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REPLACING THE LOWER CURVATURE BOUND IN TOPONOGOV'S COMPARISON THEOREM BY A WEAKER HYPOTHESIS

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Abstract. Toponogov's triangle comparison theorem and its generalizations are important tools for studying the topology of Riemannian manifolds. In these theorems, one assumes that the curvature of a given manifold is bounded from below by the curvature of a model surface. The models are either of constant curvature, or, in the generalizations, rotationally symmetric about some point. One concludes that geodesic triangles in the manifold correspond to geodesic triangles in the model surface which have the same corresponding side lengths, but smaller corresponding angles. In addition, a certain rigidity holds: Whenever there is equality in one of the corresponding angles, the geodesic triangle in the surface embeds totally geodesically and isometrically in the manifold.

In this paper, we discuss a condition relating the geometry of a Riemannian manifold to that of a model surface which is weaker than the usual curvature hypothesis in the generalized Toponogov theorems, but yet is strong enough to ensure that a geodesic triangle in the manifold has a corresponding triangle in the model with the same corresponding side lengths, but smaller corresponding angles. In contrast, it is interesting that rigidity fails in this setting.

1. Introduction. Let us briefly recall the statement of Toponogov's Triangle Comparison Theorem, also known as the Alexandrov–Toponogov Theorem.

THEOREM 1.1. *Let M be a complete Riemannian manifold whose curvature is bounded below by the constant κ, and let $\widetilde{M}$ be the complete, simply connected surface of constant curvature κ. Given a geodesic triangle $\triangle opq$ in M, there exists a geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ whose corresponding sides have the same lengths as those in $\triangle opq$ and which satisfy the following three properties:*

(1) *Angle comparison: Each of the three angles in $\triangle\tilde{o}\tilde{p}\tilde{q}$ are less than or equal to the corresponding angles of $\triangle opq$, that is $\angle\tilde{o} \le \angle o$, $\angle\tilde{p} \le \angle p$, and $\angle\tilde{q} \le \angle q$.*
(2) *Alexandrov convexity:* $\mathrm{dist}(\tilde{o}, \tilde{\sigma}(t)) \le \mathrm{dist}(o, \sigma(t))$ *for all $0 < t < l$ where σ and $\tilde{\sigma}$ are the minimal geodesics of length $l = \mathrm{dist}(p, q)$ joining p to q and $\tilde{p}$ and $\tilde{q}$ respectively.*
(3) *Rigidity: If equality holds in (1) or (2), then the interior of $\triangle\tilde{o}\tilde{p}\tilde{q}$ can be isometrically embedded as a totally geodesic surface in M.*

Over the years, numerous differential geometers have developed generalizations of Topogonov's Theorem in which the constant curvature surface is replaced by a surface $\widetilde{M}$ that is rotationally symmetric about a vertex $\tilde{o}$ with variable curvature $\kappa(r)$ depending on the dis-

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tance r from $\tilde{o}$, and in which the curvature of M at a distance r from a fixed base point $o \in M$ is bounded from below by $\kappa(r)$. For a sampling of the literature, see [4, 1, 7, 12, 13, 14, 9]. In this setting one only considers geodesic triangles in M having a vertex at the base point o. The most general version [9], for which the existence of a geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ satisfying the conclusion of Toponogov's Theorem is obtained for every geodesic triangle $\triangle opq$ in M, requires the cut points of every $\tilde{p} \in \widetilde{M}$ to lie on the meridian opposite from $\tilde{p}$. This holds, for example, when $\widetilde{M}$ is a von Mangoldt surface [18, 17]. On the other hand, some recent versions relax the restriction on cut points in $\widetilde{M}$, but then impose other restrictions [10, 6].

In this paper we replace the lower curvature bound by a weaker condition that we call $(\widetilde{M}, \tilde{o})$ has *weaker radial attraction* than (M, o). If L_o and $L_{\tilde{o}}$ denote the distance functions from o and $\tilde{o}$ respectively, having weaker radial attraction means that given any pair of geodesics, $\tilde{\sigma}$ in $\widetilde{M}$ and σ in M satisfying $(L_{\tilde{o}} \circ \tilde{\sigma})(0) = (L_o \circ \sigma)(0)$ and $(L_{\tilde{o}} \circ \tilde{\sigma})'_+(0) = (L_o \circ \sigma)'_+(0)$, then there exists an $\epsilon > 0$ such that $(L_{\tilde{o}} \circ \tilde{\sigma})(t) \geq (L_o \circ \sigma)(t)$ for all $0 \leq t < \epsilon$.

