)\,u + f(z,u,\nabla_\sharp u) & \text{in } Q, \\ \mathcal{B}u = 0 & \text{on } \partial Q. \end{cases}$$

The first corollary of this section is an improved version of Corollary 1.3. As we said in the Introduction, it generalizes a series of results involving non-well-ordered lower and upper solutions, cf. [3], [5], [7], [11]–[13].

Corollary 5.1. *Let Assumptions* A1–A6 *hold true. Let* (ψ_1,ψ_2) *be an admissible pair of functions, assume that the function* g *verifies*

$$\psi_1(z) \le g(z,u,\xi) \le \psi_2(z), \quad \text{for a.e. } z \in Q \text{ and every } (u,\xi) \in \mathbb{R} \times \mathbb{R}^\sharp,$$

and the function f *is* L^r*-bounded: there is an* $h \in L^r(Q)$ *such that*

$$|f(z,u,\xi)| \le h(z), \quad \text{for a.e. } z \in Q \text{ and every } (u,\xi) \in \mathbb{R} \times \mathbb{R}^\sharp.$$

If (P) *has a lower solution* α *and an upper solution* β*, then* (P) *has a solution* u*. Moreover, if* $\alpha \not\le \beta$*, then*

$$u \in \overline{\{v \in C^\sharp_{\mathcal{B}}(\overline{Q}) : \alpha \not\le v \text{ and } v \not\le \beta\}}.$$

Proof. Consider the change of variable $w = \lambda u$, with $\lambda \in \,]0,1]$ a small positive number. Then, (P) becomes

$$(\widehat{\mathrm{P}}_\lambda) \qquad \begin{cases} \mathcal{L}w = \widehat{g}_\lambda(z,w,\nabla_\sharp w)\,w + \widehat{f}_\lambda(z,w,\nabla_\sharp w) & \text{in } Q, \\ \mathcal{B}w = 0 & \text{on } \partial Q, \end{cases}$$

with

$$\widehat{g}_\lambda(z,w,\xi) = g\left(z,\frac{w}{\lambda},\frac{\xi}{\lambda}\right) \quad \text{and} \quad \widehat{f}_\lambda(z,w,\xi) = \lambda f\left(z,\frac{w}{\lambda},\frac{\xi}{\lambda}\right).$$

The function $\widehat{g}_\lambda$ verifies

$$\psi_1(z) \le \widehat{g}_\lambda(z,w,\xi) \le \psi_2(z), \quad \text{for a.e. } z \in Q \text{ and every } (w,\xi) \in \mathbb{R} \times \mathbb{R}^\sharp,$$

while $\widehat{f}_\lambda \in \mathcal{F}(\mathbb{R}, \lambda\|h\|_{L^r}, 0)$. We have that $\widehat{\alpha}_\lambda = \lambda\alpha$ is a lower solution and $\widehat{\beta}_\lambda = \lambda\beta$ is an upper solution for $(\widehat{\mathrm{P}}_\lambda)$. By Lemma 4.1, there are a constant $C_\alpha \geq 0$ for which $\alpha \leq C_\alpha\varphi_1$, and a constant $C_\beta \leq 0$ for which $\beta \geq C_\beta\varphi_1$. Notice that

$$C_\beta\varphi_1 - \lambda\|\alpha\|_{L^\infty} \leq \widehat{\alpha}_\lambda \leq C_\alpha\varphi_1, \quad C_\beta\varphi_1 \leq \widehat{\beta}_\lambda \leq C_\alpha\varphi_1 + \lambda\|\alpha\|_{L^\infty},$$

for every $\lambda \in\,]0,1]$. Taking $a = C_\beta$, and $b = C_\alpha$, we can apply Theorem 4.4 and we find, for λ sufficiently small, that $(\widehat{\mathrm{P}}_\lambda)$ has a solution w_λ. Moreover, if $\alpha \not\leq \beta$, then $\widehat{\alpha}_\lambda \not\leq \widehat{\beta}_\lambda$ for every $\lambda \in\,]0,1]$, hence

$$w_\lambda \in \overline{\{v \in C^\sharp_{\mathcal{B}}(\overline{Q}) : \widehat{\alpha}_\lambda \not\leq v \text{ and } v \not\leq \widehat{\beta}_\lambda\}}.$$

The proof is then easily concluded. □

As a direct consequence of Theorem 4.4, we have the following corollary, which is related to some co-bifurcation theorems, cf. [9], [10].

COROLLARY 5.2. *Let Assumptions* A1–A6 *hold true. Let $I \subseteq \mathbb{R}$ be an open interval, and Λ, K be some fixed positive numbers. Given a compact interval J, contained in I, let $\mathcal{I}$ be an open interval containing $J\varphi_1(\overline{Q})$. Then, there is a $\overline{\lambda} > 0$ with the following property: for every $\lambda \in [0, \overline{\lambda}]$ and every $f \in \mathcal{F}(\mathcal{I}, \Lambda, K)$, if there are two constants a_1, b_1 in J for which*

$$f(\,\cdot\,, b_1\varphi_1(\,\cdot\,), b_1\nabla_\sharp\varphi_1(\,\cdot\,)) \leq 0 \leq f(\,\cdot\,, a_1\varphi_1(\,\cdot\,), a_1\nabla_\sharp\varphi_1(\,\cdot\,)) \quad \textit{a.e. in } Q,$$

then the problem

$$\begin{cases} \mathcal{L}u = \lambda_1 u + \lambda f(z, u, \nabla_\sharp u) & \textit{in } Q, \\ \mathcal{B}u = 0 & \textit{on } \partial Q, \end{cases}$$

has a solution $u \in I\varphi_1$.

