

THE EQUIVALENCE OF WEAK AND VERY WEAK SUPERSOLUTIONS TO THE POROUS MEDIUM EQUATION

PEKKA LEHTELÄ AND TEEMU LUKKARI

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Abstract. We prove that various notions of supersolutions to the porous medium equation are equivalent under suitable conditions. More specifically, we consider weak supersolutions, very weak supersolutions, and m -superporous functions defined via a comparison principle. The proofs are based on comparison principles and a Schwarz type alternating method, which are also interesting in their own right. Along the way, we show that Perron solutions with merely continuous boundary values are continuous up to the parabolic boundary of a sufficiently smooth space-time cylinder.

1. Introduction. Our aim is to clarify and extend the connections between various notions of solutions and supersolutions to the porous medium equation

$$(1.1) \quad u_t - \Delta u^m = 0 \quad \text{in } \Omega_T = \Omega \times (0, T).$$

We treat both the case of prescribed boundary values and the purely local notions, and restrict our attention to the degenerate case $m > 1$. For the basic theory of the equation and numerous further references, we refer to the monographs [9], [19], [20] and [21].

There are at least two natural ways to define solutions to (1.1). *Weak solutions* are defined by multiplying the equation by a suitable test function and integrating by parts once. In this definition, the function u^m is assumed to be in a parabolic Sobolev space. In the case of the boundary value problem, the boundary values are interpreted in a Sobolev sense. The chief attraction of this notion is that a weak solution itself is an admissible test function after a mollification in the time direction, which leads to natural energy estimates. On the other hand, we may integrate by parts twice in the space variable, thus relaxing the regularity assumptions for solutions. This leads to *very weak solutions*, a notion which makes sense under the minimal assumptions that u and u^m are integrable. The boundary values are taken into account via including the appropriate integrals over the lateral boundary and at the initial time. One of the advantages of the very weak solutions is their stability under convergence. Weak and very weak solutions with fixed boundary values turn out to be the same. This result is probably known to experts, at least when the boundary values are sufficiently regular.

It is important to understand not only the solutions, but also supersolutions. Supersolutions arise naturally in obstacle problems [2, 4] and problems with measure data [3, 18].

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Furthermore, supersolutions connect the equation to potential theory, providing important tools such as the Perron method [13]. In the classical theory they also play a central role in the study of boundary regularity, removability of sets and other fine properties.

There are again various ways to define supersolutions. *Weak and very weak supersolutions* (Definition 4.1 and Definition 4.2) satisfy the inequality

$$\frac{\partial u}{\partial t} - \Delta u^m \geq 0,$$

the rigorous interpretation being analogous to the concepts of weak and very weak solutions. Another way is to use a comparison principle: supersolutions are lower semicontinuous functions which satisfy a parabolic comparison principle with respect to continuous weak solutions. We call these supersolutions *m-supersporous functions* (Definition 5.1). This is one of the ways to define superharmonic functions in classical potential theory, and it is amenable to generalization to nonlinear equations. In the case of the PME, the basic properties of this class of supersolutions have been established in [11]; see also [12]. Several nice properties follow immediately from the definition of *m-supersporous functions*. For instance, it is easy to see that the minimum of two *m-supersporous functions* is also *m-supersporous*. Moreover, the *m-supersporous functions* form a closed class under increasing convergence.

A natural question is whether the different classes of supersolutions are equivalent. The similar problem is well understood in the case of *p*-Laplace type equations, see [14, 17]. However, the question is more challenging for the porous medium equation. For example, the boundary values cannot be perturbed in the standard way, because constants cannot be added to solutions. Further difficulties arise when trying to incorporate the very weak notions to the arguments. Therefore new methods have to be developed.

Our main result is the equivalence of the above classes of supersolutions under suitable conditions:

THEOREM 1.1. *The following properties are equivalent for continuous, nonnegative functions u .*

- (1) *u is a weak supersolution,*
- (2) *u is a very weak supersolution,*
- (3) *u is m -supersporous.*

For completeness we also address the question of equivalence of the classes of solutions, as it is difficult to find a reference where this matter is treated thoroughly. Along the way, we obtain that Perron solutions with merely continuous boundary values are continuous up to the parabolic boundary of a sufficiently smooth space-time cylinder, thus complementing the results of [13].

The natural situation for Theorem 1.1 would be to consider locally bounded lower semicontinuous functions. Indeed, lower semicontinuity is the natural regularity of weak supersolutions, see [1], and local boundedness is definitely necessary. Further, the equivalence of weak supersolutions and *m-supersporous functions* under these weaker assumptions has been established in [11]. Our contribution is including very weak supersolutions to the theory. The

necessity of boundedness can be seen by considering the Barenblatt solution

$$\mathcal{B}_m(x, t) = \begin{cases} t^{-\lambda} \left(C - \frac{\lambda(m-1)}{2mn} \frac{|x|^2}{t^{2\lambda/n}} \right)_+^{1/(m-1)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where

$$\lambda = \frac{n}{n(m-1) + 2}.$$

The Barenblatt solution \mathcal{B}_m is an unbounded m -superporous function, but its gradient fails to be square integrable in any neighbourhood of the origin, and thus \mathcal{B}_m is not a weak supersolution.

It is unclear whether the classes are the same, if one only assumes lower semicontinuity. The crucial point where continuity is used is to show that very weak supersolutions are also very weak supersolutions with boundary values given by the function itself. This is needed for proving the comparison principle for very weak supersolutions. Further, there are other challenges already in the continuous case. Therefore we find the continuity assumption reasonable. Note that *equivalence holds for solutions without assuming continuity*: nonnegative very weak solutions turn out to be continuous after a redefinition on a set of zero measure, by [8].

Yet another way to define supersolutions is viscosity supersolutions, see [5, 6]. This notion uses pointwise touching test functions. In this paper we focus on the previously mentioned classes of supersolutions. A very interesting open question is whether viscosity supersolutions are equivalent to the other notions of supersolutions as well. The answer is known to affirmative for equations similar to the p -Laplacian by [10], so one would expect the same result to hold for the PME as well.

2. Weak solutions. Throughout the work we use the following notation. We work in space-time cylinders $\Omega_T = \Omega \times (0, T) \subset \mathbb{R}^{n+1}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, such that $\partial\Omega$ is sufficiently nice, for example smooth or Lipschitz. We denote the lateral boundary of Ω_T by $\Sigma_T = \partial\Omega \times [0, T]$ and the parabolic boundary by $\partial_p \Omega_T = \Omega \times \{0\} \cup \Sigma_T$. For $U \Subset \Omega$, we denote $U_{t_1, t_2} = U \times (t_1, t_2)$. The parabolic boundary of U_{t_1, t_2} is defined as $\partial_p U_{t_1, t_2} = U \times \{t_1\} \cup \partial U \times [t_1, t_2]$.

We consider the solutions to the boundary value problem

$$(2.1) \quad \begin{cases} u_t - \Delta u^m = 0 & \text{on } \Omega_T, \\ u(x, 0) = u_0(x), \\ u^m = g & \text{on } \Sigma_T, \end{cases}$$

where u_0 is in $H^1(\Omega)$ and g is a continuous function defined in $\overline{\Omega_T}$ such that $g \in L^2(0, T; H^1(\Omega))$. Further, we require that the initial and lateral boundary values are compatible in the

sense that the function $\varphi : \partial_p \Omega_T \rightarrow \mathbb{R}$ defined by

$$\varphi(x, t) = \begin{cases} g(x, t)^{1/m}, & (x, t) \in \Sigma_T, \\ u_0(x), & (x, t) \in \Omega \times \{0\} \end{cases}$$

is continuous. For simplicity, we will assume that g and u_0 are non-negative and thus the solutions will be non-negative as well by the comparison principle, which will be proved in Section 3. Hence we may assume that the solutions are always non-negative.