We motivate the terminology by imagining a geodesic to be the path of a free particle and thinking of a base point as a point of attraction. The condition on the pair of geodesics can be interpreted as saying that if the distances of the particles from the base points and the radial components of their velocities are equal at an initial time, then for a short time later, the particle experiencing the stronger attraction moves so as to be closer to the base point than the particle that is experiencing the weaker attraction.

Our main theorem, proved in Section 4, is the following:

THEOREM 1.2. *Let M be a complete Riemannian manifold with a base point o, and let $\widetilde{M}$ be a complete, simply connected surface which is rotationally symmetric about the vertex $\tilde{o}$ such that the cut locus of every point $\tilde{p}$ lies in the opposite meridian of $\tilde{p}$. Assume that $(\widetilde{M}, \tilde{o})$ has weaker radial attraction than (M, o). Then, for every geodesic triangle $\triangle opq$ in M, there exists a geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ whose corresponding sides are equal and which satisfies the angle comparison and Alexandrov convexity in Toponogov's Theorem.*

Interestingly, rigidity fails in this setting. See Example 2 in Section 6. This shows that the radial attraction condition is more flexible than the lower curvature bound and highlights the essential role that curvature plays in the rigidity portion of Toponogov's Theorem and its generalizations.

In the course of proving Theorem 1.2, we establish a Maximal Radius Theorem [Remark 4.8] when $\widetilde{M}$ is compact.

In Section 5 we relate having weaker radial attraction to a comparison of the Hessians of L_o and $L_{\tilde{o}}$, and as well as to a comparison of the principal curvatures of the geodesic spheres about o to the curvature of the geodesic circles about $\tilde{o}$ of the same radius. From this, on account of the Hessian Comparison Theorem of Green and Wu [5], we conclude that if the radial curvature of (M, o) is bounded from below by the curvature of the surface $(\widetilde{M}, \tilde{o})$, then $(\widetilde{M}, \tilde{o})$ has weaker radial attraction than (M, o). The converse in general is false. (See Remark 4.2 and Example 1 in Section 6.)

In view of Proposition 4.13, $(\widetilde{M}, \tilde{o})$ necessarily has weaker radial attraction than (M, o), if every geodesic triangle $\triangle opq$ in M has a corresponding geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ with equal corresponding sides but smaller corresponding angles. This can be regarded as a partial converse to Theorem 1.2.

2. Distance functions from a point.

If M is a complete Riemannian manifold, the distance between two points p and q in M will be denoted $\mathrm{dist}(p, q)$. The set of all minimizing geodesics joining p to q will be denoted $Geod(p, q)$. Any $\gamma \in Geod(p, q)$ has length $\mathrm{dist}(p, q)$ and will be parameterized by arclength. The distance function from p is the function $L_p : M \to \mathbb{R}^+$ defined by $L_p(q) = \mathrm{dist}(p, q)$.

2.1. The first derivative of L_p along curves.

LEMMA 2.1. *Let M be a complete Riemannian manifold, $p \in M$ and $c : (a, b) \to M$ a C^∞ curve. Then for each $s \in (a, b)$ the left and right hand derivatives of $L_p \circ c$ exist and are given by*

$$(L_p \circ c)'_+(s) = \min\left\{\langle c'(s), \gamma'(l)\rangle : \gamma \in Geod(p, c(s))\right\}$$
$$(L_p \circ c)'_-(s) = \max\left\{\langle c'(s), \gamma'(l)\rangle : \gamma \in Geod(p, c(s))\right\}$$

where $l = \mathrm{dist}(p, c(s))$.

REMARK 2.2. Let $C(p)$ denote the cut locus of p. If $c(s) \notin C(p)$, then $(L_p \circ c)'(s)$ exists and equals $\langle c'(s), \gamma'(L_p(c(s)))\rangle$ where γ is the unique minimizing geodesic from p to $c(s)$. This is a consequence of the first variation formula. Thus the lemma gives new information when $c(s) \in C(p)$. For a more general lemma of this type, see [8, Lemma 2.1].