PROOF. In this case, we have $\psi_1 = \psi_2 = \lambda_1$, so that $c_{\psi_1,\psi_2} = C_{\psi_1,\psi_2} = 1$. Writing $J = [a, b]$, we have that $\iota_a = a$ and $\kappa_b = b$. Hence, Theorem 4.4 applies by taking $\alpha = a_1\varphi_1$ and $\beta = b_1\varphi_1$. □

We now give two examples of applications of Corollary 5.2.

Let $\eta \in L^\infty(Q)$ be a function bounded below by a positive constant. Given $\mathrm{p} > 1$, consider the problem

$$\begin{cases} \mathcal{L}u = \lambda_1 u + \eta(z)|u|^{\mathrm{p}-1}u + e(z) & \text{in } Q, \\ \mathcal{B}u = 0 & \text{on } \partial Q, \end{cases} \tag{5.1}$$

Corollary 5.3. *Let Assumptions* A1–A6 *hold true. For every given* $\delta > 0$ *there is a* $\overline{\Lambda}_\delta > 0$ *such that, for every* $\mathrm{p} > 1$*, if*

$$|e(z)| \le \left(\frac{\overline{\Lambda}_\delta}{(1+\delta)^{\mathrm{p}}}\right)^{\mathrm{p}/(\mathrm{p}-1)} \varphi_1(z)^{\mathrm{p}} \quad \text{for a.e. } z \in Q, \tag{5.2}$$

then (5.1) *has a solution.*

Proof. Fix $\delta > 0$ and consider the problem

$$\begin{cases} \mathcal{L}w = \lambda_1 w + \lambda(\eta(z)|w|^{\mathrm{p}-1}w + \widetilde{e}(z)) & \text{in } Q, \\ \mathcal{B}w = 0 & \text{on } \partial Q, \end{cases} \tag{5.3}$$

where λ is a positive constant and $\widetilde{e}$ is such that

$$|\widetilde{e}(z)| \le (\operatorname{ess\,inf} \eta)\, \varphi_1(z)^{\mathrm{p}} \quad \text{for a.e. } z \in Q. \tag{5.4}$$

Take $J = [-1, 1]$ and $I = \,]-1-\delta, 1+\delta[$. Notice that, in this case, $J\varphi_1(\overline{Q}) = J$, so that we can take $\mathcal{I} = I$. Setting $f(z, w) = \eta(z)|w|^{\mathrm{p}-1}w + \widetilde{e}(z)$, we have

$$f(z, -\varphi_1(z)) \le 0 \le f(z, \varphi_1(z)),$$

for almost every $z \in Q$ and, for every $w \in I$,

$$|f(z, w)| \le \|\eta\|_{L^\infty}(1+\delta)^{\mathrm{p}} + \operatorname{ess\,inf} \eta \le 2\lambda\|\eta\|_{L^\infty}(1+\delta)^{\mathrm{p}}. \tag{5.5}$$

Take $\alpha = \varphi_1$, $\beta = -\varphi_1$, $r = +\infty$ and let $\overline{\lambda} = \overline{\lambda}_\delta > 0$ be as in the statement of Corollary 5.2. Hence, if $\lambda \in [0, \overline{\lambda}_\delta]$, then (5.3) has a solution $w \in I\varphi_1$. Set

$$\overline{\Lambda}_\delta = \overline{\lambda}_\delta(1+\delta)^{\mathrm{p}} \min\{\operatorname{ess\,inf} \eta, 1\}. \tag{5.6}$$

Consider now (5.1), and assume (5.2) with $\overline{\Lambda}_\delta > 0$ as above.

Setting $w = \overline{\lambda}_\delta^{1/(1-\mathrm{p})} u$, we see that (5.1) is equivalent to (5.3), with $\lambda = \overline{\lambda}_\delta$ and $\widetilde{e}(z) = \overline{\lambda}_\delta^{\mathrm{p}/(1-\mathrm{p})} e(z)$. Since, using (5.2), (5.6),

$$\begin{aligned} |\widetilde{e}(z)| = \overline{\lambda}_\delta^{\mathrm{p}/(1-\mathrm{p})}|e(z)| &\le \overline{\lambda}_\delta^{\mathrm{p}/(1-\mathrm{p})} \left(\frac{\overline{\Lambda}_\delta}{(1+\delta)^{\mathrm{p}}}\right)^{\mathrm{p}/(\mathrm{p}-1)} \varphi_1(z)^{\mathrm{p}} \\ &= \min\{\operatorname{ess\,inf} \eta, 1\}^{\mathrm{p}/(\mathrm{p}-1)} \varphi_1(z)^{\mathrm{p}} \le (\operatorname{ess\,inf} \eta)\, \varphi_1(z)^{\mathrm{p}}, \end{aligned}$$

we have that (5.4) holds, so that (5.3) is solvable. Therefore, (5.1) is solvable, as well. □

Remark 5.4. Notice that, taking $\delta = 1/2$, condition (5.2) is satisfied if p is sufficiently large and

$$|e(z)| \le \left(\frac{1}{2}\varphi_1(z)\right)^{\mathrm{p}} \quad \text{for a.e. } z \in Q.$$

In a similar way, we have the following result, where we use the notation

$$u^+ = \max\{u, 0\}.$$

Let $\eta \in L^\infty(Q)$ be a function bounded below by a positive constant. Given $\mathrm{p} > 1$, consider the problem

$$\begin{cases} \mathcal{L}u = \lambda_1 u + \eta(z)(u^+)^{\mathrm{p}} + e(z) & \text{in } Q, \\ \mathcal{B}u = 0 & \text{on } \partial Q, \end{cases} \tag{5.7}$$

Corollary 5.5. *Let Assumptions* A1–A6 *hold true. For every given* $\delta > 0$ *there is a* $\overline{\Lambda}_\delta > 0$ *such that, for every* $\mathrm{p} > 1$, *if* $e(z) \le 0$ *for almost every* $z \in Q$ *and* (5.2) *holds, then* (5.7) *has a solution.*