DEFINITION 2.1. We say u is a local weak solution to (1.1) if $u^m \in L^2_{\text{loc}}(0, T; H^1_{\text{loc}}(\Omega))$ and u satisfies the equality

$$\int_{\Omega_T} (-u\varphi_t + \nabla u^m \cdot \nabla \varphi) dx dt = 0$$

for any $\varphi \in C_0^\infty(\Omega_T)$.

A function u is a weak solution to the boundary value problem (2.1), if $u^m \in L^2(0, T; H^1(\Omega))$, $u^m - g \in L^2(0, T; H^1_0(\Omega))$, and

$$\int_{\Omega_T} (-u\varphi_t + \nabla u^m \cdot \nabla \varphi) dx dt = \int_{\Omega} u_0(x)\varphi(x, 0) dx$$

for all smooth test functions φ with compact support in space, vanishing at the time $t = T$.

We will show that the boundary value problem (2.1) has at most one weak solution. This follows by using a clever test function devised by Oleĭnik.

LEMMA 2.2. *Weak solutions to the boundary value problem (2.1) are unique.*

PROOF. The proof is a standard application of the Oleĭnik test function

$$\varphi = \begin{cases} \int_t^T (u^m - v^m) ds, & \text{if } 0 \leq t < T, \\ 0 & \text{otherwise.} \end{cases}$$

For a detailed proof, we refer to [19, Theorem 5.3]. □

3. Very weak solutions. In this section we consider another natural class of generalized solutions, very weak solutions. This concept is defined as follows.

DEFINITION 3.1. We say $u \in L^1_{\text{loc}}(\Omega_T)$ is a local very weak solution to (1.1) if $u^m \in L^1_{\text{loc}}(\Omega_T)$ and u satisfies the equality

$$\int_{\Omega_T} (u^m \Delta \eta + u\eta_t) dx dt = 0$$

for any $\eta \in C_0^\infty(\Omega_T)$.

A function $u \in L^1(\Omega_T)$ is a very weak solution to the boundary value problem (2.1), if $u^m \in L^1(\Omega_T)$ and

$$\int_{\Omega_T} (u^m \Delta \eta + u\eta_t) dx dt + \int_{\Omega} u_0(x)\eta(x, 0) dx = \int_{\Sigma_T} g \partial_\nu \eta dS dt$$

for all smooth η vanishing on Σ_T and at time $t = T$. Note that the test functions η are not required to have compact support in Ω_T .

We prove the comparison principle for the very weak solutions to the boundary value problem (2.1). That is, if u and v are very weak solutions to (2.1) such that $u \geq v$ on $\partial_p \Omega_T$ and $u^m, v^m \in L^2(\Omega_T)$, then $u \geq v$ in Ω_T . In fact, we only need to assume u is a very weak supersolution and v is a very weak subsolution, see Lemma 4.4. First, we present a technical lemma, which will be used in proving the comparison principles for very weak solutions and very weak supersolutions. The idea is, that in both cases the proof can be reduced to using the following lemma.

LEMMA 3.2. *Let $u, v \in L^2(\Omega_T)$ and suppose $u^m, v^m \in L^2(\Omega_T)$. If*

$$\int_{\Omega_T} ((v^m - u^m)\Delta\varphi + (v - u)\varphi_t) dx dt \geq 0$$

for every smooth φ vanishing on Σ_T , then $u \geq v$ in Ω_T .

The proof of this lemma can be found in [19, Theorem 6.5]. Next, we will show that the comparison principle for very weak solutions follows from this lemma.

LEMMA 3.3. *Let $u, v \in L^2(\Omega_T)$ be very weak solutions to the boundary value problem (2.1) with boundary and initial data g, u_0 and h, v_0 respectively. Suppose that $u^m, v^m \in L^2(\Omega_T)$. If $u_0 \geq v_0$ in Ω and $g \geq h$ on Σ_T , then $u \geq v$ in Ω_T .*

PROOF. By the definition of very weak solutions,

$$\int_{\Omega_T} (-u^m \Delta\varphi - u\varphi_t) dx dt + \int_{\Sigma_T} g^m \partial_\nu \varphi dS dt - \int_{\Omega} u_0(x)\varphi(x, 0) dx = 0$$

and

$$\int_{\Omega_T} (-v^m \Delta\varphi - v\varphi_t) dx dt + \int_{\Sigma_T} h^m \partial_\nu \varphi dS dt - \int_{\Omega} v_0(x)\varphi(x, 0) dx = 0$$

for every smooth φ vanishing on Σ_T . Subtracting the equalities gives

$$(3.1) \quad \begin{aligned} & \int_{\Omega_T} ((v^m - u^m)\Delta\varphi + (v - u)\varphi_t) dx dt - \int_{\Sigma_T} (h^m - g^m)\partial_\nu \varphi dS dt \\ & + \int_{\Omega} (v_0(x) - u_0(x))\varphi(x, 0) dx = 0. \end{aligned}$$

Now suppose $\varphi \geq 0$ in Ω_T . Since $u_0 \geq v_0$ on Ω , we see that

$$\int_{\Omega} (v_0 - u_0)\varphi dx \leq 0.$$

The function φ vanishes on the lateral boundary Σ_T , so $\partial_\nu \varphi \leq 0$, and since $g \geq h$ on Σ_T , we have

$$\int_{\Sigma_T} (h - g)\partial_\nu \varphi dS dt \geq 0.$$

Using the estimates above, we conclude

$$\int_{\Omega_T} ((v^m - u^m) \Delta \varphi + (v - u) \varphi_t) dx dt \geq 0.$$

Now we may apply Lemma 3.2 to conclude the proof. \square

REMARK 3.4. The comparison principle in Lemma 3.3 holds also for finite unions of space-time cylinders $K = \bigcup_{i=1}^N U_{t_1, t_2}^i$. This can be proved by considering an enumeration s_k of the times t_j^i , $i = 1, \dots, N$, $j = 1, 2$, where $s_1 < s_2 < \dots < s_M$ and proving the result inductively for the sets $K \cap (\mathbb{R}^n \times [s_i, s_{i+1}])$ using Lemma 3.3.

We use the following lemma from [13] to bypass the fact that we may not add constants to solutions. We will present the proof for the reader's convenience.

LEMMA 3.5. Suppose g is a continuous, non-negative function in $\overline{\Omega_T}$, such that $g \in L^2(0, T; H^1(\Omega))$ and suppose $u_0 \in H^1(\Omega)$ is non-negative. Let $\varepsilon \in (0, 1)$. Denote by $g_\varepsilon = g + \varepsilon$ and $u_{0, \varepsilon} = u_0 + \varepsilon$. Let u and u_ε be a weak solutions to (2.1) with boundary and initial data g, u_0 and $g_\varepsilon, u_{0, \varepsilon}$ respectively. Then

$$\int_{\Omega_T} (u_\varepsilon - u)(u_\varepsilon^m - u^m) dx dt \leq \varepsilon |\Omega_T| (M + 1) + \varepsilon |\Omega_T| (M + 1)^m,$$

where $M = \max\{\sup_{\overline{\Omega_T}} g, \sup_{\Omega} u_0\}$.