PROOF. Set $f = L_p \circ c$. Fix s_0. Let γ be any minimizing geodesic joining p to $c(s_0)$. Let $\bar{p}$ be a point on γ between p and $c(s_0)$. Then $c(s_0) \notin C(\bar{p})$, and setting

$$\bar{f}(s) = \mathrm{dist}(\bar{p}, c(s)) + \mathrm{dist}(p, \bar{p}), \tag{2.1}$$

one obtains

$$\bar{f}'(s_0) = \langle c'(s_0), \gamma'(f(s_0))\rangle$$

by the first variation formula (see above remark). By the triangle inequality

$$f(s) \le \bar{f}(s) \quad \text{for all} \quad s.$$

Also $\bar{f}(s_0) = f(s_0)$ because γ is minimizing. Hence, if $h > 0$,

$$\frac{f(s_0 + h) - f(s_0)}{h} \le \frac{\bar{f}(s_0 + h) - \bar{f}(s_0)}{h} \tag{2.2}$$

and

$$\frac{f(s_0 - h) - f(s_0)}{-h} \ge \frac{\bar{f}(s_0 - h) - \bar{f}(s_0)}{-h}. \tag{2.3}$$

Taking $\limsup_{h\to 0^+}$ of equation (2.2) and $\liminf_{h\to 0^+}$ of equation (2.3) give

$$D^+ f(s_0) \le D^+ \bar{f}(s_0) = \bar{f}'(s_0)$$

and

$$D_- f(s_0) \geq D_- \bar{f}(s_0) = \bar{f}'(s_0)$$

where D^+ and D_- denote the upper right and lower left Dini derivates respectively. Consequently, since $\gamma \in Geod(p, c(s_0))$ is arbitrary,

$$D^+ f(s_0) \leq \min\left\{\langle c'(s_0), \gamma'(f(s_0))\rangle : \gamma \in Geod(p, c(s_0))\right\} \tag{2.4}$$

and

$$D_- f(s_0) \geq \max\left\{\langle c'(s_0), \gamma'(f(s_0))\rangle : \gamma \in Geod(p, c(s_0))\right\} . \tag{2.5}$$

On setting

$$\check{\mu}(s) = \min\left\{\langle c'(s), \gamma'(f(s))\rangle : \gamma \in Geod(p, c(s))\right\}$$

and

$$\hat{\mu}(s) = \max\left\{\langle c'(s), \gamma'(f(s))\rangle : \gamma \in Geod(p, c(s))\right\}$$

it follows that

$$\liminf_{s\to s_0^+} \hat{\mu}(s) \geq \check{\mu}(s_0) . \tag{2.6}$$

To prove this, let s_n be a decreasing sequence converging to s_0 such that $\hat{\mu}(s_n)$ converges to $\liminf_{s\to s_0^+} \hat{\mu}(s)$. Then there exists a sequence of minimizing geodesics γ_n from p to $c(s_n)$ such that $\hat{\mu}(s_n) = \langle c'(s_n), \gamma_n'(f(s_n))\rangle$, which on passing to a subsequence may be supposed to converge to a $\gamma_0 \in Geod(p, c(s_0))$. On passing to the limit we have

$$\liminf_{s\to s_0^+} \hat{\mu}(s) = \langle c'(s_0), \gamma_0'((f(s_0))\rangle \geq \check{\mu}(s_0)$$

by definition of $\check{\mu}(s_0)$. Consequently, for every $m < \check{\mu}(s_0)$ there exists a $\delta > 0$ such that

$$\hat{\mu}(s) > m \quad \text{if} \quad s_0 < s < s_0 + \delta .$$

Thus by equation (2.5)

$$D_- f(s) > m \quad \text{if} \quad s_0 < s < s_0 + \delta .$$

Therefore, for every $s_0 < s < s_0 + \delta$, there exists an $\eta = \eta(s) > 0$ such that

$$\frac{f(s) - f(s-h)}{h} > m \quad \text{whenever} \quad 0 < h < \eta .$$

Consequently,

$$f(s) > f(s_0) + m(s - s_0) \quad \text{for all} \quad s_0 < s < s_0 + \delta$$

because f is continuous, that is,

$$\frac{f(s) - f(s_0)}{s - s_0} > m \quad \text{if} \quad s_0 < s < s_0 + \delta .$$

Therefore $D_+ f(s_0) \geq m$. But $m < \check{\mu}(s_0)$ is arbitrary, so that $D_+ f(s_0) \geq \check{\mu}(s_0)$. Therefore

$$\check{\mu}(s_0) \leq D_+ f(s_0) \leq D^+ f(s_0) \leq \check{\mu}(s_0)$$

where we used (2.4). Therefore

$$f'_+(s_0) = \check{\mu}(s_0)\,. \tag{2.7}$$

By a similar argument

$$f'_-(s_0) = \hat{\mu}(s_0)\,. \tag{2.8}$$

□

COROLLARY 2.3. *Under the hypothesis and notation of the previous lemma, for all $s \in (a, b)$,*

$$f'_-(s) \ge f'_+(s)\,.$$

Equality holds, that is, f is differentiable at s, if and only if tangent vectors $\gamma'(f(s))$ at $c(s)$ to every γ in $Geod(p, c(s))$ make the same angle with $c'(s)$, that is, they lie in the set $\{X \in T_{c(s)}M : \langle X, c'(s)\rangle = \mu\}$ where $\mu = \check{\mu}(s) = \hat{\mu}(s)$.