Proof. As in the proof of Corollary 5.3, first consider the problem

$$\begin{cases} \mathcal{L}w = \lambda_1 w + \lambda(\eta(z)(w^+)^{\mathrm{p}} + \widetilde{e}(z)) & \text{in } Q, \\ \mathcal{B}w = 0 & \text{on } \partial Q, \end{cases}$$

where $\widetilde{e}$ is such that

$$-(\operatorname{ess\,inf} \eta)\, \varphi_1(z)^{\mathrm{p}} \le \widetilde{e}(z) \le 0 \quad \text{for a.e. } z \in Q.$$

Once $\delta > 0$ is fixed, take $J = [0,1]$ and $I = \,]-\delta, 1+\delta[$. Notice that, even in this case, $J\varphi_1(\overline{Q}) = J$, so that we can take $\mathcal{I} = I$. Setting $f(z,w) = \eta(z)(w^+)^{\mathrm{p}} + \widetilde{e}(z)$, we have

$$f(z,0) \le 0 \le f(z, \varphi_1(z)),$$

and (5.5) holds, for almost every $z \in Q$ and every $w \in I$. Taking $\alpha = \varphi_1$ and $\beta = 0$, one then concludes as in the proof of Corollary 5.3. □

Let us now introduce a further assumption.

Assumption A7. There is a number $\lambda_2 > \lambda_1$ with the following property. If $q \in L^r(Q)$ satisfies

$$\lambda_1 \le q(z) \le \lambda_2 \quad \text{for a.e. } z \in Q,$$

each of the inequalities being strict on a subset of positive measure, then the problem

$$\begin{cases} \mathcal{L}u = q(z)\, u & \text{in } Q, \\ \mathcal{B}u = 0 & \text{on } \partial Q, \end{cases}$$

only has the zero solution.

The following corollary is the abstract version of Theorem 1.1.

Corollary 5.6. *Let Assumptions* A1–A7 *hold true. Let* $I \subseteq \mathbb{R}$ *be an open interval, and* $\zeta \in L^r(Q)$ *be a function such that*

$$\lambda_1 \le \zeta(z) \le \lambda_2 \quad \textit{for a.e. } z \in Q,$$

the second inequality being strict on a subset of positive measure. Given a compact interval $[a,b]$*, contained in* I*, let* $\mathcal{I}$ *be an open interval containing* $[a,b]\varphi_1(\overline{Q})$*. There is a constant* $\Lambda > 0$ *such that, for every function* g *satisfying*

$$\lambda_1 \le g(z,u,\xi) \le \zeta(z), \quad \textit{for a.e. } z \in Q \textit{ and every } (u,\xi) \in \mathcal{I} \times \mathbb{R}^\sharp,$$

and every function $f \in \mathcal{F}(\mathcal{I},\Lambda,\Lambda)$*, if there are a lower solution* α *and an upper solution* β *of* (P) *verifying*

$$a\varphi_1 - \Lambda \le \alpha \le b\varphi_1, \qquad a\varphi_1 \le \beta \le b\varphi_1 + \Lambda,$$

then problem (P) *has a solution* $u \in \mathcal{I}\varphi_1$*. Moreover, if* $\alpha \not\le \beta$*, then*

$$u \in \overline{\{v \in C^\sharp_{\mathcal{B}}(\overline{Q}) : \alpha \not\le v \textit{ and } v \not\le \beta\}}.$$

Proof. Take $\psi_1(z) = \lambda_1$ and $\psi_2(z) = \zeta(z)$. If $q \in L^r(\overline{Q})$ satisfies $\lambda_1 \le q \le \zeta$, then a solution of (4.1) is either identically zero, or, if q is equal to λ_1, it is a multiple of φ_1. Hence, the pair (ψ_1,ψ_2) is admissible, and we have $c_{\psi_1,\psi_2} = C_{\psi_1,\psi_2} = 1$, so that $\iota_a = a$ and $\kappa_b = b$. The result then follows directly from Theorem 4.4. □

6. Asymmetric nonlinearities

In this section the considered problem will be written as

$$\text{(P)} \qquad \begin{cases} \mathcal{L}u = g_+(z,u,\nabla_\sharp u)\, u^+ - g_-(z,u,\nabla_\sharp u)\, u^- + f(z,u,\nabla_\sharp u) & \text{in } Q, \\ \mathcal{B}u = 0 & \text{on } \partial Q, \end{cases}$$

where $f, g_+, g_- : Q \times \mathbb{R} \times \mathbb{R}^\sharp \to \mathbb{R}$ are L^r-Carathéodory functions, $u^+ = \max\{u,0\}$ and $u^- = \max\{-u,0\}$. We will only briefly discuss the main differences with respect to the results obtained in Section 4.

Definition 6.1. A quadruple of functions $(\psi_1,\psi_2,\chi_1,\chi_2) \in (L^r(Q))^4$ is said to be *admissible* if it satisfies $\psi_1 \le \lambda_1 \le \psi_2$, $\chi_1 \le \lambda_1 \le \chi_2$ almost everywhere in Q and, for every $q_+, q_- \in L^r(Q)$, with $\psi_1 \le q_+ \le \psi_2$, $\chi_1 \le q_- \le \chi_2$ almost everywhere in Q, if u is a solution of

$$\text{(6.1)} \qquad \begin{cases} \mathcal{L}u = q_+(z)\, u^+ - q_-(z)\, u^- & \text{in } Q, \\ \mathcal{B}u = 0 & \text{on } \partial Q, \end{cases}$$

then, either $u = 0$, or $u \ll 0$, or $u \gg 0$.

If Assumption A5 holds, the set of admissible quadruples is not empty, since it contains the quadruple of constant functions $(\lambda_1,\lambda_1,\lambda_1,\lambda_1)$. In the applications, one usually relates the admissibility of the quadruple $(\psi_1,\psi_2,\chi_1,\chi_2)$ to the non-interaction with the first curve in the Dancer–Fučik spectrum of the operator L.

