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APPROXIMATE CONTROLLABILITY OF FRACTIONAL FUNCTIONAL EQUATIONS WITH INFINITE DELAY

Ramakrishnan Ganesh
Rathinasamy Sakthivel* — Nazim I. Mahmudov

Abstract. Fractional differential equations have been used for constructing many mathematical models in science and engineering. In this paper, we study the approximate controllability results for a class of impulsive fractional differential equations with infinite delay. A new set of sufficient conditions are formulated and proved for achieving the required result. In particular, the results are established under the natural assumptions that the corresponding linear system is approximately controllable. The results are obtained by using the fractional calculus, solution operators and fixed point technique. An example is also provided to illustrate the theory. Further, as a corollary, exact controllability result is discussed without assuming compactness of characteristic solution operators.

1. Introduction

The concept of controllability plays an important role in many control problems such as stabilization of unstable systems by feedback control. The exact controllability of various kinds of nonlinear evolution equations in infinite dimensional spaces by the method of fixed point theory have been investigated by

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* The corresponding author.

many authors [1], [6], [7], [10]. The existence and controllability results for first and second order semilinear differential inclusions in Banach spaces with nonlocal conditions has been reported in [11], [12]. Klamka [15], [17] derived a set of sufficient conditions for the constrained controllability for semilinear ordinary differential state equations with multiple point delays in control by using the generalized open mapping theorem.

Exact controllability enables to steer the system to arbitrary final state while approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. The approximate controllability is more appropriate for control systems instead of exact controllability. Moreover, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications. In particular, it is difficult to realize the conditions of exact controllability for infinite-dimensional systems and thus the approximate controllability becomes a very important topic. The approximate controllability results for nonlinear evolution equations for various kind of problems have been studied in [31], [19], [20].

Fractional differential equations has emerged as a new branch of applied mathematics, which has been used for constructing many mathematical models in various fields of science and engineering [23]. The reason for this is that a realistic model of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus. The theory of existence of solutions for fractional differential equations has been extensively studied by many authors [2], [22], [29].

Recently, many researchers pay attention to study of the controllability of nonlinear fractional evolution systems [3]–[5], [16], [36], [9], [33]. Wang et al [32] established a set of sufficient conditions for nonlocal controllability of fractional evolution systems without assuming compactness of solution operators by using Mönch fixed point theorems. However, in the present literature, there are only limited number of papers on the approximate controllability of fractional differential systems [30], [26], [27]. Sakthivel et al [26], [27] studied the approximate controllability results for deterministic and stochastic fractional differential systems by using fixed point technique and fractional calculations. The approximate controllability of nonlinear control systems governed by a class of partial neutral functional differential systems of fractional order with state-dependent delay in an abstract space has been investigated in [35]. Kumar and Sukavanam [18] derived a new set of sufficient conditions for the approximate controllability of a class of semilinear delayed control systems of fractional order by using contraction principle and the Schauder fixed point theorem.

Meanwhile, an impulsive perturbation occurs very often in many practical models [28], [29]. The controllability problems for several kinds of nonlinear problems with impulses has been studied in [25], [24]. Dabas et al [8] considered the existence of mild solutions for a class of impulsive fractional equations with infinite delay. Wang et al [34] discussed the solvability and optimal controls of a class of fractional integrodifferential evolution systems with infinite delay in Banach spaces. In fact, it is important and necessary to consider the approximate controllability for fractional functional differential systems with impulses and infinite delay. However, in the present literature, no work has been reported on approximate controllability of fractional differential systems with impulses and infinite delay. Motivated by [8], [34], in this paper we study the approximate controllability of a class of fractional order functional differential equations with impulses and infinite delay in the following form

$$\begin{aligned} D_t^q x(t) &= Ax(t) + Bu(t) + f(t, x_t, Hx(t)), \quad && t \in J = [0,b],\ t \neq t_k, \\ \Delta x(t_k) &= I_k(x(t_k^-)), && k = 1, \dots, m, \\ x(t) &= \phi(t) \in \mathcal{B}_h, \end{aligned} \tag{1.1}$$

where $0 < q < 1$; D_t^q is the Caputo fractional derivative of order q; $A\colon D(A) \subset X \to X$ is an infinitesimal generator of a q-resolvent family $\{S_q(t)\}_{t\geq 0}$, the solution operator $\{T_q(t)\}_{t\geq 0}$ is defined on a Hilbert space X with the norm $\|\cdot\|_X$; the control function $u(\,\cdot\,)$ is given in $L^2(J,U)$, U is a Hilbert space; B is a bounded linear operator from U into X. The histories $x_t\colon(-\infty,0] \to X$ defined by $x_t(\theta) = x(t+\theta)$ belong to an abstract phase space $\mathcal{B}_h$. $I_k\colon X \to X$, $k = 1,\dots,m$ is continuous. Furthermore, the fixed times t_k satisfy $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$. $f\colon J \times \mathcal{B}_h \times X \to X$ is a given function; $Hx(t)$ is given by $Hx(t) = \int_0^t G(t,s)x(s)\,ds$, where $G \in C(D, \mathbb{R}^+)$ is the set of all positive continuous functions on $D = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t \leq b\}$.

2. Preliminaries

In this section, some basic definitions and lemmas are given which will be used to prove main results. Let $L(X)$ denote the Banach space of bounded linear operators from X into X with the norm $\|\cdot\|_{L(X)}$. Let $C(J,X)$ denote the space of all continuous functions from J into X with the norm $\|x\| = \sup_{t\in J}\|x(t)\|$.

Now, we present the abstract space $\mathcal{B}_h$. Let $h\colon(-\infty,0] \to (0,+\infty)$ be a continuous function with $l = \int_{-\infty}^0 h(t)\,dt < +\infty$. For any $a > 0$, define $\mathcal{B} = \{\varphi : [-a,0] \to X$ such that $\varphi(t)$ is bounded and measurable$\}$ and equip the space $\mathcal{B}$ with the norm

$$\|\varphi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\varphi(s)\|, \quad \varphi \in \mathcal{B}.$$

Further, define the space

$$\mathcal{B}_h = \Big\{\varphi\colon (-\infty,0] \to X,\ \text{for any } c>0,\ \varphi|_{[-c,0]} \in \mathcal{B} \\ \text{with } \varphi(0)=0 \text{ and } \int_{-\infty}^{0} h(s)\|\varphi\|_{[s,0]}\,ds < +\infty\Big\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s)\|\varphi\|_{[s,0]}\,ds, \quad \varphi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space.

We assume that the phase space $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a semi-normed linear space of functions mapping $(-\infty,0]$ into X and satisfying the following fundamental axioms [14].

(A1) If $x\colon(-\infty,b] \to X$, $b>0$, is continuous on J and $x_0 \in \mathcal{B}_h$, then for every $t \in J$, the following conditions hold:
  - (i) $x_t \in \mathcal{B}_h$,
  - (ii) $\|x(t)\| \le L\|x_t\|_{\mathcal{B}_h}$,
  - (iii) $\|x_t\|_{\mathcal{B}_h} \le C_1(t)\sup_{0\le s\le t}\|x(s)\| + C_2(t)\|x_0\|_{\mathcal{B}_h}$, where $L>0$ is a constant; $C_1\colon[0,b]\to[0,\infty)$ is continuous, $C_2\colon[0,\infty)\to[0,\infty)$ is locally bounded and C_1, C_2 are independent of $x(\,\cdot\,)$.