PROOF. Since u and u_ε are weak solutions, the equalities

$$\begin{aligned} \int_{\Omega_T} (-u \varphi_t + \nabla u^m \cdot \nabla \varphi) dx dt &= \int_{\Omega} u_0(x) \varphi(x, 0) dx \quad \text{and} \\ \int_{\Omega_T} (-u_\varepsilon \varphi_t + \nabla u_\varepsilon^m \cdot \nabla \varphi) dx dt &= \int_{\Omega} u_{0, \varepsilon}(x) \varphi(x, 0) dx \end{aligned}$$

hold. Now a subtraction gives

$$\int_{\Omega_T} (-(u_\varepsilon - u) \varphi_t + \nabla (u_\varepsilon^m - u^m) \cdot \nabla \varphi) dx dt = \int_{\Omega} (u_{0, \varepsilon} - u_0) \varphi(x, 0) dx.$$

We will use an Oleĭnik type test function defined as

$$\varphi(x, t) = \begin{cases} \int_t^T (u_\varepsilon^m - u^m - \varepsilon) ds, & t \in [0, T], \\ 0 & \text{otherwise.} \end{cases}$$

Now φ has the properties

$$\varphi_t = -(u_\varepsilon^m - u^m - \varepsilon) \quad \text{and} \quad \nabla \varphi = \int_t^T \nabla (u_\varepsilon^m - u^m) ds.$$

Thus

$$\int_{\Omega_T} \left((u_\varepsilon - u)(u_\varepsilon^m - u^m - \varepsilon) + \nabla (u_\varepsilon^m - u^m) \cdot \left(\int_t^T \nabla (u_\varepsilon^m - u^m) ds \right) \right) dx dt$$

$$= \int_{\Omega} (u_{0,\varepsilon} - u_0) \left(\int_0^T (u_{\varepsilon}^m - u^m - \varepsilon) ds \right) dx.$$

We observe that

$$\begin{aligned} & \int_{\Omega_T} \nabla(u_{\varepsilon}^m - u^m) \cdot \left(\int_t^T \nabla(u_{\varepsilon}^m - u^m) ds \right) dx dt \\ &= \frac{1}{2} \int_{\Omega} \left(\int_0^T (\nabla u_{\varepsilon}^m - \nabla u^m) ds \right)^2 dx \geq 0 \quad \text{and} \\ & -\varepsilon T \int_{\Omega} (u_{0,\varepsilon} - u_0) dx \leq 0. \end{aligned}$$

Hence, we have the estimate

$$\begin{aligned} (3.2) \quad & \int_{\Omega_T} (u_{\varepsilon} - u)(u_{\varepsilon}^m - u^m) dx dt \\ & \leq \varepsilon \int_{\Omega_T} (u_{\varepsilon} - u) dx dt + \int_{\Omega} (u_{0,\varepsilon} - u_0) \left(\int_0^T (u_{\varepsilon}^m - u^m) ds \right) dx. \end{aligned}$$

By the comparison principle $u \leq M$ in Ω_T and thus by construction of g_{ε} and $u_{0,\varepsilon}$, the comparison principle gives $u_{\varepsilon} \leq M + 1$ in Ω_T . Then the right hand side of (3.2) can be bounded from above using

$$\begin{aligned} u_{\varepsilon} - u &\leq M + 1, \\ u_{0,\varepsilon} - u_0 &\leq \varepsilon \quad \text{and} \\ u_{\varepsilon}^m - u^m &\leq (u_{\varepsilon} - u)^m \leq (M + 1)^m. \end{aligned}$$

We have

$$\int_{\Omega_T} (u_{\varepsilon} - u)(u_{\varepsilon}^m - u^m) dx dt \leq \varepsilon(M + 1)|\Omega_T| + \varepsilon(M + 1)^m|\Omega_T|. \quad \square$$

For proving the equivalence of local weak and very weak supersolutions, we need to consider solutions to the boundary value problem (2.1) when the functions u_0 and g are only assumed to be continuous. In such a case, the previous interpretation of the boundary and initial conditions is no longer available, so we use the notion of Perron solutions [13] instead. Perron solutions are weak solutions in the interior, but the question whether they attain the correct boundary values was left open in [13]. Next we show that this is indeed the case in sufficiently smooth cylinders, by using a barrier argument. This justifies calling the Perron solution the solution to the boundary value problem (2.1).

In order to construct a suitable lower barrier, we need to show the existence of signed solutions to the boundary value problem with smooth boundary values. This will be done in the next lemma. The proof follows the ideas outlined in Chapter 5 of [19].

LEMMA 3.6. *Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Let g be a smooth function defined in a neighbourhood of Σ_T and let $u_0 \in C^\infty(\Omega)$. Then there exists a weak solution to the boundary value problem*

$$\begin{cases} u_t - \Delta(|u|^{m-1}u) = 0 & \text{on } \Omega_T, \\ u(x, 0) = u_0(x), \\ u^m = g & \text{on } \Sigma_T. \end{cases}$$

PROOF. Let $\phi(s) = |s|^{m-1}s$. Define a smooth function ϕ_1 such that

$$\phi_1(s) = \begin{cases} \phi(s) & \text{for } |s| \geq 1, \\ cs & \text{for } |s| \leq \frac{1}{2}, \end{cases}$$

ϕ_1 is convex for $s \geq 0$ and $\phi_1(-s) = -\phi_1(s)$. Now define $\phi_n(s) = n^{-m}\phi_1(ns)$, where $n = 1, 2, \dots$. Then $\phi_n(s) = \phi(s)$ for $|s| \geq \frac{1}{n}$ and $\phi'_n(s) > 0$ for every $s \in \mathbb{R}$. Moreover $\phi_n \rightarrow \phi$ uniformly on compact sets. We consider the approximate problem

$$(3.3) \quad \begin{cases} u_t - \Delta\phi_n(u) = 0 & \text{on } \Omega_T, \\ u(x, 0) = u_0(x), \\ \phi(u) = g & \text{on } \Sigma_T. \end{cases}$$

By the quasilinear regularity theory (see [15]), there exists a smooth solution u_n to (3.3). Moreover, by the maximum principle we have

$$-N \leq u_n(x, t) \leq M \quad \text{in } \Omega_T,$$

where $N = \max\{\sup(-u_0), \sup(-g)\}$ and $M = \max\{\sup(u_0), \sup(g)\}$. We multiply the equation $(u_n)_t - \Delta\phi_n(u_n) = 0$ by a test function $\phi_n(u_n) - g \in L^2(0, T; H_0^1(\Omega))$ and integrate by parts to get

$$\int_{\Omega_T} (u_n)_t (\phi_n(u_n) - g) dx dt + \int_{\Omega_T} \nabla\phi_n(u_n) (\nabla\phi_n(u_n) - \nabla g) dx dt = 0.$$

Therefore

$$(3.4) \quad \begin{aligned} \int_{\Omega_T} |\nabla\phi_n(u_n)|^2 dx dt &= \int_{\Omega_T} \nabla\phi_n(u_n) \cdot \nabla g dx dt - \int_{\Omega_T} (u_n)_t \phi_n(u_n) dx dt \\ &\quad + \int_{\Omega_T} (u_n)_t g dx dt. \end{aligned}$$

Let Ψ_n denote the primitive of ϕ_n , defined as

$$\Psi_n(s) = \int_0^s \phi_n(t) dt.$$

We observe that

$$(\Psi_n(u_n))_t = (u_n)_t \Psi'_n(u_n) = (u_n)_t \phi_n(u_n),$$

and thus

$$(3.5) \quad \int_{\Omega_T} (u_n)_t \phi_n(u_n) dx dt = \int_{\Omega} \Psi_n(u_n(x, T)) dx - \int_{\Omega} \Psi_n(u_0(x)) dx.$$

To control the last term on the right hand side of (3.4), we integrate by parts to get

$$(3.6) \quad \begin{aligned} \int_{\Omega_T} (u_n)_t g dx dt &= - \int_{\Omega_T} u_n g_t dx dt + \int_{\Omega} u_n(x, T) g(x, T) dx \\ &\quad - \int_{\Omega} u_0(x) g(x, 0) dx. \end{aligned}$$

Collecting the facts from (3.4), (3.5) and (3.6) and using Young's inequality gives us an upper bound for the L^2 -norm of the gradient

$$\begin{aligned} \int_{\Omega_T} |\nabla \phi_n(u_n)|^2 dx dt &\leq C \left(\int_{\Omega_T} |\nabla g|^2 dx dt + \int_{\Omega} |\Psi_n(u_0(x))| dx \right. \\ &\quad + \int_{\Omega} |\Psi_n(u_n(x, T))| dx + \int_{\Omega_T} |u_n| |g_t| dx dt \\ &\quad \left. + \int_{\Omega} |u_n(x, T)| |g(x, T)| dx + \int_{\Omega} |u_0(x)| |g(x, 0)| dx \right). \end{aligned}$$