LEMMA 6.2. *Let Assumptions* A2, A4 *and* A5 *hold. Given an admissible quadruple* $(\psi_1,\psi_2,\chi_1,\chi_2)$*, there are two positive constants* $c_{\psi_1,\psi_2,\chi_1,\chi_2}$ *and* $C_{\psi_1,\psi_2,\chi_1,\chi_2}$ *such that, for every* $q_+, q_- \in L^r(Q)$*, with* $\psi_1 \le q_+ \le \psi_2$*,* $\chi_1 \le q_- \le \chi_2$ *almost everywhere in* Q*, if* u *is a solution of* (6.1)*, then*

$$c_{\psi_1,\psi_2,\chi_1,\chi_2}\|u\|_{L^\infty}\varphi_1 \le |u| \le C_{\psi_1,\psi_2,\chi_1,\chi_2}\|u\|_{L^\infty}\varphi_1.$$

PROOF. Just follow the lines of the proof of Lemma 4.3. The only difference is the use of the following well-known fact: if $(q_n)_n$ is a sequence in $L^r(Q)$ such that $q_n \rightharpoonup q$ weakly in $L^r(Q)$, and $(u_n)_n$ is a sequence in $C(\overline{Q})$ such that $u_n \to u$ uniformly, then $q_n u_n^+ \rightharpoonup qu^+$ weakly in $L^r(Q)$. □

Let $c_{\psi_1,\psi_2,\chi_1,\chi_2}$ and $C_{\psi_1,\psi_2,\chi_1,\chi_2}$ be the two positive constants given by Lemma 4.3. Given two numbers a, b, set

$$(6.2)\qquad \iota_a = \begin{cases} \dfrac{C_{\psi_1,\psi_2,\chi_1,\chi_2}}{c_{\psi_1,\psi_2,\chi_1,\chi_2}}\,a & \text{if } a \le 0,\\[2mm] \dfrac{c_{\psi_1,\psi_2,\chi_1,\chi_2}}{C_{\psi_1,\psi_2,\chi_1,\chi_2}}\,a & \text{if } a \ge 0,\end{cases} \qquad \kappa_b = \begin{cases} \dfrac{c_{\psi_1,\psi_2,\chi_1,\chi_2}}{C_{\psi_1,\psi_2,\chi_1,\chi_2}}\,b & \text{if } b \le 0,\\[2mm] \dfrac{C_{\psi_1,\psi_2,\chi_1,\chi_2}}{c_{\psi_1,\psi_2,\chi_1,\chi_2}}\,b & \text{if } b \ge 0.\end{cases}$$

We have the following analogue of Theorem 4.4.

THEOREM 6.3. *Let Assumptions* A1–A6 *hold true. Let* $(\psi_1,\psi_2,\chi_1,\chi_2)$ *be an admissible quadruple of functions, and* a, b *be two real numbers, with* $a \le b$*. Let* $I \subseteq \mathbb{R}$ *be an open interval containing* $[\iota_a,\kappa_b]$*, and* $\mathcal{I} \subseteq \mathbb{R}$ *be an open interval containing* $[\iota_a,\kappa_b]\varphi_1(\overline{Q})$*, where* ι_a *and* κ_b *are given by* (6.2)*. There is a constant* $\Lambda > 0$ *such that, for every two functions* g_+*,* g_- *satisfying*

$$\psi_1(z) \le g_+(z,u,\xi) \le \psi_2(z),$$
$$\chi_1(z) \le g_-(z,u,\xi) \le \chi_2(z),$$

for almost every $z \in Q$*, every* $(u,\xi) \in \mathcal{I}\times\mathbb{R}^\sharp$*, and every function* $f \in \mathcal{F}(\mathcal{I},\Lambda,\Lambda)$*, if there are a lower solution* α *and an upper solution* β *of* (P) *verifying*

$$a\varphi_1 - \Lambda \le \alpha \le b\varphi_1, \qquad a\varphi_1 \le \beta \le b\varphi_1 + \Lambda,$$

then problem (P) *has a solution* $u \in I\varphi_1$*. Moreover, if* $\alpha \not\le \beta$*, then*

$$u \in \overline{\{v \in C^\sharp_{\mathcal{B}}(\overline{Q}) : \alpha \not\le v \text{ and } v \not\le \beta\}}.$$

PROOF. We follow the lines of the proof of Theorem 4.4. Let $\mu := \min\varphi_0$, and fix $\varepsilon > 0$ such that $[\iota_a - \varepsilon,\ \kappa_b + \varepsilon] \subseteq I$ and $[\iota_a - \varepsilon,\ \kappa_b + \varepsilon]\varphi_1(\overline{Q}) \subseteq \mathcal{I}$. Arguing by contradiction, we modify the functions $f_m(z,u,\xi)$, $g_{+,m}(z,u,\xi)$, $g_{-,m}(z,u,\xi)$ outside the set

$$\Big\{(z,u,\xi) : (\iota_a - \varepsilon)\varphi_1(z) - \frac{1}{m\mu}\varphi_0(z) \le u \le (\kappa_b + \varepsilon)\varphi_1(z) + \frac{1}{m\mu}\varphi_0(z)\Big\},$$

so that both $g_{+,m}$ and $g_{-,m}$ will be equal to λ_1 outside the set

$$\left\{(z,u,\xi) : (\iota_a-\varepsilon)\varphi_1(z)-\frac{2}{m\mu}\varphi_0(z)\le u\le(\kappa_b+\varepsilon)\varphi_1(z)+\frac{2}{m\mu}\varphi_0(z)\right\}.$$