(A2) For the function $x(\,\cdot\,)$ in (A1), x_t is a $\mathcal{B}_h$-valued function on $[0,b]$.

(A3) The space $\mathcal{B}_h$ is complete.

Definition 2.1 ([23]). The Caputo derivative of order q for a function $f\colon[0,\infty)\to R$ can be written as

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1} f^{(n)}(s)\,ds = I^{n-q} f^n(t),$$

for $n-1<q<n$, $n\in N$. If $0<q\le 1$, then

$$D_t^q f(t) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q} f^{(1)}(s)\,ds.$$

The Laplace transform of the Caputo derivative of order $q>0$ is given as

$$L\{D_t^q f(t) : \lambda\} = \lambda^q f(\lambda) - \sum_{k=0}^{n-1} \lambda^{q-k-1} f^{(k)}(0); \quad n-1<q<n.$$

The Mittag–Lefler type function in two arguments is defined by the series expansion

$$E_{q,p}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+p)} = \frac{1}{2\pi i}\int_C \frac{\mu^{q-p} e^{\mu}}{\mu^q - z}\,d\mu, \quad q,p>0,\ z\in\mathbb{C},$$

where C is a contour which starts and ends at $-\infty$ and encircles the disc $\|\mu\| \leq |z|^{1/2}$ counter clockwise. The Laplace transform of the Mittag–Lefller function is given as follows

$$\int_0^\infty e^{-\lambda t} t^{p-1} E_{q,p}(\omega t^q)\, dt = \frac{\lambda^{q-p}}{\lambda^q - \omega}, \quad \operatorname{Re}\lambda > \omega^{1/q},\ \omega > 0,$$

and for more details (see [23]).

Definition 2.2 ([13]). A closed and linear operator A is said to be sectorial if there are constants $\omega \in \mathbb{R}$, $\theta \in [\pi/2, \pi]$, $M > 0$, such that the following two conditions are satisfied:

(a) $\rho(A) \subset \sum_{(\theta,\omega)} = \{\lambda \in C : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$,

(b) $\|R(\lambda, A)\|_{L(X)} \leq \dfrac{M}{|\lambda - \omega|}, \quad \lambda \in \sum_{(\theta,\omega)}$.

Definition 2.3 ([13]). Let A be a linear closed operator with domain $D(A)$ defined on X. Let $\rho(A)$ be the resolvent set of A. We call A is the generator of a q-resolvent family if there exists $\omega \geq 0$ and a strongly continuous functions $S_q\colon \mathbb{R}^+ \to L(x)$ such that $\{\lambda^q : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$(\lambda^q I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_q(t) x\, dt, \quad \operatorname{Re}\lambda > \omega,\ x \in X.$$

In this case, S_q is called the q-resolvent family generated by A.

Definition 2.4 ([2]). Let A be a linear closed operator with domain $D(A)$ defined on X. We call A is the generator of a solution operator if there exists $\omega \geq 0$ and a strongly continuous functions $S_q\colon \mathbb{R}^+ \to L(x)$ such that $\{\lambda^q : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{q-1}(\lambda^q I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_q(t) x\, dt, \quad \operatorname{Re}\lambda > \omega,\ x \in X.$$

In this case, S_q is called the solution operator generated by A.

Lemma 2.5 ([8]). *If f satisfies the uniform Hölder condition with the exponent $\beta \in (0, 1]$ and A is a sectorial operator, then the unique solution of the Cauchy problem*

$$\begin{aligned} D_t^q x(t) &= Ax(t) + f(t, x_t, Hx(t)), && t > t_0,\ t_0 \in \mathbb{R},\ 0 < q < 1, \\ x(t) &= \phi(t), && t \leq t_0, \end{aligned}$$

is given by

$$x(t) = T_q(t - t_0)(x(t_0^+)) + \int_{t_0}^t S_q(t - s) f(s, x_s, Hx(s))\, ds,$$

where

$$\begin{aligned} T_q(t) &= E_{q,1}(At^q) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{\lambda^{q-1}}{\lambda^q - A}\,d\lambda,\\ S_q(t) &= t^{q-1}E_{q,q}(At^q) = \frac{1}{2\pi i}\int_{\widehat{B}_r} e^{\lambda t}\frac{1}{\lambda^q - A}\,d\lambda, \end{aligned}$$

$\widehat{B}_r$ denotes the Bromwich path. $S_q(t)$ is called the q-resolvent family and $T_q(t)$ is the solution operator generated by A.

Consider the space

$$\begin{aligned} \mathcal{B}_b = \{x : (-\infty, b] \to X \text{ such that } x|_{J_k} \in C(J_k, X)\\ \text{and there exist } x(t_k^-) \text{ and } x(t_k^+)\\ \text{with } x(t_k) = x(t_k^-),\ x_0 = \phi \in \mathcal{B}_h,\ k = 0, \ldots, m\}. \end{aligned}$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, \ldots, m$. Set $\|\cdot\|_{\mathcal{B}_b}$ be a seminorm in $\mathcal{B}_b$ defined by

$$\|x\|_{\mathcal{B}_b} = \sup_{s\in J}\|x(s)\| + \|\phi\|_{\mathcal{B}_h}, \quad x \in \mathcal{B}_b.$$

According to [8], we give the following definition of the mild solution of (1.1).

Definition 2.6. A function $x\colon (-\infty, b] \to X$ is said to be a mild solution for the system (1.1) if the following holds: $x_0 = \phi \in \mathcal{B}_h$ on $(-\infty, 0]$ with $\phi(0) = 0$; $\Delta x(t_k) = I_k(x(t_k^-)), k = 1, \ldots, m$, the restriction of $x(\cdot)$ to the interval $[0, b) \setminus \{t_1, \ldots, t_m\}$ is continuous and satisfies the following integral equation:

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0],\\ \int_0^t S_q(t-s)Bu(s)\,ds \\ \qquad + \int_0^t S_q(t-s)f(s, x_s, H(x(s)))\,ds, & t \in [0, t_1],\\ T_q(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) \\ \qquad + \int_{t_1}^t S_q(t-s)Bu(s)\,ds \\ \qquad + \int_{t_1}^t S_q(t-s)f(s, x_s, H(x(s)))\,ds, & t \in (t_1, t_2],\\ \cdots\cdots\cdots \\ T_q(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) \\ \qquad + \int_{t_m}^t S_q(t-s)Bu(s)\,ds \\ \qquad + \int_{t_m}^t S_q(t-s)f(s, x_s, H(x(s)))ds, & t \in (t_m, b]. \end{cases} \tag{2.2}$$

If $q \in (0, 1)$ and $A \in A^q(\theta_0, \omega_0)$, then for any $x \in X$ and $t > 0$, we have $\|T_q(t)\|_{L(X)} \le Me^{\omega t}$ and $\|S_q(t)\|_{L(X)} \le Ce^{\omega t}(1 + t^{q-1})$, $t > 0$, $\omega > \omega_0$. Let

$$\widetilde{M}_T = \sup_{0\le t\le b}\|T_q(t)\|_{L(X)}, \qquad \widetilde{M}_T = \sup_{0\le t\le b} Ce^{\omega t}(1 + t^{1-q}).$$

Hence, we have $\|T_q(t)\|_{L(X)} \leq \widetilde{M}_T$, $\|S_q(t)\|_{L(X)} \leq t^{q-1}\widetilde{M}_S$ (see [29]).