Thus $\nabla \phi_n(u_n)$ is uniformly bounded in $L^2(\Omega_T)$. In order to control the time derivative $(\phi_n(u_n))_t$, we multiply the equation $(u_n)_t - \Delta \phi_n(u_n) = 0$ by the test function $\zeta(t)(\phi_n(u_n) - g)_t$, where $\zeta(t)$ is a smooth cut-off function, such that $0 \leq \zeta \leq 1$, $\zeta(t) = 1$ for $t \in (\varepsilon, T - \varepsilon)$, and $\zeta(0) = \zeta(T) = 0$. Integrating by parts gives

$$\int_{\Omega_T} \zeta (\phi_n(u_n) - g)_t (u_n)_t dx dt = - \int_{\Omega_T} \zeta \nabla \phi_n(u_n) \cdot \nabla (\phi_n(u_n) - g)_t dx dt,$$

which can be written as

$$\begin{aligned} \int_{\Omega_T} \zeta \phi_n(u_n)_t (u_n)_t dx dt &= \int_{\Omega_T} \zeta g_t u_t dx dt - \int_{\Omega_T} \zeta \nabla \phi_n(u_n) \cdot \nabla \phi_n(u_n)_t dx dt \\ &\quad + \int_{\Omega_T} \zeta \nabla \phi_n(u_n) \cdot \nabla g dx dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We integrate I_1 by parts in the time variable to get

$$I_1 = \int_{\Omega_T} \zeta g_t u_t dx dt = - \int_{\Omega_T} (\zeta g_t)_t u_n dx dt.$$

Integrating I_2 by parts gives

$$\begin{aligned} I_2 &= - \int_{\Omega_T} \zeta \nabla \phi_n(u_n) \cdot \nabla \phi_n(u_n)_t dx dt \\ &= \int_{\Omega_T} (\zeta \nabla \phi_n(u_n))_t \cdot \nabla \phi_n(u_n) dx dt \end{aligned}$$

$$= \int_{\Omega_T} \zeta' |\nabla \phi_n(u_n)|^2 dx dt + \int_{\Omega_T} \zeta \nabla \phi_n(u_n) \cdot \nabla \phi_n(u_n)_t dx dt,$$

and therefore

$$I_2 = \frac{1}{2} \int_{\Omega_T} \zeta' |\nabla \phi_n(u_n)|^2 dx dt.$$

Finally, I_3 can be bounded by

$$\begin{aligned} |I_3| &= \left| \int_{\Omega_t} \zeta \nabla \phi_n(u_n) \cdot \nabla g dx dt \right| \\ &\leq \left(\int_{\Omega_T} \zeta^2 |\nabla g|^2 dx dt \right)^{1/2} \left(\int_{\Omega_T} |\nabla \phi_n(u_n)|^2 dx dt \right)^{1/2}. \end{aligned}$$

Since u_n is bounded, $\phi'_n(u_n) \leq C$ for some C . Thus by (3.7), we get

$$\begin{aligned} \int_{\Omega_{\varepsilon, T-\varepsilon}} |\phi_n(u_n)_t|^2 dx dt &= \int_{\Omega_{\varepsilon, T-\varepsilon}} |(u_n)_t \phi'_n(u_n)|^2 dx dt \leq \int_{\Omega_T} \zeta |(u_n)_t|^2 \phi'_n(u_n) dx dt \\ &\leq C(|I_1| + |I_2| + |I_3|). \end{aligned}$$

Hence, $\phi_n(u_n)_t$ is uniformly bounded in $L^2(\Omega \times (\varepsilon, T - \varepsilon))$. In conclusion, $\phi_n(u_n)$ is uniformly bounded in $H^1(\Omega \times (\varepsilon, T - \varepsilon))$. By compactness, there exists a subsequence $\phi_{n_j}(u_{n_j}) \rightarrow w \in L^2(\Omega \times (\varepsilon, T - \varepsilon))$ almost everywhere. It follows that $w \in L^2((0, T), H^1(\Omega))$. The sequence u_{n_j} is uniformly bounded, so it converges to some u almost everywhere (taking a subsequence, if necessary) and $\phi_{n_j}(u_{n_j}) \rightarrow \phi(u)$ almost everywhere. Therefore $w = \phi(u)$ almost everywhere. Since u_n is a classical solution to (3.3), it satisfies

$$\int_{\Omega_T} (-u_n \varphi_t + \nabla \phi_n(u_n) \cdot \nabla \varphi) dx dt = \int_{\Omega} u_0(x) \varphi(x, 0) dx.$$

By weak compactness, $\nabla \phi_n(u_n) \rightarrow \nabla \phi(u)$ weakly, thus showing that indeed u is a weak solution to the problem. \square

We are now ready to show that Perron solutions attain the correct boundary values in the classical sense.

LEMMA 3.7. *Let the functions $g \in C(\overline{\Omega_T})$ and $u_0 \in C(\overline{\Omega})$ be non-negative and compatible. Then the Perron solution to the boundary value problem (2.1) attains the correct boundary values continuously.*

PROOF. We will show the claim by a barrier type argument. To simplify notation we write

$$\varphi(x, t) = \begin{cases} g(x, t)^{1/m}, & (x, t) \in \Sigma_T, \\ u_0(x), & (x, t) \in \Omega \times \{0\}. \end{cases}$$

Fix $\xi \in \partial_p \Omega_T$ and take $\varepsilon > 0$. We will show that there exists a supersolution $v^+ \in \mathfrak{U}_\varphi$, such that $\lim_{z \rightarrow \xi} v^+(z) = \varphi(\xi) + \varepsilon$ and a subsolution $v^- \in \mathfrak{L}_\varphi$, such that $\lim_{z \rightarrow \xi} v^-(z) = \varphi(\xi) - \varepsilon$. Here \mathfrak{U}_φ and \mathfrak{L}_φ denote the upper and lower Perron classes respectively.

The upper barrier v^+ can be constructed by solving the boundary value problem (2.1) with boundary values $\varphi + \varepsilon$. A continuous solution exists by the quasilinear theory, as described in the proof of the previous lemma. Moreover v^+ is continuous up to the boundary by [22]. In order to construct the lower barrier v^- , we will consider a small neighbourhood E of ξ . Let f be a smooth function, such that $f(\xi) = \varphi(\xi) - \varepsilon$ and $f = -k$ on $\partial_p(E \cap \Omega_T)$ outside a neighbourhood of ξ . By Lemma 3.6, there exists a weak solution \tilde{v} in $E \cap \Omega_T$ with boundary values f . We extend \tilde{v} to the whole Ω_T by defining

$$v^- = \begin{cases} \max\{\tilde{v}, -k\} & \text{in } E \cap \Omega_T, \\ -k & \text{in } \Omega_T \setminus E. \end{cases}$$

By choosing k large enough, we have $v^- \in \mathfrak{L}_\varphi$ and $v^- = \tilde{v}$ in $E \cap \Omega_T$. Again, the continuity of v^- up to the boundary is provided by [22].

By the definition of the Perron solution, $v^- \leq u \leq v^+$ and thus

$$\varphi(\xi) - \varepsilon \leq \liminf_{z \rightarrow \xi} u(z) \leq \limsup_{z \rightarrow \xi} u(z) \leq \varphi(\xi) + \varepsilon.$$

Since this holds for every $\varepsilon > 0$, we conclude that $\lim_{z \rightarrow \xi} u(z) = \varphi(\xi)$. \square

We are now ready to prove the first of the main results, the equivalence of the different notions of solutions to the boundary value problem. We emphasize the fact that the boundary and initial values are only assumed to be continuous.