We thus obtain the new functions $\widetilde f_m(z,u,\xi)$, $\widetilde g_{+,m}(z,u,\xi)$, $\widetilde g_{-,m}(z,u,\xi)$, and consider the modified problem $(\widetilde{\mathrm{P}}_m)$:

$$\begin{cases} \mathcal{L}u=\widetilde g_{m,+}(z,u,\nabla_\sharp u)\,u^+-\widetilde g_{m,-}(z,u,\nabla_\sharp u)\,u^-+\widetilde f_m(z,u,\nabla_\sharp u) & \text{in } Q,\\ \mathcal{B}u=0 & \text{on } \partial Q.\end{cases}$$

Defining the functions

$$\widetilde\alpha_m=(\iota_a-\varepsilon)\varphi_1-\frac{3}{m\mu}\varphi_0,\qquad \widetilde\beta_m=(\kappa_b+\varepsilon)\varphi_1+\frac{3}{m\mu}\varphi_0,$$

we can prove that $\widetilde\alpha_m$ is a strict lower solution, and $\widetilde\beta_m$ is a strict upper solution, and (4.8) holds. We then prove the following claim.

Claim. There is a solution u_m of $(\widetilde{\mathrm{P}}_m)$ such that

$$\widetilde\alpha_m\ll u_m\ll\widetilde\beta_m, \tag{6.3}$$

and

$$u_m\in\overline{\{v\in C^\sharp_{\mathcal{B}}(\overline{Q}):\alpha_m\not\le v \text{ and } v\not\le\beta_m\}}. \tag{6.4}$$

Let $(u_m)_m$ be the sequence provided by the above Claim. There are a subsequence, still denoted by $(u_m)_m$, and a function $\overline u\in C^\sharp_{\mathcal{B}}(\overline{Q})$ such that $u_m\to\overline u$ in $C^\sharp(\overline{Q})$. Moreover,

$$\widetilde f_m(\,\cdot\,,u_m(\,\cdot\,),\nabla_\sharp u_m(\,\cdot\,))\to 0\quad\text{in } L^r(Q),$$

and, for some subsequence,

$$\begin{aligned}\widetilde g_{m,+}(\,\cdot\,,u_m(\,\cdot\,),\nabla_\sharp u_m(\,\cdot\,))&\rightharpoonup\overline q_+(\,\cdot\,)\quad\text{(weakly) in } L^r(Q),\\ \widetilde g_{m,-}(\,\cdot\,,u_m(\,\cdot\,),\nabla_\sharp u_m(\,\cdot\,))&\rightharpoonup\overline q_-(\,\cdot\,)\quad\text{(weakly) in } L^r(Q).\end{aligned}$$

By a standard argument, we can conclude that

$$L\overline u=\overline q_+(\,\cdot\,)\overline u^+-\overline q_-(\,\cdot\,)\overline u^-,$$

and

$$\psi_1(z)\le\overline q_+(z)\le\psi_2(z),\qquad \chi_1(z)\le\overline q_-(z)\le\chi_2(z),$$

for almost every $z\in Q$. We can then prove that $\iota_a\varphi_1\le\overline u\le\kappa_b\varphi_1$.

By Assumption A5, we have $(\iota_a-\varepsilon)\varphi_1\ll\overline u\ll(\kappa_b+\varepsilon)\varphi_1$. Since $u_m\to\overline u$ in $C^\sharp(\overline{Q})$, by Assumption A4 we have that, for m sufficiently large,

$$(\iota_a-\varepsilon)\varphi_1\ll u_m\ll(\kappa_b+\varepsilon)\varphi_1,$$

so that u_m is a solution to problem (P_m). By the choice of ε, we have that $u_m \in I\varphi_1$, and since (6.4) holds, we get a contradiction, which ends the proof.□

7. The case $M = 1$. An application to the parabolic equation

Let Ω be a bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\partial\Omega$. Given $T > 0$, set $Q = \Omega \times \,]0,T[$.

Define the elliptic differential operator

$$\mathcal{A}u := -\sum_{i,j=1}^{N} a_{ij}(x,t)\partial^2_{x_ix_j}u + \sum_{i=1}^{N} a_i(x,t)\partial_{x_i}u + a_0(x,t)u.$$

Here $a_{ij} \in C(\overline{Q})$, $a_{ij} = a_{ji}$, $a_{ij}(x,0) = a_{ij}(x,T)$ in $\overline{\Omega}$, for $i,j = 1,\dots,N$, there exists $\overline{a} > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geq \overline{a}\|\xi\|^2, \quad \text{for every } (x,t,\xi) \in \overline{Q} \times \mathbb{R}^N,$$

$a_i \in L^\infty(Q)$, for $i = 0,\dots,N$.

We choose $C^\sharp(\overline{Q}) = C^{1,0}(\overline{Q})$, and $W(Q) = W^{2,1}_r(Q)$, the space of functions u such that

$$u,\ \partial_t u,\ \partial_{x_i}u,\ \partial^2_{x_ix_j}u \in L^r(Q),$$

for $i,j = 1,\dots,N$, with the usual norm

$$\|u\|_W = \|u\|_{L^r} + \|\partial_t u\|_{L^r} + \sum_{i=1}^{N} \|\partial_{x_i}u\|_{L^r} + \sum_{i,j=1}^{N} \|\partial^2_{x_ix_j}u\|_{L^r}.$$

Taking $r > N+2$, we have that $W^{2,1}_r(Q)$ is compactly imbedded into $C^{1,0}(\overline{Q})$.