REMARK 2.4. Since the function f is locally Lipschitz, $f'(s)$ exists for almost all s [15, pp.105–108]. Moreover, f' is integrable, $f'_+ = f'$ almost everywhere, and

$$f(s_1) - f(s_0) = \int_{s_0}^{s_1} f'_+(s)ds\,.$$

2.2. The second derivative of L_p along geodesics. Let $\sigma : (a, b) \to M$ be a unit speed geodesic in M. If $\sigma(s_0) \notin C(p) \cup \{p\}$, then $L_p \circ \sigma$ is smooth near s_0, and its second derivative at s_0 can be computed by means of the second variation formula. Let $\gamma : [0, l] \to M$ be the unique minimizing geodesic joining p to $\sigma(s_0)$, and let J be the Jacobi field along γ satisfying $J(0) = 0$ and $J(l) = \sigma'(s_0)$. We have

$$(L_p \circ \sigma)''(s_0) = \int_0^l \langle \nabla_T J, \nabla_T J\rangle - \langle R(J, T)T, J\rangle - \langle \nabla_T J, T\rangle^2 dt \tag{2.9}$$

where $T = \gamma'$ is the tangent velocity vector field along γ. It might be noted that the term $\langle \nabla_T J, T\rangle$ is a constant equal to $(L_p \circ \sigma)'(0)/l$. (See [3, p. 19].)

Since L_p is smooth in the open complement of $C(p) \cup \{p\}$, its Hessian $\nabla^2 L_p$ is the symmetric tensor field defined by

$$\nabla^2 L_p(X, Y) = X(Y(L_p)) - dL_p(\nabla_X Y) \tag{2.10}$$

for two smooth vector fields X and Y in the complement of $C(p) \cup \{p\}$. Thus if σ is a geodesic in M with $\sigma(s_0)$ in $M \backslash C(p) \cup \{p\}$, then

$$(L_p \circ \sigma)''(s_0) = \nabla^2 L_p(\sigma'(s_0), \sigma'(s_0))\,. \tag{2.11}$$

On the other hand, when $\sigma(s_0)$ is in the cut locus of p, $L_p \circ \sigma$ may fail to be smooth near s_0. However in this case we can construct smooth upper support functions for $L_p \circ \sigma$ near s_0, that is, smooth functions F with $F(s) \ge L_p \circ \sigma(s)$ for all s near s_0 and which satisfy $F(s_0) = L_p \circ \sigma(s_0)$. The construction goes as follows: Let $\gamma : [0, l] \to M$ be a minimizing geodesic joining p to $\sigma(s_0)$ and let V be a piecewise smooth vector field along γ that satisfies $V(0) = 0$ and $V(l) = \sigma'(s_0)$. Let $v_s(t)$ be the variation of γ defined by

$$v_s(t) = \exp_{\gamma(t)}(sV(t)) \tag{2.12}$$

and set

$$F_V(s) = \mathcal{L}(v_s) = \int_0^l \langle v_s'(t), v_s'(t)\rangle^{\frac{1}{2}} dt \tag{2.13}$$

where $\mathcal{L}$ denotes the arclength. Obviously, F_V satisfies the conditions of being an upper support function for $L_p \circ \sigma$ near s_0. Applying the second variation formula gives

PROPOSITION 2.5.

$$F_V''(s_0) = \int_0^l \langle \nabla_T V, \nabla_T V\rangle - \langle R(V,T)T, V\rangle - (\langle \nabla_T V, T\rangle)^2 dt\,.$$

PROOF. The boundary terms in the second variation formula [3, p. 20] vanish since the transverse curves are geodesics. □

By choosing nice vector fields V we can obtain useful formulas for the second derivative of certain upper support functions. For example, let P denote the parallel transport of $\sigma'(s_0)$ along γ. For any t_0 between 0 and l, $\gamma(t_0) \notin C(p)$. Thus it is possible to define a unique Jacobi field J satisfying $J(0) = 0$ and $J(t_0) = P(t_0)$.

COROLLARY 2.6. *Let $V(t) = J(t)$ for $0 \le t \le t_0$ and $V(t) = P(t)$ for $t_0 \le t \le l$. Then V is piecewise smooth and*

$$F_V''(s_0) = \nabla^2 L_p(P(t_0), P(t_0)) - \left(1 - \langle P, T\rangle^2\right) \int_{t_0}^l \kappa(P \wedge T) dt$$

where $\kappa(P \wedge T)$ is the sectional curvature of the 2-plane spanned by P and T.