THEOREM 3.8. *Let u be the Perron solution and v a very weak solution to the boundary value problem*

$$\begin{cases} u_t - \Delta u^m = 0 & \text{on } \Omega_T, \\ u(x, 0) = u_0(x), \\ u^m = g & \text{on } \Sigma_T, \end{cases}$$

with continuous, compatible boundary values u_0 and g . If $v^m \in L^2(\Omega_T)$, then $u = v$.

PROOF. The claim follows from the comparison principle for very weak solutions (Lemma 3.3) as soon as we show that the Perron solution u is also a very weak solution to the boundary value problem. For smooth boundary values, this follows by Green's formula from the fact that the Perron solution also attains the correct boundary values in the Sobolev sense, see Theorem 5.8 in [13].

It remains to reduce the general case to the smooth case. We do this by an approximation argument. Define as before

$$\varphi(x, t) = \begin{cases} g(x, t)^{1/m}, & (x, t) \in \Sigma_T, \\ u_0(x), & (x, t) \in \Omega \times \{0\}, \end{cases}$$

extend φ continuously to the whole space and choose smooth functions φ_j converging to φ uniformly and such that

$$\varphi_j \leq \varphi \leq \varphi_j + 1/j.$$

Further, let u_j and v_j be the Perron solutions with boundary values φ_j and $\varphi_j + 1/j$, respectively. Since $u_j \leq u$ and

$$u - u_j \leq v_j - u_j \rightarrow 0$$

as $j \rightarrow \infty$ by Lemma 3.5, we have that $u_j \rightarrow u$ pointwise in Ω_T . Now u_j is a very weak solution to the boundary value problem with boundary values given by φ_j , and passing to the limit $j \rightarrow \infty$ in the very weak formulation for u_j shows that u is a very weak solution to the boundary value problem with boundary values given by φ . \square

The previous theorem together with the continuity result in [8] implies the equivalence of local weak and very weak solutions.

COROLLARY 3.9. *A nonnegative function u is a local very weak solution to the PME if and only if u is a local weak solution to the PME.*

PROOF. By [8], local very weak solutions are continuous in the interior of Ω_T . Thus, following the proof of Lemma 4.3 below, we may show that local very weak solutions are solutions to the boundary value problem (2.1) in space-time cylinders $B_{t_1, t_2} \Subset \Omega_T$ where the base is a ball, with boundary values defined by the function itself. Therefore the result follows from Theorem 3.8 and the fact that being a weak solution is a local property. \square

4. Supersolutions. In this section, we turn our attention to supersolutions. The definitions of weak supersolutions and very weak supersolutions are analogous to those of weak solutions and very weak solutions.

DEFINITION 4.1. A function u is a (local) weak supersolution to (1.1) if $u^m \in L^2_{\text{loc}}(0, T; H^1_{\text{loc}}(\Omega))$

$$\int_{\Omega_T} (-u\varphi_t + \nabla u^m \cdot \nabla \varphi) dx dt \geq 0$$

for all non-negative, compactly supported smooth test functions φ .

As in the case of weak solutions, it is natural to consider also very weak supersolutions.

DEFINITION 4.2. A function $u \in L^1_{\text{loc}}(\Omega_T)$ is a (local) very weak supersolution to (1.1), if $u^m \in L^1_{\text{loc}}(\Omega_T)$ and

$$\int_{\Omega_T} (-u\varphi_t - u^m \Delta \varphi) dx dt \geq 0$$

for all non-negative, compactly supported smooth test functions φ .

As the first step in relating the various classes of supersolutions we will show that continuous very weak supersolutions can be seen as supersolutions to the boundary value problem in a space-time cylinder, whose base is a ball, with boundary values defined by the function itself. The known argument for solutions (see e.g. [9]) carries over to supersolutions without serious difficulties. However, the continuity assumption is essential in the proof.

LEMMA 4.3. *Let u be a non-negative, continuous very weak supersolution in Ω_T . Then for any $B_r \times (t_1, t_2) \Subset \Omega_T$, u is a very weak supersolution in $B_r \times (t_1, t_2)$ with boundary values $u|_{\partial_p B_r \times (t_1, t_2)}$.*

PROOF. Let η be a smooth function in $\overline{B_r \times (t_1, t_2)}$ vanishing on $\partial B_r \times (t_1, t_2)$. For $\varepsilon \in (0, r)$ and $\theta \in [0, \varepsilon]$ let $\Psi_{\varepsilon\theta}$ be the radial, continuous function satisfying

$$\Psi_{\varepsilon\theta}(\rho) = \begin{cases} 1 & \text{for } 0 \leq \rho \leq r - \varepsilon, \\ 0 & \text{for } \rho \geq r - \theta, \end{cases}$$

and

$$\Delta \Psi_{\varepsilon\theta}(x) = \frac{n-1}{|x|} \Psi'_{\varepsilon\theta}(|x|) + \Psi''_{\varepsilon\theta}(|x|) = 0 \quad \text{in } B_{r-\theta} \setminus B_{r-\varepsilon}.$$

By solving the equation we obtain

$$\begin{aligned} \Psi_{\varepsilon\theta}(\rho) &= \frac{\rho}{\theta - \varepsilon} + \frac{r - \theta}{\varepsilon - \theta}, \quad n = 1, \\ \Psi_{\varepsilon\theta}(\rho) &= \frac{\ln(\rho) - \ln(r - \theta)}{\ln(r - \varepsilon) - \ln(r - \theta)}, \quad n = 2, \\ \Psi_{\varepsilon\theta}(\rho) &= \frac{(r - \varepsilon)^n (\rho^{n-2} (r - \theta)^2 - (r - \theta)^n)}{(r - \varepsilon)^n (r - \theta)^2 - (r - \varepsilon)^2 (r - \theta)^n} \frac{1}{\rho^{n-2}}, \quad n > 2. \end{aligned}$$

From now on, we will assume that $n > 2$ for simplicity. A similar reasoning can be carried out also in the cases $n = 1, 2$. We observe that

$$\nabla \Psi_{\varepsilon\theta}(x) = \begin{cases} \frac{(n-2)(r-\varepsilon)^{n-2}(r-\theta)^{n-2}}{(r-\varepsilon)^{n-2} - (r-\theta)^{n-2}} \frac{x}{|x|^n} = -W_{\varepsilon\theta} \frac{x}{|x|^n} & \text{in } B_{r-\theta} \setminus B_{r-\varepsilon}, \\ 0 & \text{otherwise.} \end{cases}$$

Now $\Delta \Psi_{\varepsilon\theta}$ can be seen as the distribution

$$\int_B \varphi \Delta \Psi_{\varepsilon\theta} dx = W_{\varepsilon\theta} \left(\int_{\partial B_{r-\theta}} \varphi dS - \int_{\partial B_{r-\varepsilon}} \varphi dS \right).$$

Let K_v be a standard mollifier, i.e. a smooth, positive, radially symmetric function supported in $B_v(0)$ with the property $\int K_v dx = 1$. Define

$$\Psi_{\varepsilon\theta}^v(x) = \Psi_{\varepsilon\theta} * K_v(x).$$

Let $\phi_\lambda(t)$ be smooth functions with compact support in $(0, T)$, converging to

$$H_{t_1}(t) = \begin{cases} 0 & \text{for } t < t_1, \\ 1 & \text{for } t \geq t_1, \end{cases}$$

as $\lambda \rightarrow 0$.