We define the operator $\mathcal{L}\colon W^{2,1}_r(Q) \to L^r(Q)$ as follows:

$$\mathcal{L}u = \partial_t u + \mathcal{A}u.$$

Assume that $\partial\Omega$ is the disjoint union of two closed sets Γ_1 and Γ_2 (the cases $\Gamma_1 = \emptyset$ or $\Gamma_2 = \emptyset$ are admitted). Let τ_s be the operator defined by

$$(\tau_s u)(x,t) = u(x,t+s),$$

and consider the boundary operator

$$\mathcal{B}u := \begin{cases} u & \text{on } \Gamma_1 \times [0,T], \\ \displaystyle\sum_{i=1}^{N} b_i(x,t)\partial_{x_i}u + b_0(x,t)u & \text{on } \Gamma_2 \times [0,T], \\ u - \tau_T u & \text{in } \Omega \times \{0\}, \\ \tau_{(-T)}u - u & \text{in } \Omega \times \{T\}. \end{cases}$$

Here $b_i \in C^1(\partial\Omega \times [0,T])$, $b_i(x,0) = b_i(x,T)$ in $\partial\Omega$, for $i = 0,\dots,N$, and there exists $\overline{b} > 0$ such that

$$b_0(x,t) \geq 0 \quad \text{and} \quad \sum_{i=1}^{N} b_i(x,t)\nu_i(x) \geq \overline{b}, \quad \text{for every } (x,t) \in \partial\Omega \times]0,T[.$$

The vector $\nu(x) = (\nu_1(x),\dots \nu_N(x))$ is the unit outer normal to Ω at $x \in \partial\Omega$.

In this setting, problem (P) coincides with (1.1). We then have that u is a *solution* of problem (P) if u belongs to $W_r^{2,1}(Q)$, it satisfies the differential equation almost everywhere in Q and the boundary conditions pointwise. A function with these properties is usually called "strong solution" in the literature.

We can assume without loss of generality that

$$a_0(x,t) \geq 0 \quad \text{for a.e. } (x,t) \in Q.$$

Indeed, if it were not so, we could add to both sides of the differential equation the term ρu, with $\rho = \operatorname{essinf} a_0$.

Definition 7.1. Given two functions $u, v \in C^{1,0}(\overline{Q})$, we will write $u \ll v$ if the following two conditions hold:

(a) for every $(x,t) \in \Omega \times [0,T]$, $u(x,t) < v(x,t)$,
(b) for every $(x,t) \in \partial\Omega \times [0,T]$, either $u(x,t) < v(x,t)$, or $u(x,t) = v(x,t)$ and $\partial_\nu u(x,t) > \partial_\nu v(x,t)$.

Here ν denotes the outer unit normal to $\partial\Omega$ at the point (x,t).

We need to check whether our Assumptions A1–A7 are verified.

Assumption A1: Let $v \in W_r^{2,1}(Q)$ be such that

$$m := \min v < 0 \quad \text{and} \quad \mathcal{B}v \geq 0.$$

Let $(x_0,t_0) \in \overline{Q}$ satisfy $v(x_0,t_0) = m$. If $t_0 = 0$, since $v(x_0,0) - v(x_0,T) \geq 0$, we have that $v(x_0,T) = m$, as well. We can then assume without loss of generality that $t_0 > 0$. Assume by contradiction that there is a neighbourhood U of (x_0,t_0) such that $\mathcal{L}v > 0$ a.e. on $U \cap Q$. Restricting to a smaller neighbourhood, if necessary, we can assume that $U = \Omega_1 \times]t_1,t_2[$, where Ω_1 is an open set with a C^2-boundary, and $0 < t_1 < t_0 < t_2$. We distinguish between three different cases.

If x_0 belongs to Ω, we can take Ω_1 contained in Ω, and the Strong Maximum Principle (cf. [7, Proposition I.1.1]) implies that v is constant in $\Omega_1 \times]t_1,t_0]$, which is impossible, since $\mathcal{L}v > 0$ there.

If x_0 belongs to Γ_1, then $v(x_0,t_0) = 0 > m$, contrary to the assumption.

If x_0 belongs to Γ_2, we take an open set Ω_2, with a C^2-boundary, such that $\Omega_2 \subseteq \Omega_1 \cap \Omega$, and $\overline{\Omega}_2 \cap \partial\Omega = \{x_0\}$. Notice that the outer normal ν at x_0 is the same for Ω and for Ω_2. By the Strong Maximum Principle (cf. [7, Proposition I.1.1]), since v is not constant in $\Omega_2\times]t_1,t_0]$, it has to be $\partial_\mu v(x_0,t_0) < 0$ for every $\mu\in\mathbb{R}^N$ such that $\mu\cdot\nu>0$. Let $\mu=(b_1(x_0,t_0),\dots,b_N(x_0,t_0))$, the functions b_i coming from the boundary condition. Then, since $\mathcal{B}v\geq 0$, we would have

$$\partial_\mu v(x_0,t_0)\geq -b_0(x_0,t_0)v(x_0,t_0)\geq 0,$$

a contradiction.

Assumption A2 is verified for every $\sigma<0$, see [7, Proposition I.1.3].

Assumption A3 is a classical estimate, with

$$\mathcal{G}(y)=cy^2,$$

for any constant $c\geq 0$, see [7, Proposition III.1.4].

Assumption A4: The properties (2.2) can be directly verified. By contradiction, let $(v_n)_n$ and v in $C^{1,0}_{\mathcal{B}^+}(\overline{Q})$ be such that $v\gg 0$, $v_n\to v$ in $C^{1,0}(\overline{Q})$ and $v_n\not\gg 0$, for every n. Two cases are possible.

Case 1. There is a sequence $(x_n,t_n)_n$ in Q such that $v_n(x_n,t_n)\leq 0$. Then, there are a subsequence, still denoted by $(x_n,t_n)_n$, and a point $(\overline{x},\overline{t})\in\overline{Q}$ such that $(x_n,t_n)\to(\overline{x},\overline{t})$. Since $v\gg 0$ and $v_n\to v$ in $C^{1,0}(\overline{Q})$, it has to be $v(\overline{x},\overline{t})=0$, and hence $\overline{x}\in\partial\Omega$ and $\partial_\nu v(\overline{x},\overline{t})<0$. We distinguish two subcases.