Lemma 2.7 ([22]). *Let*

$$C_1^* = \sup_{0<\tau<b} C_1(\tau), \quad C_2^* = \sup_{0<\tau<b} C_2(\tau), \quad \mu_1^* = \sup_{0<\tau<b} \mu_1(\tau), \quad \mu_2^* = \sup_{0<\tau<b} \mu_2(\tau).$$

Then, for any $s \in J$,

$$\mu_1(s)\|y_s + \overline{z}_s\|_{\mathcal{B}_h} + \mu_2(s)\|H(y(s) + \overline{z}(s))\|_X \\ \leq \mu_1^* \Big[C_1^* \sup_{0\leq\tau\leq s} \|z(\tau)\|_X + C_2^*\|\phi\|_{\mathcal{B}_h}\Big] + \mu_2^* \int_0^s G(s,\tau)\|z(\tau)\|_X \, d\tau.$$

If $\|z\|_X < r$, $r > 0$, then

$$\mu_1(s)\|y_s + \overline{z}_s\|_{\mathcal{B}_h} + \mu_2(s)\|H(y(s) + \overline{z}(s))\|_X \leq \mu_1^*\,[C_1^* r + C_2^*\|\phi\|_{\mathcal{B}_h}] + \mu_2^* H^* r = \delta,$$

where $H^* = \sup_{t\in[0,b]} \int_0^t G(t,s)\,ds < \infty$.

The main result of this paper is established by using the following fixed point theorem.

Lemma 2.8 ([8]). *Let S be a bounded closed and convex subset of a Banach space X. Let P and Q maps E into X such that*:

(a) *$Px + Qy \in E$ for every $x, y \in E$,*
(b) *P is compact and continuous,*
(c) *Q is a contraction mapping,*

then $Px + Qy = x$ has a solution on S.

3. Approximate controllability

In this section, we prove the approximate controllability of nonlinear impulsive fractional-order functional differential equations with infinite delay under suitable conditions. Consider the linear fractional control system:

$$\begin{aligned} D_t^q x(t) &= Ax(t) + (Bu)(t), \quad t \in [0,b], \\ x(0) &= \phi(0). \end{aligned} \tag{3.1}$$

Let us now introduce the following operators:

Define the operator $\Gamma_0^b \colon X \to X$ associated with (3.1) as

$$\Gamma_0^b = \int_0^b S_q(b-s)BB^*S_q^*(b-s)\,ds \colon X \to X, \qquad R(\alpha,\Gamma_0^b) = (\alpha I + \Gamma_0^b)^{-1} \colon X \to X,$$

where B^* denotes the adjoint of B and $S_q^*(t)$ is the adjoint of $S_q(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

Definition 3.1. The control system (1.1) is said to be approximately controllable on J if for every $\phi \in \mathcal{B}_h$, there is some control $u \in L_2(J,U)$, the closure of the reachable set $\overline{R(b,\phi)}$ is dense in X, i.e. $\overline{R(b,\phi)} = X$, where $R(b,\phi) = \{x_b(\phi;u)(0): u(\,\cdot\,) \in L_2(J,U)\}$ is the reachable set of system (1.1) with the initial value ϕ at terminal time b.

Remark 3.2. The approximate controllability of (3.1) is equivalent to the convergence of function $\alpha(\alpha I + \Gamma_0^b)^{-1}: X \to X$ to zero as $\alpha \to 0^+$ in the strong operator topology (see [21]).

In order to establish the result, we need the following hypothesis:

(H1) The function $f: J \times \mathcal{B}_h \times X \to X$ is continuous and there exists two continuous functions $\mu_1, \mu_2: J \to (0,\infty)$ such that

$$\|f(t,\varphi,x)\|_X \le \mu_1(t)\|\varphi\|_{\mathcal{B}_h} + \mu_2(t)\|x\|_X, \quad (t,\varphi,x) \in J \times \mathcal{B}_h \times X.$$

(H2) $I_k \in C(X,X)$ and there exist constants $\Omega > 0$, $\rho_k > 0$ such that

$$\Omega = \max_{1\le k\le m} \{\|I_k(x)\|_X\}, \quad \|I_k(x) - I_k(y)\|_X \le \rho_k\|x-y\|_X,$$

for $x, y \in X$, $(k = 1,\ldots,m)$.

(H3) There exists constants $N_1 > 0$ and $N_2 > 0$ such that

$$\|f(t,\varphi,x) - f(t,\psi,y)\|_X \le N_1\|\varphi-\psi\|_{\mathcal{B}_h} + N_2\|x-y\|_X,$$

for $t \in J$, $\varphi, \psi \in \mathcal{B}_h$, $x, y \in X$.

(H4) The linear system (3.1) is approximately controllable.

It will be shown that the system (1.1) is approximately controllable, if for all $\alpha > 0$, there exists a continuous function $x(\,\cdot\,) \in \mathcal{B}_b$ such that

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty,0], \\ \int_0^t S_q(t-s)Bu_x(s)\,ds & \\ \qquad + \int_0^t S_q(t-s)f(s,x_s,H(x(s)))\,ds, & t \in [0,t_1], \\ T_q(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) & \\ \qquad + \int_{t_1}^t S_q(t-s)Bu_x(s)\,ds & \\ \qquad + \int_{t_1}^t S_q(t-s)f(s,x_s,H(x(s)))\,ds, & t \in (t_1,t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ T_q(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) & \\ \qquad + \int_{t_m}^t S_q(t-s)Bu_x(s)\,ds & \\ \qquad + \int_{t_m}^t S_q(t-s)f(s,x_s,H(x(s)))\,ds, & t \in (t_m,b], \end{cases} \tag{3.2}$$

$$
(3.3)\qquad u_x(t) = \begin{cases} B^* S_q^*(t_1 - t) R(\alpha, \Gamma_0^{t_1}) p(x(\,\cdot\,)), & t \in [0, t_1], \\ B^* S_q^*(t_2 - t) R(\alpha, \Gamma_{t_1}^{t_2}) p(x(\,\cdot\,)), & t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ B^* S_q^*(b - t) R(\alpha, \Gamma_{t_m}^{b}) p(x(\,\cdot\,)), & t \in (t_m, b], \end{cases}
$$

where

$$
p(x(\,\cdot\,)) = \begin{cases} x_{t_1} - \int_0^{t_1} S_q(t_1 - s) f(s, x_s, H(x(s)))\, ds, & t \in [0, t_1], \\ x_{t_2} - T_q(t_2 - t_1)(x(t_1^-) + I_1(x(t_1^-))) & \\ \qquad\qquad - \int_{t_1}^{t_2} S_q(t_2 - s) f(s, x_s, H(x(s)))\, ds, & t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ x_b - T_q(b - t_m)(x(t_m^-) + I_m(x(t_m^-))) & \\ \qquad\qquad - \int_{t_m}^{b} S_q(b - s) f(s, x_s, H(x(s)))\, ds, & t \in (t_m, b]. \end{cases}
$$