Define $\varphi(x, t) = \Psi_{\varepsilon\theta}^v(x) \phi_\lambda(t) \eta(x, t)$. Now φ is a smooth, compactly supported function in Ω_T and thus

$$(4.1) \quad \int_{\Omega_T} (-u \varphi_t - u^m \Delta \varphi) dx dt \geq 0.$$

Since

$$\begin{aligned}\Delta\varphi &= \phi_\lambda(\Psi_{\varepsilon\theta}^\nu\Delta\eta + 2\nabla\Psi_{\varepsilon\theta}^\nu \cdot \nabla\eta + \eta\Delta\Psi_{\varepsilon\theta}^\nu) \quad \text{and} \\ \varphi_t &= \Psi_{\varepsilon\theta}^\nu(\phi_\lambda\eta_t + (\phi_\lambda)_t\eta),\end{aligned}$$

inequality (4.1) can be written as

$$\begin{aligned}(4.2) \quad 0 &\leq \int_{t_1}^{t_2} \int_B (-u^m \phi_\lambda \Psi_{\varepsilon\theta}^\nu \Delta\eta - u \Psi_{\varepsilon\theta}^\nu \phi_\lambda \eta_t) dx dt - \int_{t_1}^{t_2} \int_B 2u^m \phi_\lambda \nabla\Psi_{\varepsilon\theta}^\nu \cdot \nabla\eta dx dt \\ &\quad - \int_{t_1}^{t_2} \int_B u^m \phi_\lambda \eta \Delta\Psi_{\varepsilon\theta}^\nu dx dt - \int_{t_1}^{t_2} \int_B u \Psi_{\varepsilon\theta}^\nu (\phi_\lambda)_t \eta dx dt = I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Letting $\nu \rightarrow 0$, $\theta \rightarrow 0$, $\varepsilon \rightarrow 0$ and $\lambda \rightarrow 0$ gives us

$$I_1 \rightarrow \int_{t_1}^{t_2} \int_B (-u^m \Delta\eta - u \eta_t) dx dt.$$

Letting $\nu \rightarrow 0$ and $\lambda \rightarrow 0$ and taking $\text{supp}(\nabla\Psi_{\varepsilon\theta})$ into account,

$$\begin{aligned}I_2 &= -2 \int_{t_1}^{t_2} \int_{B_{r-\theta} \setminus B_{r-\varepsilon}} u^m \nabla\Psi_{\varepsilon\theta} \cdot \nabla\eta dx dt \\ &= 2W_{\varepsilon\theta} \int_{t_1}^{t_2} \int_{B_{r-\theta} \setminus B_{r-\varepsilon}} u^m \frac{x}{|x|^n} \cdot \nabla\eta dx dt \\ &= 2W_{\varepsilon\theta} \int_{t_1}^{t_2} \int_{r-\varepsilon}^{r-\theta} \int_{S^{n-1}} u^m \partial_\nu \eta|_{|x|=\rho} dS d\rho dt \\ &\rightarrow 2W_{\varepsilon 0} \int_{t_1}^{t_2} \int_{r-\varepsilon}^r \int_{S^{n-1}} u^m \partial_\nu \eta|_{|x|=\rho} dS d\rho dt\end{aligned}$$

as $\theta \rightarrow 0$. Since

$$W_{\varepsilon 0} = \frac{(n-2)(r-\varepsilon)^{n-2}r^{n-2}}{r^{n-2} - (r-\varepsilon)^{n-2}} = \frac{(r-\varepsilon)^{n-2}r^{n-2}}{\varepsilon\xi^{n-3}},$$

where $\xi \in (r-\varepsilon, r)$, we get

$$\begin{aligned}2W_{\varepsilon 0} &\int_{t_1}^{t_2} \int_{r-\varepsilon}^r \int_{S^{n-1}} u^m \partial_\nu \eta|_{|x|=\rho} dS d\rho dt \\ &= 2 \frac{(r-\varepsilon)^{n-2}r^{n-2}}{\xi^{n-3}} \int_{t_1}^{t_2} \frac{1}{\varepsilon} \int_{r-\varepsilon}^r \int_{S^{n-1}} u^m \partial_\nu \eta|_{|x|=\rho} dS d\rho dt \\ &\rightarrow 2r^{n-1} \int_{t_1}^{t_2} \int_{S^{n-1}} u^m \partial_\nu \eta|_{|x|=r} dS dt.\end{aligned}$$

Since u is continuous,

$$I_3 \rightarrow -W_{\varepsilon\theta} \int_{t_1}^{t_2} \left(\int_{\partial B_{r-\theta}} u^m \eta dS - \int_{\partial B_{r-\varepsilon}} u^m \eta dS \right) dt,$$

as $\nu \rightarrow 0$ and $\lambda \rightarrow 0$, due to the weak convergence of the measures $\Delta \Psi_{\varepsilon\theta}^\nu$. Note that the continuity assumption is essential here, as we use weak convergence for signed measures. Now, as $\theta \rightarrow 0$ we get

$$\begin{aligned} & -W_{\varepsilon 0} \int_{t_1}^{t_2} \int_{S^{n-1}} u^m \left(\eta|_{|x|=r} - \eta|_{|x|=r-\varepsilon} \right) dS dt \\ &= -\frac{(r-\varepsilon)^{n-2} r^{n-2}}{\varepsilon^{n-3}} \int_{t_1}^{t_2} \int_{S^{n-1}} u^m \left(\frac{\eta|_{|x|=r} - \eta|_{|x|=r-\varepsilon}}{\varepsilon} \right) dS dt \\ &\rightarrow -r^{n-1} \int_{t_1}^{t_2} \int_{S^{n-1}} u^m \partial_\nu \eta|_{|x|=r} dS dt. \end{aligned}$$

Finally, letting $\nu \rightarrow 0$, $\theta \rightarrow 0$ and $\varepsilon \rightarrow 0$ gives us

$$I_4 \rightarrow - \int_{t_1}^{t_2} \int_B u(\phi_\lambda)_t \eta dx dt \rightarrow - \int_B u(x, 0) \eta(x, 0) dx$$

as $\lambda \rightarrow 0$. Now we may conclude from inequality (4.2) that

$$\begin{aligned} & \int_{t_1}^{t_2} \int_B (-u^m \Delta \eta - u \eta_t) dx dt + \int_{t_1}^{t_2} \int_{\partial B} u^m \partial_\nu \eta dS dt \\ & - \int_B u(x, 0) \eta(x, 0) dx \geq 0. \end{aligned} \quad \square$$

The next step is to show that continuous very weak supersolutions satisfy the comparison principle with continuous very weak solutions in the special case where we look at a cylinder whose base is a ball. Since weak solutions are also very weak solutions, this lemma is the key to showing that continuous very weak supersolutions are indeed m -superporous functions in the sense of Definition 5.1 below.

LEMMA 4.4. *Let u be a continuous very weak supersolution and let v be a continuous very weak solution in Ω_T . Let $U_{t_1, t_2} = B_r \times [t_1, t_2] \Subset \Omega_T$. Then if $u \geq v$ on $\partial_p U_{t_1, t_2}$, then $u \geq v$ in U_{t_1, t_2} .*

PROOF. Since u is a continuous very weak supersolution, Lemma 4.3 gives

$$\begin{aligned} & \int_{t_1}^{t_2} \int_B (-u^m \Delta \varphi - u \varphi_t) dx dt + \int_{t_1}^{t_2} \int_{\partial B} u^m \partial_\nu \varphi dS dt \\ & - \int_B u(x, 0) \varphi(x, 0) dx \geq 0 \end{aligned}$$

for every smooth φ vanishing on $\partial B_r \times (t_1, t_2)$. By definition of very weak solutions

$$\begin{aligned} & \int_{t_1}^{t_2} \int_B (-v^m \Delta \varphi - v \varphi_t) dx dt + \int_{t_1}^{t_2} \int_{\partial B} v^m \partial_\nu \varphi dS dt \\ & - \int_B v(x, 0) \varphi(x, 0) dx = 0. \end{aligned}$$

Subtracting the inequalities gives

$$\begin{aligned} & \int_{t_1}^{t_2} \int_B ((v^m - u^m) \Delta \varphi + (v - u) \varphi_t) dx dt - \int_{t_1}^{t_2} \int_{\partial B} (v^m - u^m) \partial_\nu \varphi dS dt \\ & + \int_B (v(x, 0) - u(x, 0)) \varphi(x, 0) dx \geq 0. \end{aligned}$$

In fact, we could have assumed v is only a very weak subsolution to get the same inequality. Now we are at similar situation as in (3.1) with inequality instead of equality. However, the same reasoning still applies, and thus we may use Lemma 3.2 to conclude that $u \geq v$ in U_{t_1, t_2} . Note that u and v are continuous functions and thus $u, v, u^m, v^m \in L^2(U_{t_1, t_2})$ so the assumptions of Lemma 3.3 hold for u and v . \square

The following lemma extends the comparison property to finite unions of space-time cylinders whose bases are balls. We utilize a Schwarz type alternating method. The proof is delicate since we need to work around the fact that constants cannot be added to solutions.