If $\overline{x}\in\Gamma_1$, we can find a sufficiently small ball $U=B(\overline{x},\rho)$ such that, on $(U\cap\partial\Omega)\times[0,T]$, all u_n vanish. Moreover, since $\partial_\nu v(\overline{x},\overline{t})<0$, if ρ is small enough we will have, for the same direction ν, that $\partial_\nu v_n(x,t)<0$, for every $(x,t)\in U\times]\overline{t}-\rho,\overline{t}+\rho[$. Taking ρ still smaller, if necessary, we have that the segment $\{x_n+s\nu : s\geq 0\}\cap U$ meets the boundary $\partial\Omega$ at a single point x_n'. For n large enough, $t_n\in]\overline{t}-\rho,\overline{t}+\rho[$, so that $v_n(x_n,t_n)\leq 0\leq v_n(x_n',t_n)$, and this is impossible since $\partial_\nu v_n$ is negative on the segment joining x_n with x_n'.

If $\overline{x}\in\Gamma_2$, since $v(\overline{x},\overline{t})=0$ and $\mathcal{B}v\geq 0$, letting $\mu=(b_1(\overline{x},\overline{t}),\dots,b_N(\overline{x},\overline{t}))$, it has to be $\partial_\mu v(\overline{x},\overline{t})\geq 0$. Since $v\gg 0$, recalling that $\mu\cdot\nu>0$, it cannot be that $\partial_\mu v(\overline{x},\overline{t})>0$, so $\partial_\mu v(\overline{x},\overline{t})=0$. Since $\partial_\nu v(\overline{x},\overline{t})<0$, taking $\delta>0$ sufficiently small, we have that $(\mu-\delta\nu)\cdot\nu>0$ and $\partial_{(\mu-\delta\nu)}v(\overline{x},\overline{t})=\partial_\mu v(\overline{x},\overline{t})-\delta\partial_\nu v(\overline{x},\overline{t})>0$, which is impossible, since $v\gg 0$.

Case 2. There is a sequence $(x_n,t_n)_n$ in $\partial\Omega\times[0,T]$ such that, either $v_n(x_n,t_n)<0$, or $v_n(x_n,t_n)=0$, and $\partial_\nu v_n(x_n,t_n)\geq 0$. For a subsequence, $(x_n,t_n)\to(\overline{x},\overline{t})\in\partial\Omega\times[0,T]$. As $v\gg 0$, either $v(\overline{x},\overline{t})>0$, or $v(\overline{x},\overline{t})=0$ and $\partial_\nu v(\overline{x},\overline{t})<0$. Since $v_n(x_n,t_n)\leq 0$, it cannot be that $v(\overline{x},\overline{t})>0$. So, $v(\overline{x},\overline{t})=0$ and $\partial_\nu v(\overline{x},\overline{t})<0$. If $v_n(x_n,t_n)<0$, since $\mathcal{B}v_n\geq 0$, letting $\mu_n=(b_1(x_n,t_n),\dots,b_N(x_n,t_n))$, it has to be $\partial_{\mu_n}v_n(x_n,t_n)\geq 0$. Passing to the limit, $\mu_n\to\mu=(b_1(\overline{x},\overline{t}),\dots,b_N(\overline{x},\overline{t}))$ and $\partial_\mu v(\overline{x},\overline{t})\geq 0$. We met the same situation in Case 1, and we saw that this is not possible. Then, $v_n(x_n,t_n)=0$, and $\partial_\nu v_n(x_n,t_n)\geq 0$. Since $\partial_\nu v(\overline{x},\overline{t})<0$ and $v_n\to v$ in $C^{1,0}(\overline{Q})$, this is also impossible, finishing the proof.

Assumption A5 is rather standard for this kind of problems. Let us sketch its proof. Fix $\sigma < 0$ and consider the operator $S\colon C^{1,0}_{\mathcal{B}}(\overline{Q}) \to C^{1,0}_{\mathcal{B}}(\overline{Q})$ such that $Sy = (L - \sigma I)^{-1} y$. By the compact imbedding of $W^{2,1}_r(Q)$ in $C^{1,0}(\overline{Q})$, we have that S is a compact linear operator. Consider the cone $K = \{u \in C^{1,0}_{\mathcal{B}}(\overline{Q}) : u \geq 0\}$. It can be shown, by an argument similar to the one used to verify Assumption A4, that the interior of K is the set $\{u \in C^{1,0}_{\mathcal{B}}(\overline{Q}) : u \gg 0\}$. By the Strong Maximum Principle, one can see that S is strongly positive with respect to K. The proof then follows from the Krein–Rutman theorem, see [15, Theorem 7C].

Assumption A6 is directly verified, taking as $\varphi_0(x,t)$ the function with constant value 1, by the fact that $a_0(x,t) \geq 0$.

Assumption A7: We follow [7, Proposition I.1.8]. By contradiction, assume that, for every n, there is a $q_n \in L^r(Q)$ such that $\lambda_1 \leq q_n \leq \lambda_1 + 1/n$, each of the inequalities being strict on a subset of positive measure, and a solution u_n of $Lu_n = q_n u_n$, with $\|u_n\|_{L^\infty} = 1$. For a subsequence, $u_n \to \overline{u}$, for some $\overline{u}$ verifying $L\overline{u} = \lambda_1 \overline{u}$. Then, $\|\overline{u}\|_{L^\infty} = 1$, so that either $\overline{u} = \varphi_1$, or $\overline{u} = -\varphi_1$. Assume for instance $\overline{u} = \varphi_1$. Then, $u_n \gg 0$, for n large enough. Let c_n be the minimal constant for which $c_n u_n \geq \varphi_1$, and set $v_n = c_n u_n - \varphi_1$. Then, $\min v_n = 0$, $\mathcal{L} v_n \geq 0$ and $\mathcal{B} v_n = 0$. By the Strong Maximum Principle, it has to be, either $v_n = 0$, or $v_n \gg 0$. This second possibility is excluded by the fact that c_n is minimal. So, $v_n = 0$, which implies $\mathcal{L} v_n = (q_n - \lambda_1)\varphi_1 = 0$, hence $q_n = \lambda_1$, contrary to the assumption.