THEOREM 3.3. *Let the hypothesis* (H1)–(H3) *hold and if* $A \in A^q(\theta_0, \omega_0)$, *then the system* (1.1) *has at least one mild solution on* $(-\infty, b]$ *provided that*

$$
\widehat{L}_0 = \max_{i \le i \le m} \bigg\{ \frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \frac{b^{2q-1}}{2q-1} \widetilde{M}_T (1 + \rho_i) + \bigg(\frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \frac{b^{2q-1}}{2q-1} + 1 \bigg) \frac{b^q}{q} \widetilde{M}_S (N_1 C_1^* + N_2 H^*) \bigg\} < 1.
$$

PROOF. The main aim in this section is to find conditions for solvability of system (3.2) and (3.3) for $\alpha > 0$. Now it will be shown that for $\alpha > 0$, the operator $\Phi\colon \mathcal{B}_b \to \mathcal{B}_b$ defined by

$$
(\Phi x)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \int_0^t S_q(t - s) B u_x(s)\, ds & \\ \qquad + \int_0^t S_q(t - s) f(s, x_s, H(x(s))) ds, & t \in [0, t_1], \\ T_q(t - t_1)(x(t_1^-) + I_1(x(t_1^-))) & \\ \qquad + \int_{t_1}^t S_q(t - s) B u_x(s)\, ds & \\ \qquad + \int_{t_1}^t S_q(t - s) f(s, x_s, H(x(s))) ds, & t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ T_q(t - t_m)(x(t_m^-) + I_m(x(t_m^-))) & \\ \qquad + \int_{t_m}^t S_q(t - s) B u_x(s)\, ds & \\ \qquad + \int_{t_m}^t S_q(t - s) f(s, x_s, H(x(s)))\, ds, & t \in (t_m, b]. \end{cases}
$$

has a fixed point, which is then a solution of system (1.1).

Let $y(\,\cdot\,)\colon (-\infty, b] \to X$ be the function defined by

$$
y(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in J, \end{cases}
$$

then $y_0 = \phi$. For each $z \in C(J, R)$ with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ z(t), & t \in J. \end{cases}$$

If $x(\cdot)$ satisfies (2.2), then we can decompose $x(\cdot)$ as $x(t) = y(t) + \overline{z}(t)$ for $t \in J$, which implies $x_t = y_t + \overline{z}_t$ for $t \in J$ and the function $z(\cdot)$ satisfies

$$z(t) = \begin{cases} \int_0^t S_q(t-s) B u_{y+\overline{z}}(s)\, ds \\ \qquad + \int_0^t S_q(t-s) f(s, y_s + \overline{z}_s, H(y(s) + \overline{z}(s)))\, ds, & t \in [0, t_1], \\ T_q(t - t_1)(y(t_1^-) + \overline{z}(t_1^-) + I_1(y(t_1^-) + \overline{z}(t_1^-))) \\ \qquad + \int_{t_1}^t S_q(t-s) B u_{y+\overline{z}}(s)\, ds \\ \qquad + \int_{t_1}^t S_q(t-s) f(s, y_s + \overline{z}_s, H(y(s) + \overline{z}(s)))\, ds, & t \in (t_1, t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ T_q(t - t_m)(y(t_m^-) + \overline{z}(t_m^-) + I_m(y(t_m^-) + \overline{z}(t_m^-))) \\ \qquad + \int_{t_m}^t S_q(t-s) B u_{y+\overline{z}}(s)\, ds \\ \qquad + \int_{t_m}^t S_q(t-s) f(s, y_s + \overline{z}_s, H(y(s) + \overline{z}(s)))\, ds, & t \in (t_m, b], \end{cases}$$

where

$$u_{y+\overline{z}}(t) = \begin{cases} B^* S_q^*(t_1 - t) R(\alpha, \Gamma_0^{t_1}) \\ \quad \cdot \big[x_{t_1} - \int_0^{t_1} S_q(t_1 - s) f(s, y_s + \overline{z}_s, H(y(s) + \overline{z}(s)))\, ds\big](t), \\ \hfill t \in [0, t_1], \\ B^* S_q^*(t_2 - t) R(\alpha, \Gamma_{t_1}^{t_2}) \\ \quad \cdot \big[x_{t_2} - T_q(t_2 - t_1)((y(t_1^-) + \overline{z}(t_1^-)) + I_1(y(t_1^-) + \overline{z}(t_1^-))) \\ \quad - \int_{t_1}^{t_2} S_q(t_2 - s) f(s, y_s + \overline{z}_s, H(y(s) + \overline{z}(s)))\, ds\big](t), & t \in (t_1, t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ B^* S_q^*(b - t) R(\alpha, \Gamma_{t_m}^{b}) \\ \quad \cdot \big[x_b - T_q(b - t_m)((y(t_m^-) + \overline{z}(t_m^-)) + I_m(y(t_m^-) + \overline{z}(t_m^-))) \\ \quad - \int_{t_m}^{b} S_q(b - s) f(s, y_s + \overline{z}_s, H(y(s) + \overline{z}(s)))\, ds\big](t), & t \in (t_m, b]. \end{cases}$$

Let $\mathcal{B}_b^0 = \{z \in \mathcal{B}_b : z_0 = 0 \in \mathcal{B}_h\}$. For any $z \in \mathcal{B}_b^0$, we have

$$\|z\|_{\mathcal{B}_b^0} = \sup_{s \in J} \|z(s)\|_X + \|z_0\|_{\mathcal{B}_h} = \sup_{s \in J} \|z(s)\|_X, \quad z \in \mathcal{B}_b^0.$$

Thus, $(\mathcal{B}_b^0, \|\cdot\|_{\mathcal{B}_b^0})$ is a Banach space.

Let the operator $\Pi\colon \mathcal{B}_b^0 \to \mathcal{B}_b^0$ be defined by

$$(3.4)\quad \Pi z(t)=\begin{cases} \int_0^t S_q(t-s)Bu_{y+\overline{z}}(s)\,ds \\ \quad +\int_0^t S_q(t-s)f(s,y_s+\overline{z}_s,H(y(s)+\overline{z}(s)))\,ds, & t\in[0,t_1], \\ T_q(t-t_1)(y(t_1^-)+\overline{z}(t_1^-)+I_1(y(t_1^-)+\overline{z}(t_1^-))) \\ \quad +\int_{t_1}^t S_q(t-s)Bu_{y+\overline{z}}(s)\,ds \\ \quad +\int_{t_1}^t S_q(t-s)f(s,y_s+\overline{z}_s,H(y(s)+\overline{z}(s)))\,ds, & t\in(t_1,t_2], \\ \cdots\cdots\cdots \\ T_q(t-t_m)(y(t_m^-)+\overline{z}(t_m^-)+I_m(y(t_m^-)+\overline{z}(t_m^-))) \\ \quad +\int_{t_m}^t S_q(t-s)Bu_{y+\overline{z}}(s)\,ds \\ \quad +\int_{t_m}^t S_q(t-s)f(s,y_s+\overline{z}_s,H(y(s)+\overline{z}(s)))\,ds, & t\in(t_m,b]. \end{cases}$$

It is clear that the operator Φ has a fixed point if and only if Π has a fixed point. Now we will show that Π has a fixed point.