LEMMA 4.5. *Let $B_i \subset \mathbb{R}^n$, $i = 1, \dots, N$ be a collection of balls and let $U_i = B_i \times (t_1, t_2)$. Set $K = \bigcup_{i=1}^N U_i$. Suppose that u satisfies the comparison principle for cylinders whose base is a ball in a neighbourhood of \overline{K} . That is, if h is a continuous weak solution such that $h \leq u$ on $\partial_p U$, where U is a cylinder whose base is a ball, then $h \leq u$ in U .*

Then the comparison principle for u holds also in K . That is, if h is a solution of the PME in K , which is continuous up to the boundary of K , then $h \leq u$ on $\partial_p K$ implies $h \leq u$ in K .

PROOF. Let $\delta > 0$. Take $\varphi \in C^\infty(K) \cap C(\overline{K})$ such that

$$\begin{aligned} \varphi &\leq u \quad \text{in } K \cup \partial_p K, \\ h - \delta &\leq \varphi \quad \text{on } \partial_p K. \end{aligned}$$

Let Ψ_0 be a continuous weak subsolution to the PME in K satisfying

$$\begin{aligned} \Psi_0 &= \varphi \quad \text{on } \partial_p K, \\ \Psi_0 &\leq \varphi \quad \text{in } K. \end{aligned}$$

Such a subsolution can be constructed by the arguments leading to Theorem 2.6 of [4]. We want to construct an increasing sequence of continuous weak subsolutions v_k such that $v_k \rightarrow w$, where w is a continuous weak solution. Set $v_0 = \Psi_0$. For $1 \leq i \leq N$ and $j \geq 0$ we define the functions recursively by

$$v_{Nj+i} = \begin{cases} \tilde{v}_{Nj+i-1} & \text{in } U_i, \\ v_{Nj+i-1} & \text{in } K \setminus U_i, \end{cases}$$

where \tilde{v}_{Nj+i-1} is the continuous weak solution in U_i with boundary values v_{Nj+i-1} on $\partial_p U_i$. Existence and continuity of \tilde{v}_{Nj+i-1} are provided by [13]. Thus v_k is a continuous weak subsolution for each k .

We want to show that the sequence v_k converges to a continuous weak solution. Since v_{Nj+i-1} is a continuous weak subsolution in U_i and v_{Nj+i} is a continuous weak solution in U_i , we may use the comparison principle for subsolutions in U_i . By construction, v_{Nj+i-1} and v_{Nj+i} coincide on $\partial_p U_i$ and thus $v_{Nj+i-1} \leq v_{Nj+i}$ in U_i . Hence v_k is an increasing sequence.

The function v_0 has been chosen in such a way, that $v_0 \leq \varphi \leq u$ in K . Suppose $v_{Nj+i-1} \leq u$ in K . Now $v_{Nj+i} \leq u$ on $\partial_p U_i$ by construction and therefore since u satisfies the comparison principle in U by assumption, $v_{Nj+i} \leq u$ in U_i . It follows by induction, that $v_k \leq u$ in K for all k . Now v_k is bounded and increasing and thus $v_k \rightarrow w \leq u$ for some w in \overline{K} .

Weak solutions are locally Hölder continuous (see [7]), so for each $z \in K$, there are $i_z \in \mathbb{N}$ and $r_z > 0$ such that $B(z, r_z) \subset U_{i_z}$ and v_{Nj+i_z} is Hölder continuous in $B(z, r_z)$ for every j . Therefore the subsequence v_{Nj+i_z} converges to a continuous function in $B(z, r_z)$. Since $v_k \rightarrow w$, we conclude that w is continuous in K . To show the continuity of w up to the boundary, let h_φ be the continuous weak solution in K , with boundary values φ on $\partial_p K$. By construction $v_0 \leq w$ and by the comparison principle $w \leq h_\varphi$ in $K \cup \partial_p K$. Since $v_0 = h_\varphi$ on $\partial_p K$ and v_0, h_φ are continuous, we conclude that w is continuous in $K \cup \partial_p K$.

Finally, we need to show that w is indeed a continuous weak solution in K . It suffices to show that w is a continuous weak solution in $U_i^\rho = B_{i_0} \times (t_1, t_2 - \rho)$ for every $1 \leq i_0 \leq N$ and $\rho > 0$. The sequence v_k is increasing, each v_k is continuous in \overline{K} and w is continuous in $K \cup \partial_p K$. Therefore $v_k \rightarrow w$ uniformly in $K \cap \{t \leq t_2 - \rho\}$. Thus for every $\varepsilon > 0$ there is j_ε such that for $j \geq j_\varepsilon$, we have

$$|w - v_{Nj+i_0}| < \varepsilon \quad \text{in } \overline{U_{i_0}^\rho}.$$

Let w' be a continuous weak solution in $U_{i_0}^\rho$ with boundary values w on $\partial_p U_{i_0}^\rho$. Now

$$v_{Nj+i_0} \leq w' + \varepsilon, \quad \text{and} \quad w' \leq v_{Nj+i_0} + \varepsilon \quad \text{on } \partial_p U_{i_0}^\rho.$$

Let w_ε be the continuous weak solution in $U_{i_0}^\rho$ with boundary values $w' + \varepsilon$ on $\partial_p U_{i_0}^\rho$ and let v_ε be the continuous weak solution in $U_{i_0}^\rho$ with boundary values $v_{Nj+i_0} + \varepsilon$ on $\partial_p U_{i_0}^\rho$. Then by the comparison principle for weak solutions $v_{Nj+i_0} \leq w_\varepsilon$ and $w' \leq v_\varepsilon$ in $U_{i_0}^\rho$. Since

$$\begin{aligned} |w' - v_{Nj+i_0}| &= (w' - v_{Nj+i_0})_+ + (v_{Nj+i_0} - w')_+ \\ &\leq (w_\varepsilon - v_{Nj+i_0}) + (v_\varepsilon - w'), \end{aligned}$$

we may use Lemma 3.5 to conclude $|v_{Nj+i_0} - w'| \rightarrow 0$ uniformly in $U_{i_0}^\rho$ (passing to a subsequence, if necessary). Therefore we may assume

$$v_{Nj+i_0} - \varepsilon \leq w' \leq v_{Nj+i_0} + \varepsilon \quad \text{in } \overline{U_{i_0}^\rho}.$$

Now

$$|w - w'| \leq |w - v_{Nj+i_0}| + |v_{Nj+i_0} - w'| < 2\varepsilon \quad \text{in } \overline{U_{i_0}^\rho}.$$

Letting $\varepsilon \rightarrow 0$ shows that $w' = w$ and thus w is a continuous weak solution in $U_{i_0}^\rho$. Denote by w_δ the continuous weak solution in K with boundary values $\varphi + \delta$ on $\partial_p K$. Then $w_\delta \geq h$ on $\partial_p K$ and thus by comparison principle for the continuous weak solutions, the inequality holds in K . Therefore

$$0 \leq (h - w)_+ (h^m - w^m)_+ \leq (w_\delta - w)(w_\delta^m - w^m).$$

Lemma 3.5 gives us

$$\begin{aligned} 0 &\leq \int_K (h - w)_+ (h^m - w^m)_+ dx dt \leq \int_K (w_\delta - w)(w_\delta^m - w^m) dx dt \\ &\leq \delta |K| (\sup \varphi + 1) + \delta |K| (\sup \varphi + 1)^m. \end{aligned}$$

Since this holds for any $\delta > 0$, letting $\delta \rightarrow 0$ shows that $h \leq w$ in K . On the other hand, $w \leq u$ by construction and thus $h \leq u$ as we wanted. \square

5. m -superporous functions. Another important class of supersolutions is the class of m -superporous functions, defined in terms of a comparison principle with respect to continuous weak solutions. This class is analogous to superharmonic functions in classical potential theory, where the definition is due to Riesz.