Having verified that Assumptions A1–A7 are satisfied, we may conclude that Theorem 1.1 follows directly from Corollary 5.6, and Corollary 1.3 from Corollary 5.1.

8. The case $M = 0$. An application to the elliptic equation

Let $Q = \Omega$ be a bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\partial\Omega$. In this section we use the standard notation for the Sobolev space $W^{2,r}(\Omega)$, which should not be confused with the notations used in the previous section. We choose $C^\sharp(\overline{Q}) = C^1(\overline{\Omega})$ and $W(Q) = W^{2,r}(\Omega)$.

Taking $r > N$, we have that $W^{2,r}(\Omega)$ is compactly imbedded into $C^1(\overline{\Omega})$. Define the elliptic differential operator $\mathcal{L}\colon W^{2,r}(\Omega) \to L^r(\Omega)$ as follows:

$$\mathcal{L}u := -\sum_{i,j=1}^{N} a_{ij}(x)\partial^2_{x_i x_j} u + \sum_{i=1}^{N} a_i(x)\partial_{x_i} u + a_0(x)u.$$

Here $a_{ij} \in C(\overline{\Omega})$, $a_{ij} = a_{ji}$, for $i,j = 1,\dots,N$, there exists $\overline{a} > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \overline{a}\|\xi\|^2, \quad \text{for every } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N,$$

$a_i \in L^\infty(\Omega)$, for $i = 0, \dots, N$.

Assume that $\partial\Omega$ is the disjoint union of two closed sets Γ_1 and Γ_2 (the cases $\Gamma_1 = \emptyset$ or $\Gamma_2 = \emptyset$ are admitted). Consider the boundary operator

$$\mathcal{B}u := \begin{cases} u & \text{on } \Gamma_1, \\ \displaystyle\sum_{i=1}^N b_i(x)\partial_{x_i} u + b_0(x)u & \text{on } \Gamma_2. \end{cases}$$

Here $b_i \in C^1(\partial\Omega)$, for $i = 0, \dots, N$, and there exists $\bar{b} > 0$ such that

$$b_0(x) \geq 0 \quad \text{and} \quad \sum_{i=1}^N b_i(x)\nu_i(x) \geq \bar{b}, \quad \text{for every } x \in \partial\Omega.$$

The vector $\nu(x) = (\nu_1(x), \dots \nu_N(x))$ is the unit outer normal to Ω at $x \in \partial\Omega$.

We say that u is a *solution* of problem (P) if u belongs to $W^{2,r}(\Omega)$, it satisfies the differential equation almost everywhere in Ω and the boundary condition pointwise.

As in Section 7, we can assume without loss of generality that $a_0(x) \geq 0$ almost everywhere.

Definition 8.1. Given two functions $u, v \in C^1(\overline{\Omega})$, we will write $u \ll v$ if the following two conditions hold:

(a) for every $x \in \Omega$, $u(x) < v(x)$,
(b) for every $x \in \partial\Omega$, either $u(x) < v(x)$, or $u(x) = v(x)$ and $\partial_\nu u(x) > \partial_\nu v(x)$.

Here ν denotes the outer unit normal to $\partial\Omega$ at the point x.

We need to check whether our Assumptions A1–A7 are verified.

Assumption A1 is proved in the same way as in the previous section. For the Strong Maximum Principle, see e.g. [14, Lemma 3.26 and Theorem 3.27].

Assumption A2 is verified for every $\sigma < 0$, see [14, Theorems 3.28 and 3.29].

Assumption A3 is a classical estimate, with $\mathcal{G}(y) = cy^2$, for any constant $c \geq 0$, see [14, Lemma 5.10].

Assumption A4 is proved in the same way as in the previous section.

Assumption A5 follows from the Krein–Rutman theorem, as explained in the previous section.

Assumption A6 is directly verified, taking as $\varphi_0(x,t)$ the function with constant value 1, by the fact that $a_0(x) \geq 0$.

Assumption A7 is proved in the same way as in the previous section.

Let us conclude by stating the following direct consequence of Corollary 5.3 and Remark 5.4.

Corollary 8.2. *The Neumann problem*

$$\begin{cases} \Delta u + |u|^{\mathrm{p}-1}u = e(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution, provided that p *is sufficiently large and*

$$\|e\|_{L^\infty} \leq \frac{1}{2^{\mathrm{p}}}.$$

9. Final remarks

1. As further examples of application, we could consider the periodic problem associated to a scalar first order ordinary differential equation, like

$$\begin{cases} u' = F(t,u), \\ u(0) = u(T), \end{cases}$$

(see e.g. [6] and references therein) or, to a second order equation, like

$$\begin{cases} u'' = F(t,u,u'), \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases}$$

Notice however that, for such problems, the Bernstein–Nagumo condition can be considerably weakened, cf. [4].

2. The choice of dealing with the space $C^\sharp(\overline{Q})$ was made in view of the applications we had in mind (see Sections 7 and 8). Other types of function spaces could be considered, in order to be able to treat different kinds of equations and boundary value problems.

3. It could be possible to deal with more general definitions of lower and upper solutions, cf. [4]–[7], [12]. This could be useful, e.g. in the well-ordered setting of Theorem 3.1, in order to prove the existence of a greater and a least solution in $[\alpha,\beta]$, cf. [6, Theorem 2.15]. We did not focus on this point, not to complicate too much the exposition.

4. As already mentioned, our abstract setting well fits to the search of *strong* solutions. In order to deal with the case of *weak* solutions, a different approach would be needed.

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