Choose

$$r\ge \widetilde{M_T}(r+\Omega)+\frac{1}{\alpha}\,M_B^2\widetilde{M_S^2}\,\frac{b^{2q-1}}{2q-1}\left[\|x_{t_{i+1}}\|+\widetilde{M_T}(r+\Omega)+\widetilde{M_S}\delta\,\frac{b^q}{q}\right]+\widetilde{M_S}\delta\,\frac{b^q}{q}$$

and consider a set $B_r=\{z\in\mathcal{B}_b^0:\|z\|_{\mathcal{B}_b^0}\le r\}$, then $B_r\subset\mathcal{B}_b^0$, is clearly a bounded closed convex set. Now for $t\in J$, we decompose Π as $\Pi=\Pi_1+\Pi_2$, where Π_1 and Π_2 are defined on B_r, by

$$\Pi_1 z(t)=\begin{cases} 0, & t\in[0,t_1], \\ T_q(t-t_1)(z(t_1^-)+I_1(z(t_1^-))), & t\in(t_1,t_2], \\ \cdots\cdots\cdots \\ T_q(t-t_m)(z(t_m^-)+I_m(z(t_m^-))), & t\in(t_m,b]. \end{cases}$$

$$\Pi_2 z(t)=\begin{cases} \int_0^t S_q(t-\eta)BB^*S_q^*(t_1-\eta)R(\alpha,\Gamma_0^{t_1}) \\ \quad\cdot\big[x_{t_1}-\int_0^{t_1} S_q(t_1-s)f(s,y_s+\overline{z}_s,H(y(s)+\overline{z}(s)))\,ds\big](\eta)\,d\eta \\ \quad+\int_0^t S_q(t-s)f(s,y_s+\overline{z}_s,H(y(s)+\overline{z}(s)))\,ds, & t\in[0,t_1], \\ \int_{t_1}^t S_q(t-\eta)BB^*S_q^*(t_2-\eta)R(\alpha,\Gamma_{t_1}^{t_2}) \\ \quad\cdot\big[x_{t_2}-T_q(t_2-t_1)((z(t_1^-))+I_1(z(t_1^-))) \\ \quad-\int_{t_1}^{t_2} S_q(t_2-s)f(s,y_s+\overline{z}_s,H(y(s)+\overline{z}(s)))\,ds\big](\eta)\,d\eta \\ \quad+\int_{t_1}^t S_q(t-s)f(s,y_s+\overline{z}_s,H(y(s)+\overline{z}(s)))\,ds, & t\in(t_1,t_2], \\ \cdots\cdots\cdots \\ \int_{t_m}^t S_q(t-\eta)BB^*S_q^*(b-\eta)R(\alpha,\Gamma_{t_m}^{b})\big[x_b-T_q(b-t_m)((z(t_m^-)) \\ +I_m(z(t_m^-)))-\int_{t_m}^b S_q(b-s)f(s,y_s+\overline{z}_s,H(y(s)+\overline{z}(s)))\,ds\big](\eta)\,d\eta \\ +\int_{t_m}^t S_q(t-s)f(s,y_s+\overline{z}_s,H(y(s)+\overline{z}(s)))\,ds, & t\in(t_m,b]. \end{cases}$$

First we prove that $\Pi_1 z + \Pi_2 z^* \in B_r$, whenever $z, z^* \in B_r$. For $t \in [0, t_1]$, we have

$$\begin{aligned}
\|(\Pi_1 z)(t) + (\Pi_2 z^*)(t)\|_X &\le \int_0^t \|S_q(t-\eta) BB^* S_q^*(t_1-\eta) R(\alpha, \Gamma_0^{t_1})\|_X \\
&\quad \cdot \bigg[\|x_{t_1}\| + \int_0^{t_1} \|S_q(t_1-s)\|_{L(X)} \|f(s, y_s + \overline{z}_s^*, H(y(s)+\overline{z}^*(s)))\|_X\, ds\bigg](\eta)\, d\eta \\
&\quad + \int_0^t \|S_q(t-s)\|_{L(X)} \|f(s, y_s + \overline{z}_s^*, H(y(s)+\overline{z}^*(s)))\|_X\, ds \\
&\le \frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \int_0^t (t-\eta)^{2(q-1)} \bigg[\|x_{t_1}\| \\
&\quad + \widetilde{M}_S \int_0^{t_1} (t_1-s)^{q-1} (\mu_1(s)\|y_s + \overline{z}_s^*\|_{\mathcal{B}_h} + \mu_2(s)\|H(y(s)+\overline{z}^*(s))\|_X)\, ds\bigg](\eta)\, d\eta \\
&\quad + \widetilde{M}_S \int_0^t (t-s)^{q-1} (\mu_1(s)\|y_s + \overline{z}_s^*\|_{\mathcal{B}_h} + \mu_2(s)\|H(y(s)+\overline{z}^*(s))\|_X)\, ds.
\end{aligned}$$

By using Lemma 2.7, we deduce that

$$\|(\Pi_1 z) + (\Pi_2 z^*)\|_b \le \frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \frac{b^{2q-1}}{2q-1} \bigg[\|x_{t_1}\| + \widetilde{M}_S \delta \frac{b^q}{q}\bigg] + \widetilde{M}_S \delta \frac{b^q}{q} < r.$$

Similarly, when $t \in (t_i, t_{i+1}]$, $i = 1, \ldots, m$, we have

$$\begin{aligned}
\|(\Pi_1 z)(t) + (\Pi_2 z^*)(t)\|_X &\le \|T_q(t-t_1)(z(t_i^-) + I_i(z(t_i^-)))\|_X \\
&\quad + \int_{t_i}^t \|S_q(t-\eta) BB^* S_q^*(t_{i+1}-\eta) R(\alpha, \Gamma_{t_i}^{t_{i+1}})\|_X \\
&\quad \cdot \bigg[\|x_{t_{i+1}}\| + \|T_q(t_{i+1}-t_i)(z^*(t_i^-) + I_i(z^*(t_i^-)))\|_X \\
&\quad + \int_{t_i}^{t_{i+1}} \|S_q(t_{i+1}-s)\|_{L(X)} \|f(s, y_s + \overline{z}_s^*, H(y(s)+\overline{z}^*(s)))\|_X\, ds\bigg](\eta)\, d\eta \\
&\quad + \int_{t_i}^t \|S_q(t-s)\|_{L(X)} \|f(s, y_s + \overline{z}_s^*, H(y(s)+\overline{z}^*(s)))\|_X\, ds \\
&\le \widetilde{M}_T(\|z\|_b + \|I_i(z(t_i^-))\|) + \frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \frac{b^{2q-1}}{2q-1} \\
&\quad \cdot \bigg[\|x_{t_{i+1}}\| + \widetilde{M}_T(\|z^*\|_b + \|I_i(z^*(t_i^-))\|) + \widetilde{M}_S \delta \frac{b^q}{q}\bigg] + \widetilde{M}_S \delta \frac{b^q}{q} \\
&\le \widetilde{M}_T(r + \Omega) + \frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \frac{b^{2q-1}}{2q-1} \\
&\quad \cdot \bigg[\|x_{t_{i+1}}\| + \widetilde{M}_T(r + \Omega) + \widetilde{M}_S \delta \frac{b^q}{q}\bigg] + \widetilde{M}_S \delta \frac{b^q}{q} < r.
\end{aligned}$$

Hence for all $t \in [0, b]$, $\|(\Pi_1 z) + (\Pi_2 z^*)\|_{\mathcal{B}_b^0} \le r$. Using the same argument as in [8], we can obtain that Π_1 is continuous and equicontinuous. Finally, by using Ascoli's theorem, we conclude that Π_1 is compact.