PROOF. Using additivity, write the integral in the Proposition 2.5 as the sum $\int_0^{t_0} + \int_{t_0}^l$. The first integral reduces to $\nabla^2 L_p(P(t_0), P(t_0))$ by (2.9) and (2.11) because J is a Jacobi field. Since P is parallel, the second integral reduces to $\int_{t_0}^l -\langle R(P,T)T, P\rangle dt$. Finally recall that

$$\kappa(P \wedge T) = \frac{\langle R(P,T)T, P\rangle}{|P|^2|T|^2 - \langle P, T\rangle^2} = \frac{\langle R(P,T)T, P\rangle}{1 - \langle P, T\rangle^2}\,.$$

It is important to note that the denominator is constant for $t_0 \le t \le l$ because P and T are parallel unit vector fields. □

The integral of the curvature in Corollary 2.6 can be made arbitrarily small by choosing t_0 sufficiently close to l.

As an aside we point out that the upper support function $\bar{f}$ defined in equation (2.1) can be obtained through the construction of Proposition 2.5 by taking V to be equal to 0 between p and $\bar{p}$, and equal to the Jacobi field J with $J(\bar{p}) = 0$ and $J(\sigma(s_0)) = \sigma'(s_0)$ between $\bar{p}$ and $\sigma(s_0)$.

2.3. Geodesic spheres and the Hessian of L_p. If $q \in M$ is not in the cut locus of $p \in M$, there is a unique geodesic $\gamma \in Geod(p, q)$. We will let T be the tangent vector to γ at q. Through such points q there passes the geodesic sphere centered at p of radius $\operatorname{dist}(p, q)$ which is a smooth hypersurface in a neighborhood of q with unit normal T at q. We will let II_q denote the second fundamental form of that hypersurface at q.

PROPOSITION 2.7. *If X and Y are tangent at q to the geodesic sphere centered at p passing through q then*

$$\nabla^2 L_p(X, Y) = -\langle T, II_q(X, Y)\rangle .$$

PROOF. Extend X and Y to smooth vector fields denoted by the same letters in a neighborhood of q which are tangent to the geodesic spheres centered at p in that neighborhood. By definition of the Hessian we have

$$\begin{aligned}\nabla^2 L_p(X, Y) &= X(Y(L_p)) - dL_p(\nabla_X Y) = -\langle T, \nabla_X Y\rangle \\ &= -\langle T, II_q(X, Y)\rangle\end{aligned}$$

when evaluated at q, since $Y(L_p) = 0$ because Y is tangent to the geodesic spheres on which L_p is constant, and since T is equal to the gradient of L_p at q. □

3. Model surfaces. Let $\widetilde{M}$ be a simply connected, complete surface which is rotationally symmetric about the vertex $\tilde{o}$. If $\widetilde{M}$ is compact, then $\widetilde{M}$ is diffeomorphic to the sphere S^2. In this case, we let $\tilde{o}'$ denote the point which is at the maximum distance ℓ from $\tilde{o}$. If $\widetilde{M}$ is not compact, then $\widetilde{M}$ is diffeomorphic to the plane $\mathbb{R}^2$, and we set $\ell = \infty$.

In the normal polar coordinate system centered at $\tilde{o}$, the Riemannian metric on $\widetilde{M}$ takes the form:

$$ds^2 = dr^2 + y(r)^2 d\theta^2$$

where y is a smooth function that satisfies $y(0) = 0$ and $y'(0) = 1$, and is strictly positive for r in the open interval $(0, \ell)$. Since the Riemannian metric on $\widetilde{M}$ is smooth, the function y extends to a smooth odd function $y : \mathbb{R} \to \mathbb{R}$, that is, $y(-r) = -y(r)$. In addition, when ℓ is finite, y satisfies $y(\ell) = 0$ and $y'(\ell) = -1$, and is antisymmetric about ℓ, that is, $y(r - \ell) = -y(\ell - r)$. Consequently, the extended function y is periodic with period 2ℓ.