DEFINITION 5.1. A function $u : \Omega_T \rightarrow [0, \infty]$ is m -superporous, if

- (1) u is lower semicontinuous,
- (2) u is finite in a dense subset of Ω_T , and
- (3) the following parabolic comparison principle holds: Let $U_{t_1, t_2} \Subset \Omega_T$, and let h be a weak solution to the PME which is continuous in $\overline{U_{t_1, t_2}}$. Then, if $h \leq u$ on $\partial_p U_{t_1, t_2}$, $h \leq u$ also in U_{t_1, t_2} .

Our aim in this section is to connect m -superporous functions to the notions of weak and very weak supersolutions, i.e. to prove Theorem 1.1. The first step is the next lemma, which shows that continuous very weak supersolutions are m -superporous. This is essentially a consequence of Lemma 4.5, but some care is again required due to the fact that constants may not be added to solutions.

LEMMA 5.2. Let u be a continuous very weak supersolution to (1.1) in Ω_T . Then u is m -superporous.

PROOF. Let $U_{t_1, t_2} \Subset \Omega_T$ and let h be a continuous weak solution such that $h \leq u$ on $\partial_p U_{t_1, t_2}$. We want to show, that $h \leq u$ in U_{t_1, t_2} . Take $\varepsilon > 0$ and define the set

$$D = \{(x, t) \in \overline{U_{t_1, t_2}} : h \geq u + \varepsilon\}.$$

Now D is compact and by the assumption $D \subset U \times [t_1, t_2]$. Thus D has a finite covering

$$K = \bigcup_{i=1}^N B_i \times [t_1, t_2],$$

where B_i are balls, such that $\overline{B_i} \subset U$. Since $D \subset K$, we have $\partial_p K \subset U_{t_1, t_2} \setminus D$ and therefore $h < u + \varepsilon$ on $\partial_p K$. Let u_ε be the continuous weak solution with boundary values $u + \varepsilon$ on $\partial_p K$. Then by the comparison principle $h \leq u_\varepsilon$ in K . By Lemma 4.4 and Lemma 4.5, u satisfies the comparison principle in K and thus $u \leq u_\varepsilon$ in K . Now

$$0 \leq (h - u)_+ (h^m - u^m)_+ \leq (u_\varepsilon - u)(u_\varepsilon^m - u^m)$$

and so by Lemma 3.5

$$\begin{aligned} 0 &\leq \int_K (h - u)_+ (h^m - u^m)_+ dx dt \leq \int_K (u_\varepsilon - u)(u_\varepsilon^m - u^m) dx dt \\ &\leq \varepsilon |K| (\sup u + 1) + \varepsilon |K| (\sup u + 1)^m. \end{aligned}$$

By construction of the set D , we have $h \leq u + \varepsilon$ in $U_{t_1, t_2} \setminus D$. Thus letting $\varepsilon \rightarrow 0$ shows that $h \leq u$ in U_{t_1, t_2} . We conclude that u is m -superporous in Ω_T . \square

The other nontrivial fact needed for Theorem 1.1 is that locally bounded m -superporous functions are weak supersolutions. For this purpose, we next present a Caccioppoli type estimate for the weak supersolutions.

LEMMA 5.3. *Let $u^m \in L^2(0, T; H^1(\Omega))$ be a weak supersolution, such that $u \leq M$ in Ω_T for some $M > 0$. Then*

$$\int_{\Omega_T} \zeta^2 |\nabla u^m|^2 dx dt \leq 16M^{2m} T \int_{\Omega} |\nabla \zeta|^2 dx + 4M^{m+1} \int_{\Omega} \zeta^2 dx,$$

for every non-negative $\zeta \in C_0^\infty(\Omega)$. Note that ζ depends only on x .

PROOF. Formally, we use the test function $\varphi = (M^m - u^m)\zeta^2$ in the definition of weak supersolutions. However, since no regularity for u is assumed in the time variable, we need to use a time-regularized inequality to avoid the appearance of the possibly nonexistent quantity u_t . The proof is then just a straightforward computation. For the details, we refer to [11, Lemma 2.15]. \square

The next step is to show that locally bounded m -superporous functions are weak supersolutions. The idea of the proof is from [16]: one approximates a given m -superporous function pointwise by solutions to the obstacle problem. The approximants are weak supersolutions, so the claim then follows from the Caccioppoli estimate. For the PME, this argument has been carried out in [11].

LEMMA 5.4. *Let u be a locally bounded m -superporous function in Ω_T . Then u is a weak supersolution in Ω_T .*

PROOF. We give the main points of the argument, referring to the proof of Theorem 3.2 in [11] for the full details. Since u is lower semicontinuous, there exists a sequence of functions $\psi_k \in C^\infty(\Omega_T)$, such that

$$\psi_i < \psi_{i+1} \quad \text{for every } i,$$

and $\lim_{k \rightarrow \infty} \psi_k(x, t) = u(x, t)$ for every $(x, t) \in \Omega_T$. Without loss of generality, we may consider a set $Q_{t_1, t_2} \Subset \Omega_T$. For each k , let u_k be the solution to the obstacle problem with obstacle function ψ_k . By Theorem 2.6 in [4], a solution u_k exist, such that

$$\begin{aligned} u_k &= \psi_k \text{ on } \partial_p Q_{t_1, t_2}, \\ u_k &\geq \psi_k \text{ in } Q_{t_1, t_2} \quad \text{and} \\ u_k^m &\in L^2(t_1, t_2; H^1(Q)). \end{aligned}$$

Moreover, u_k is a continuous weak supersolution in Q_{t_1, t_2} , and a weak solution in the open set $\{u_k > \psi_k\}$. The latter fact and the comparison principle of Remark 3.4 imply that

$$\begin{aligned} u_1 &\leq u_2 \leq \dots \quad \text{and} \\ u_k &\leq u \quad \text{for every } k. \end{aligned}$$

Now we have $u_k \rightarrow u$ due to the inequalities

$$\psi_k \leq u_k \leq u;$$

recall that $\lim_{k \rightarrow \infty} \psi_k(x, t) = u(x, t)$.

Finally, the fact that u is indeed a weak supersolution to the porous medium equation follows from the Caccioppoli estimate (Lemma 5.3) and weak compactness. \square

We now have everything we need to prove the second main result.

PROOF OF THEOREM 1.1. Let u be a continuous weak supersolution in Ω_T . By the definition of weak derivatives, it is clear that u is also a very weak supersolution. Let then u be a continuous very weak supersolution. The comparison property with respect to continuous weak solutions for any space-time cylinder $U_{t_1, t_2} \Subset \Omega_T$ is the content of Lemma 5.2. Thus continuous very weak supersolutions are m -superporous. Finally, a continuous m -superporous function u is locally bounded, and hence a weak supersolution by Lemma 5.4. \square

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DEPARTMENT OF MATHEMATICS
P.O. BOX 11100, FI-00076
AALTO UNIVERSITY
FINLAND

E-mail addresses: pekka.lehtela@aalto.fi
teemu.lukkari@aalto.fi