Next we show that Π_2 is a contraction mapping. Let $z, z^* \in B_r$ and for $t \in [0, t_1]$, we have

$$\begin{aligned}
\|(\Pi_2 z)(t) - (\Pi_2 z^*)(t)\|_X \le & \int_0^t \|S_q(t-\eta)BB^*S_q^*(t_1-\eta)R(\alpha, \Gamma_0^{t_1})\|_X \\
& \cdot \Big[\int_0^{t_1} \|S_q(t_1-s)\|_{L(X)} \|f(s, y_s + \overline{z}_s, H(y(s)+\overline{z}(s))) \\
& - f(s, y_s + \overline{z}_s^*, H(y(s)+\overline{z}^*(s)))\|_X \, ds\Big](\eta)\, d\eta \\
& + \int_0^t \|S_q(t-s)\|_{L(X)} \|f(s, y_s + \overline{z}_s, H(y(s)+\overline{z}(s))) \\
& - f(s, y_s + \overline{z}_s^*, H(y(s)+\overline{z}^*(s)))\|_X \, ds \\
\le & \frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \int_0^t (t-\eta)^{2(q-1)} \Big[\widetilde{M}_S \int_0^{t_1} (t_1-s)^{q-1} (N_1 \|\overline{z}_s - \overline{z}_s^*\|_{\mathcal{B}_h} \\
& + N_2 \|H(y(s)+\overline{z}(s)) - H(y(s)+\overline{z}^*(s))\|_X)\, ds\Big](\eta)\, d\eta \\
& + \widetilde{M}_S \int_0^t (t-s)^{q-1} (N_1 \|\overline{z}_s - \overline{z}_s^*\|_{\mathcal{B}_h} \\
& + N_2 \|H(y(s)+\overline{z}(s)) - H(y(s)+\overline{z}^*(s))\|_X)\, ds \\
\le & \Big\{\frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \frac{b^{2q-1}}{2q-1} \Big[\frac{b^q}{q} \widetilde{M}_S (N_1 C_1^* + N_2 H^*)\Big] \\
& + \frac{b^q}{q} \widetilde{M}_S (N_1 C_1^* + N_2 H^*)\Big\} \|z - z^*\|_{\mathcal{B}_b^0} \\
\le & \Big(\frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \frac{b^{2q-1}}{2q-1} + 1\Big) \frac{b^q}{q} \widetilde{M}_S (N_1 C_1^* + N_2 H^*) \|z - z^*\|_{\mathcal{B}_b^0}.
\end{aligned}$$

For $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$,

$$\begin{aligned}
\|(\Pi_2 z)(t) - (\Pi_2 z^*)(t)\|_X \le & \int_{t_i}^t \|S_q(t-\eta)BB^*S_q^*(t_1-\eta)R(\alpha, \Gamma_{t_i}^{t_{i+1}})\|_X \\
& \cdot \Big[\|T_q(t_{i+1}-t_i)\|_{L(X)} (\|z(t_i^-) - z^*(t_i^-)\|_X + \|I_i(z(t_i^-)) - I_i(z^*(t_i^-))\|_X) \\
& + \int_{t_i}^{t_{i+1}} \|S_q(t_{i+1}-s)\|_{L(X)} \|f(s, y_s + \overline{z}_s, H(y(s)+\overline{z}(s))) \\
& - f(s, y_s + \overline{z}_s^*, H(y(s)+\overline{z}^*(s)))\|_X \, ds\Big](\eta)\, d\eta \\
& + \int_{t_i}^t \|S_q(t-s)\|_{L(X)} \|f(s, y_s + \overline{z}_s, H(y(s)+\overline{z}(s))) \\
& - f(s, y_s + \overline{z}_s^*, H(y(s)+\overline{z}^*(s)))\|_X \, ds \\
\le & \frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \int_{t_i}^t (t-\eta)^{2(q-1)} \Big[\widetilde{M}_T (\|z(t_i^-) - z^*(t_i^-)\|_X + \rho_i \|z(t_i^-) - z^*(t_i^-)\|_X)
\end{aligned}$$

$$
\begin{aligned}
&+ \widetilde{M}_S \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{q-1} (N_1 \|\overline{z}_s - \overline{z}_s^*\|_{\mathcal{B}_h} \\
&+ N_2 \|H(y(s)+\overline{z}(s)) - H(y(s)+\overline{z}^*(s))\|_X)\, ds \bigg] (\eta) d\eta \\
&+ \widetilde{M}_S \int_{t_i}^{t} (t-s)^{q-1} (N_1 \|\overline{z}_s - \overline{z}_s^*\|_{\mathcal{B}_h} \\
&+ N_2 \|H(y(s)+\overline{z}(s)) - H(y(s)+\overline{z}^*(s))\|_X)\, ds \\
\le& \bigg\{ \frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \frac{b^{2q-1}}{2q-1} \widetilde{M}_T (1+\rho_i) \\
&+ \bigg(\frac{1}{\alpha} M_B^2 \widetilde{M}_S^2 \frac{b^{2q-1}}{2q-1} + 1 \bigg) \frac{b^q}{q} \widetilde{M}_S (N_1 C_1^* + N_2 H^*) \bigg\} \|z - z^*\|_{\mathcal{B}_b^0}.
\end{aligned}
$$

Thus, for all $t \in [0,b]$, we have $\|(\Pi_2 z) - (\Pi_2 z^*)\|_X \le \widehat{L}_0 \|z - z^*\|_{\mathcal{B}_b^0}$. Hence Π_2 is a contraction mapping. Hence, by the Krasnosel'skiĭ fixed-point theorem, we deduce that Π has a fixed point $z \in B_r$ which is a mild solution of (1.1). □

Theorem 3.4. *Suppose that the assumptions* (H1)–(H4) *are satisfied and the operator family* $(S_q(t))_{t\ge 0}$ *is compact. Moreover, if f is uniformly bounded then the fractional system* (1.1) *is approximately controllable on* $[0,b]$.