In polar coordinates, geodesics $\gamma(t) = (r(t), \theta(t))$ satisfy the differential equations:

$$\begin{aligned}\ddot{r} &= y(r)y'(r)\dot{\theta}^2 \\ \ddot{\theta} &= -2\frac{y'(r)}{y(r)}\dot{\theta}\dot{r}\end{aligned}$$

where the dot indicates differentiation with respect to t. Since γ is assumed to be unit speed, we also have $\dot{r}^2 + y(r)^2\dot{\theta}^2 = 1$. Besides this, it is well known that the quantity

$$\left\langle \dot{\gamma}(t), \frac{\partial}{\partial\theta} \right\rangle = y(r(t))^2\dot{\theta} \tag{3.1}$$

is constant along γ. It is known as Clairaut's constant. ([16, pp. 212–213] is one reference for this material.)

We will be concerned with the set $\widetilde{M}^+$ of points in $\widetilde{M}$ that satisfy $0 \leq \theta \leq \pi$ and $0 \leq r \leq \ell$ and its interior $\mathrm{int}(\widetilde{M}^+)$ of points satisfying $0 < \theta < \pi$ and $0 < r < \ell$.

DEFINITION 3.1. For $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi$, define the functions

$$D_\theta : (0, \ell) \times [0, \ell) \to \mathbb{R}^+$$
$$R_\phi : (0, \ell) \times [0, \infty) \to \mathbb{R}^+$$

as follows: $D_\theta(r_1, r_2) = L_{\tilde{p}}(\tilde{q})$ where $\tilde{p}$ has coordinates $(r_1, 0)$ and $\tilde{q} \in \widetilde{M}^+$ has coordinates (r_2, θ). $R_\phi(r_1, t) = L_{\tilde{o}}(\sigma(t))$ where σ is the unit speed geodesic starting at $\tilde{p} = (r_1, 0)$ making an angle ϕ with the meridian $\theta = 0$. In particular $\cos\phi = \langle \frac{\partial}{\partial r}, \sigma'(0)\rangle$.

PROPOSITION 3.2. *D_θ and R_ϕ have the following monotonicity properties.*

(1) *For fixed $(r_1, r_2) \in (0, \ell) \times (0, \ell)$, $D_\theta(r_1, r_2)$ is strictly increasing for $0 \leq \theta \leq \pi$. (cf. [9, Lemma 2.1] or [16, Lemma 7.3.2].)*
(2) *For fixed $r_1 \in (0, \ell)$, if t is less than the injectivity radius of $\tilde{p}$, then $R_\phi(r_1, t)$ is strictly decreasing for $0 \leq \phi \leq \pi$. (cf. [11, Lemma 5.1].)*

PROOF. (1) Fix $(r_1, r_2) \in (0, \ell) \times (0, \ell)$ and set $f(\theta) = D_\theta(r_1, r_2)$. Let $\tilde{p} = (r_1, 0)$, and let $\tilde{q}(\theta) = (r_2, \theta)$ for $0 \leq \theta \leq \pi$. Then $f(\theta) = L_{\tilde{p}}(\tilde{q}(\theta))$ and $f'(\theta) = \langle \tilde{q}'(\theta), \gamma_\theta'(f(\theta))\rangle$ for some minimizing geodesic γ_θ joining $\tilde{p}$ to $\tilde{q}(\theta)$. Thus $f_+'(\theta) = \langle \frac{\partial}{\partial\theta}, \gamma_\theta'(f(\theta))\rangle$ is equal to Clairaut's constant for the geodesic γ_θ. Hence $f_+'(\theta) > 0$ when $0 < \theta < \pi$, because by (3.1) the Clairaut constant of any geodesic from $\tilde{p}$ entering $\mathrm{int}(\widetilde{M}^+)$ is strictly positive. Therefore f is strictly increasing on $[0, \pi]$.

(2) Setting $\tilde{p} = (r_1, 0)$ and σ_ϕ equal to the unit speed geodesic from $\tilde{p}$ making the angle ϕ with the meridian $\theta = 0$, it is clear that if t is less than the injectivity radius of $\tilde{p}$, then $\sigma_\phi(t)$, $0 \leq \phi \leq \pi$, traces out a smooth geodesic semicircle centered at $\tilde{p}$ in $\widetilde{M}^+$. Obviously the only critical points of $L_{\tilde{o}}$ restricted to this semicircle are a maximum at $\phi = 0$ and a minimum at $\phi = \pi$. □

The above proof shows that for fixed $(r_1, r_2) \in (0, \ell) \times (0, \ell)$, the function D_θ has a positive right-hand partial derivative $\partial_\theta^+ D_\theta(r_1, r_2)$. We can say more.