Proof. Let $\widetilde{x}^\alpha(\cdot)$ be a fixed point of Π in B_r. By Theorem 3.3, any fixed point of Π is a mild solution of (1.1) under the control

$$
\widetilde{u}^\alpha(t) = \begin{cases} B^* S_q^*(t_1 - t) R(\alpha, \Gamma_0^{t_1}) p(\widetilde{x}^\alpha), & t \in [0, t_1], \\ B^* S_q^*(t_2 - t) R(\alpha, \Gamma_{t_1}^{t_2}) p(\widetilde{x}^\alpha), & t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots & \\ B^* S_q^*(b - t) R(\alpha, \Gamma_{t_m}^{b}) p(\widetilde{x}^\alpha), & t \in (t_m, b] \end{cases}
$$

and satisfies

$$
\begin{cases} \widetilde{x}^\alpha(t_1) = x_{t_1} + \alpha R(\alpha, \Gamma_0^{t_1}) p(\widetilde{x}^\alpha), & t \in [0, t_1], \\ \widetilde{x}^\alpha(t_2) = x_{t_2} + \alpha R(\alpha, \Gamma_{t_1}^{t_2}) p(\widetilde{x}^\alpha), & t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots & \\ \widetilde{x}^\alpha(b) = x_b + \alpha R(\alpha, \Gamma_{t_m}^{b}) p(\widetilde{x}^\alpha), & t \in (t_m, b]. \end{cases} \tag{3.4}
$$

Moreover, by the assumption that f is uniformly bounded, there exists $N > 0$ such that

$$
\int_0^b \|f(s, \widetilde{x}_s^\alpha, H\widetilde{x}^\alpha(s))\|^2\, ds \le bN^2
$$

and consequently, the sequence $\{f(s, \widetilde{x}_s^\alpha, H\widetilde{x}^\alpha(s))\}$ is bounded in $L^2(J, X)$. Then there is a subsequence denoted by $\{f(s, \widetilde{x}_s^\alpha, H\widetilde{x}^\alpha(s))\}$, that converges weakly to say $f(s)$ in $L^2(J,X)$. Define

$$w=\begin{cases} x_{t_1}-\int_0^{t_1} S_q(t_1-s)f(s)\,ds, & t\in[0,t_1],\\ x_{t_2}-T_q(t_2-t_1)(x(t_1^-)+I_1(x(t_1^-)))-\int_{t_1}^{t_2} S_q(t_2-s)f(s)\,ds, & t\in(t_1,t_2],\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ x_b-T_q(b-t_m)(x(t_m^-)+I_m(x(t_m^-)))-\int_{t_m}^{b} S_q(b-s)f(s)\,ds, & t\in(t_m,b]. \end{cases}$$

Now, for $t\in[0,t_1]$, we have

$$\begin{aligned} \text{(3.5)}\quad \|p(\widetilde{x}^\alpha)-w\| &= \left\| \int_0^{t_1} S_q(t_1-s)[f(s,\widetilde{x}_s^\alpha,H\widetilde{x}^\alpha(s))-f(s)]\,ds \right\| \\ &\le \sup_{t\in[0,t_1]} \left\| \int_0^{t} S_q(t-s)[f(s,\widetilde{x}_s^\alpha,H\widetilde{x}^\alpha(s))-f(s)]\,ds \right\|. \end{aligned}$$

For $t\in[t_i,t_{i+1}]$, $i=1,\dots,m$, we have

$$\begin{aligned} \text{(3.6)}\quad \|p(\widetilde{x}^\alpha)-w\| &= \left\| \int_{t_i}^{t_{i+1}} S_q(t_{i+1}-s)[f(s,\widetilde{x}_s^\alpha,H\widetilde{x}^\alpha(s))-f(s)]\,ds \right\| \\ &\le \sup_{t\in(t_i,t_{i+1}]} \left\| \int_{t_i}^{t} S_q(t-s)[f(s,\widetilde{x}_s^\alpha,H\widetilde{x}^\alpha(s))-f(s)]\,ds \right\|. \end{aligned}$$

By using infinite-dimensional version of the Ascoli–Arzela theorem, one can show that an operator

$$l(\,\cdot\,)\to\int_0^{\cdot}(\,\cdot\,-s)^{q-1}S_q(\,\cdot\,-s)\,l(s)\,ds\colon L^1(J,X)\to C(J,X)$$

is compact. Hence, for all $t\in[0,b]$, we obtain that $\|p(\widetilde{x}^\alpha)-w\|\to 0$ as $\alpha\to 0^+$. Moreover, from (3.4) we get, for $t\in[0,t_1]$,

$$\begin{aligned} \|\widetilde{x}^\alpha(t_1)-x_{t_1}\| &\le \|\alpha R(\alpha,\Gamma_0^{t_1})(w)\|+\|\alpha R(\alpha,\Gamma_0^{t_1})\|\|p(\widetilde{x}_\alpha)-w\| \\ &\le \|\alpha R(\alpha,\Gamma_0^{t_1})(w)\|+\|p(\widetilde{x}_\alpha)-w\|. \end{aligned}$$

It follows from assumption Remark 3.2 and the estimation (3.5) that $\|\widetilde{x}_\alpha(t_1)-x_{t_1}\|\to 0$ as $\alpha\to 0^+$. Similarly, in the view of (3.6), for $t\in(t_i,t_{i+1}]$, $i=1,\dots,m$, $\|\widetilde{x}_\alpha(t_{i+1})-x_{t_{i+1}}\|\to 0$ as $\alpha\to 0^+$. Thus, for all $t\in[0,b]$, we get $\|\widetilde{x}_\alpha(b)-x_b\|\to 0$ as $\alpha\to 0^+$. This proves the approximate controllability of (1.1). □

EXAMPLE 3.5. Let $X=L^2(0,\pi)$, and $A=d^2/dy^2$ with $D(A)$ consisting of all $x\in X$ with d^2x/dy^2 and $x(0)=0=x(\pi)$. Put $e_n(y)=\sqrt{2/\pi}\sin ny$, $n=1,2,\dots$, then $\{e_n,\ n=1,2,\dots\}$ is an orthonormal base for X and e_n is the eigenfunction corresponding to the eigenvalue $\lambda_n=-n^2$ of the operator A, $n=1,2,\dots$ Define an infinite dimensional control space U by

$$U=\left\{u\;\middle|\; u=\sum_{n=2}^{\infty}u_ne_n \text{ with } \sum_{n=2}^{\infty}u_n^2<\infty\right\}$$

with norm defined by

$$\|u\|_U = \bigg(\sum_{n=2}^{\infty} u_n^2\bigg)^{1/2}.$$

Define a continuous linear map B from U to X as

$$Bu = 2u_2 e_1 + \sum_{n=2}^{\infty} u_n e_n \quad \text{for } u = \sum_{n=2}^{\infty} u_n e_n \in U.$$

Let us consider the following fractional partial integro-differential equation with infinite delay of the form

$$\tag{3.7} \begin{aligned} {}^cD_t^q x(t,y) &= \frac{\partial^2}{\partial y^2} x(t,y) + \mu(t,y) + \int_{-\infty}^{t} K(t,x,s-t) Q(x(s,y))\,ds \\ &\qquad + \int_0^t g(s,t) e^{-x(s,y)}\,ds, \qquad t \in J = [0,1],\ y \in [0,\pi],\ t \neq t_k, \\ x(t,0) &= x(t,\pi) = 0, \\ x(t,y) &= \phi(t,y), \qquad\qquad\qquad t \in (-\infty,0],\ y \in [0,\pi], \\ \Delta x(t_i)(y) &= \int_{-\infty}^{t_i} q_i(t_i - s) x(s,y)\,ds, \quad y \in [0,\pi], \end{aligned}$$

where ${}^cD_t^q$ is the Caputo fractional derivative of order $0 < q < 1$, $\phi(t,y)$ is continuous, $q_i \colon R \to R$ are continuous.