LEMMA 3.3. *Fix $r_1 \in (0, \ell)$, and let K be a compact subset of $\mathrm{int}(\widetilde{M}^+)$. Then there exists a constant $C > 0$ depending on r_1 and K such that $\partial_\theta^+ D_\theta(r_1, r_2) \geq C$ whenever $(r_2, \theta) \in K$.*

PROOF. The constant C is the infimum of the set of Clairaut constants of the minimizing geodesics joining the point $\tilde{p} = (r_1, 0)$ to the points of K. The infimum is positive because no sequence of minimizing geodesics from $\tilde{p}$ to points of K converges to a geodesic contained in the union of the meridians $\theta = 0$ and $\theta = \pi$. □

COROLLARY 3.4. *Let $K = \{\tilde{q} \in \widetilde{M}^+ : \tilde{q} = (r, \theta) \in I_1 \times I_2\}$ where $I_1 \subset (0, \ell)$ and $I_2 \subset (0, \pi)$ are closed bounded intervals. Suppose $\tilde{p} = (r_1, 0)$, $\tilde{q}_1 = (r_2, \theta_1)$, and $\tilde{q}_2 = (r_2, \theta_2)$. If $\tilde{q}_1, \tilde{q}_2 \in K$ and $\theta_2 > \theta_1$, then*

$$L_{\tilde{p}}(\tilde{q}_2) - L_{\tilde{p}}(\tilde{q}_1) \geq C(\theta_2 - \theta_1)$$

where $C > 0$ *is the constant associated with* K *in Lemma* 3.3. (*cf.* [10, Lemma 4.2].)

PROOF. The curve $c(\theta) = (r_2, \theta)$ for $0 < \theta < \pi$ is smooth in $\widetilde{M}$. By definition $(L_{\tilde{p}} \circ c)'_+(\theta) = \partial_\theta^+ D_\theta(r_1, r_2)$. Thus, by Remark 2.5 and Lemma 3.3,

$$L_{\tilde{p}}(\tilde{q}_2) - L_{\tilde{p}}(\tilde{q}_1) = L_{\tilde{p}}(c(\theta_2)) - L_{\tilde{p}}(c(\theta_1)) = \int_{\theta_1}^{\theta_2} (L_{\tilde{p}} \circ c)'_+(\theta) d\theta \geq \int_{\theta_1}^{\theta_2} C\, d\theta\,.$$

□

4. A generalized Toponogov theorem. The sides of a geodesic triangle $\triangle opq$ in a Riemannian manifold will always be minimizing geodesics joining the vertices, which are assumed to be distinct points. Although the notation $\triangle opq$ is ambiguous when there are more than one minimizing geodesic joining a pair of the vertices, there should be no confusion as it will always be clear which geodesic is part of the triangle. If the lengths of the sides of a geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in the model surface are equal to the lengths of the corresponding sides of $\triangle opq$, that is, $\mathrm{dist}(\tilde{o}, \tilde{p}) = \mathrm{dist}(o, p)$, $\mathrm{dist}(\tilde{o}, \tilde{q}) = \mathrm{dist}(o, q)$, and $\mathrm{dist}(\tilde{p}, \tilde{q}) = \mathrm{dist}(p, q)$, then $\triangle \tilde{o}\tilde{p}\tilde{q}$ will be called a *corresponding triangle* of $\triangle opq$.

DEFINITION 4.1. The model surface $(\widetilde{M}, \tilde{o})$ is said to have *weaker radial attraction* than the pointed complete Riemannian manifold (M, o), if, for any unit speed geodesics σ, $\tilde{\sigma}$ in M, $\widetilde{M}$ respectively satisfying $L_o \circ \sigma(0) = L_{\tilde{o}} \circ \tilde{\sigma}(0) < \ell$ and $(L_o \circ \sigma)'_+(0) = (L_{\tilde{o}} \circ \tilde{\sigma})'_+(0)$, then there exists an $\epsilon > 0$ such that $L_o \circ \sigma(t) \leq L_{\tilde{o}} \circ \tilde{\sigma}(t)$ for all $0 \leq t < \epsilon$. Recall $\ell = \max_{\tilde{x} \in \widetilde{M}} \mathrm{dist}(\tilde{o}, \tilde{x})$ in the case that $\widetilde{M}$ is compact, and $\ell = \infty$ if $\widetilde{M}$ is not compact.

REMARK 4.2. If the radial curvature of M is bounded below by $\widetilde{M}$, then $\widetilde{M}$ has weaker radial attraction than M by the corollary to the Berger-Rauch Theorem [3]. An alternative argument is given in Section 5. It is possible for $\widetilde{M}$ to have weaker radial attraction than M but not to bound the radial curvature from below. For this see Remark 5.2 and Example 1 in Section 6.