It is well known that A generates a analytic semigroup $\{T(t), t > 0\}$ in X and it is given by

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}(x, e_n) e_n, \quad x \in X.$$

From these expression it follows that $\{T(t),\, t > 0\}$ is uniformly bounded compact semigroup, so that, $R(\lambda, A) = (\lambda I - A)^{-1}$ is compact operator for $\lambda \in \rho(A)$.

Let $h(s) = e^{2s}$, $s < 0$, then $l = \int_{\infty}^{0} h(s)\,ds = 1/2$ and define

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} \|\phi(\theta)\|_{L^2}\,ds.$$

For $(t,\phi) \in J \times \mathcal{B}_h$, where $\phi(\theta)(y) = \phi(\theta,y)$, $(\theta,y) \in (-\infty,0] \times [0,\pi]$.

Let $x(t)(y) = x(t,y)$, and define the bounded linear operator $B \colon U \to X$ by $(Bu)(t)(y) = \mu(t,y)$, $0 \le y \le \pi$ and

$$f(t,\phi,Hx(t))(y) = \int_{-\infty}^{0} K(t,y,\theta) Q(\phi(\theta)(y))\,d\theta + Hx(t)(y),$$

where

$$Hx(t)(y) = \int_0^t g(s,t) e^{-x(s,y)}\,ds, \qquad I_k(x(t_i^-))(y) = \int_{-\infty}^{t_i} q_i(t_i - s) x(s,y)\,ds.$$

On the otherhand, the linear fractional control system corresponding to (3.7) is approximately controllable. Then, the system (3.7) can be written in the abstract form of (1.1) and all the conditions of the Theorem 3.4 are satisfied. Further, if we impose suitable conditions on K, Q, g and q_i to verify assumptions on Theorem 3.4, then we can conclude that the fractional control system (3.7) is approximately controllable on $[0,1]$.

Definition 3.6. The control system (1.1) is said to be exactly controllable on J if for every $\phi \in \mathcal{B}_h$, there is some control $u \in L_2(J,U)$, the reachable set, $R(b,\phi)$ is dense in X, i.e. $R(b,\phi) = X$.

Assume that the linear fractional differential system

$$\begin{aligned} D_t^q x(t) &= Ax(t) + (Bu)(t), \quad t \in [0,b], \\ x(0) &= \phi(0). \end{aligned} \tag{3.8}$$

is exactly controllable. It is convenient at this point to introduce the controllability operator associated with (3.8) as

$$\Gamma_0^b = \int_0^b S_q(b-s)BB^*S_q^*(b-s)\,ds.$$

Lemma 3.7. *If the linear fractional system* (3.8) *is exactly controllable if and only then for some* $\gamma > 0$ *such that* $\langle \Gamma_0^b x, x\rangle \geq \gamma\|x\|^2$, *for all* $x \in X$ *and consequently* $\|(\Gamma_0^b)^{-1}\| \leq 1/\gamma$.

Corollary 3.8. *Assume that the hypotheses* (H_2) *and* (H_3) *are hold. If the linear system associated with the system* (1.1) *is exactly controllable on all* $[0,t]$, $t > 0$, *then the nonlinear system* (1.1) *is exactly controllable on* $[0,b]$ *provided that*

$$\max_{1\leq i\leq m}\left\{\left(\frac{1}{\gamma}\,M_B^2\widetilde{M}_S^2\,\frac{b^{2q-1}}{2q-1}+1\right)\left[\widetilde{M}_T(1+\rho_i)+\frac{b^q}{q}\,\widetilde{M}_S(N_1C_1^*+N_2H^*)\right]\right\} < 1.$$

Proof. Choose the feedback control function

$$\widehat{u}_x(t) = \begin{cases} B^*S_q^*(t_1-t)(\Gamma_0^{t_1})^{-1} \\ \quad \cdot[x_{t_1} - \int_0^{t_1} S_q(t_1-s)f(s,x_s,H(x(s)))\,ds](t), & t\in[0,t_1], \\ B^*S_q^*(t_2-t)(\Gamma_{t_1}^{t_2})^{-1} \\ \quad \cdot[x_{t_2} - T_q(t_2-t_1)(x(t_1^-)+I_1(x(t_1^-))) \\ \quad - \int_{t_1}^{t_2} S_q(t_2-s)f(s,x_s,H(x(s)))\,ds](t), & t\in(t_1,t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ B^*S_q^*(b-t)(\Gamma_{t_m}^{b})^{-1} \\ \quad \cdot[x_b - T_q(b-t_m)(x(t_m^-)+I_m(x(t_m^-))) \\ \quad - \int_{t_m}^{b} S_q(b-s)f(s,x_s,H(x(s)))\,ds](t), & t\in(t_m,b]. \end{cases} \tag{3.9}$$

and define the operator $\widehat{\Phi}\colon \mathcal{B}_b \to \mathcal{B}_b$ by

$$(\widehat{\Phi}x)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \int_0^t S_q(t-s)B\widehat{u}_x(s)\,ds \\ \qquad + \int_0^t S_q(t-s)f(s, x_s, H(x(s)))\,ds, & t \in [0, t_1], \\ T_q(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) \\ \qquad + \int_{t_1}^t S_q(t-s)B\widehat{u}_x(s)\,ds \\ \qquad + \int_{t_1}^t S_q(t-s)f(s, x_s, H(x(s)))\,ds, & t \in (t_1, t_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ T_q(t-t_m)(x(t_m^-) + I_m(x(t_m^-))) \\ \qquad + \int_{t_m}^t S_q(t-s)B\widehat{u}_x(s)\,ds \\ \qquad + \int_{t_m}^t S_q(t-s)f(s, x_s, H(x(s)))\,ds, & t \in (t_m, b]. \end{cases}$$

Note that the control (3.9) transfers the system (1.1) from the initial state ϕ to the final state x_b provided that the operator $\widehat{\Phi}$ has a fixed point.

To prove the exact controllability, it is enough to show that the operator $\widehat{\Phi}$ has a fixed point in B_b. From Lemma 3.7 and the assumptions on the data, one can easily prove that $\widehat{\Phi}$ has a fixed point. The proof of this corollary is similar to that of Theorem 3.3 with some changes and hence it is omitted. □

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Ramakrishnan Ganesh
Department of Mathematics
Anna University Regional Centre
Coimbatore 641 047, INDIA

Rathinasamy Sakthivel
Department of Mathematics
Sungkyunkwan University
Suwon 440-746, SOUTH KOREA

E-mail address: krsakthivel@yahoo.com

Nazim I. Mahmudov
Department of Mathematics
Eastern Mediterranean University
Gazimagusa, Mersin 10, TURKEY