The portion of a geodesic ray emanating from $o \in M$ up to and including the cut point to o is a *maximal minimizing geodesic* emanating from o. Of course, if there is no cut point on the ray, then the entire ray is a maximal minimizing geodesic.

DEFINITION 4.3. We define an *axis* of (M, o) to be the union of two maximal minimizing geodesics emanating from o whose initial tangent vectors are the negatives of each other. Clearly, given a minimizing geodesic segment γ in M joining o to a point $p \neq o$, then there exists a unique axis A of (M, o) containing γ.

From now on assume that the model surface $(\widetilde{M}, \tilde{o})$ has weaker radial attraction than the complete pointed manifold (M, o) and that for each $\tilde{p}$ in $\widetilde{M}$ the cut locus $C(\tilde{p})$ of $\tilde{p}$ is contained in the opposite meridian. Thus for each $\tilde{q} \in \mathrm{int}(\widetilde{M}^+)$ there exists exactly one minimizing geodesic from $\tilde{p}$ to $\tilde{q}$ contained in $\widetilde{M}^+$. Fix $p \in M$ with $\mathrm{dist}(o, p) < \ell$, and let $\triangle opq$ be a geodesic triangle in M. Let A denote the axis containing the side op of $\triangle opq$.

LEMMA 4.4. *Assume $q \notin A$, and suppose $L_o(q) < \ell$ and* $\mathrm{dist}(p, q) < D_\pi(L_o(p), L_o(q))$. *Then there exists a corresponding triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ which satisfies Alexandrov convexity.*

PROOF. The hypotheses on q imply that there exists a unique corresponding triangle $\tilde{\triangle} = \triangle\tilde{o}\tilde{p}\tilde{q}$ with $\theta(\tilde{p}) = 0$ and $\tilde{q} \in \mathrm{int}(\widetilde{M}^+)$. Following the argument in (Section 4.1, [12]), we deform $\tilde{\triangle}$ through a family $\triangle^s$ for $0 \le s \le 1$ with $\triangle^1 = \tilde{\triangle}$.

During the first stage of the deformation, we shorten the two sides meeting $\tilde{o}$, but keep the length of $\tilde{p}\tilde{q}$ fixed. In detail, set $l_0 = \mathrm{dist}(p, q) = \mathrm{dist}(\tilde{p}, \tilde{q})$. Since $l_0 < D_\pi(L_o(p), L_o(q))$, there exists an $\bar{s} \in (0, 1)$ such that for all $\bar{s} \le s \le 1$, $l_0 < D_\pi(sL_o(p), sL_o(q))$. Consequently, if $\bar{s} \le s \le 1$, there exists a unique geodesic triangle $\triangle\tilde{o}\tilde{p}_s\tilde{q}_s$ such that $\theta(\tilde{p}_s) = 0$, $\tilde{q}_s \in \mathrm{int}(\widetilde{M}^+)$, $L_{\tilde{o}}(\tilde{p}_s) = sL_o(p)$, $L_{\tilde{o}}(\tilde{q}_s) = sL_o(q)$, and $\mathrm{dist}(\tilde{p}_s, \tilde{q}_s) = l_0$. For such s, set $\triangle^s = \triangle\tilde{o}\tilde{p}_s\tilde{q}_s$, and let $\sigma^s : [0, l_0] \to \widetilde{M}^+$ be the minimizing geodesic joining $\tilde{p}_s$ to $\tilde{q}_s$.

During the second stage of the deformation, the lengths of the two sides meeting $\tilde{o}$ are fixed while the base angle decreases to 0, and the side opposite $\tilde{o}$ is a broken geodesic of total length $\mathrm{dist}(p, q)$. In detail, for every $0 \le s \le \bar{s}$, let $\tilde{p}_s = \tilde{p}_{\bar{s}}$, and let $\tilde{q}_s$ be the unique point in $\widetilde{M}^+$ satisfying $L_{\tilde{o}}(\tilde{q}_s) = L_{\tilde{o}}(\tilde{q}_{\bar{s}})$ and $\theta(\tilde{q}_s) = \frac{s}{\bar{s}}\theta(\tilde{q}_{\bar{s}})$. If $\tilde{\gamma}_s$ denotes the minimizing geodesic joining $\tilde{o}$ to $\tilde{q}_s$, then by the triangle inequality, the function $F_s(t) = \mathrm{dist}(\tilde{p}_s, \tilde{\gamma}_s(t)) + \mathrm{dist}(\tilde{q}_s, \tilde{\gamma}_s(t))$ is strictly decreasing. Since by Proposition 3.2

$$F_s(0) = L_{\tilde{o}}(\tilde{p}_s) + L_{\tilde{o}}(\tilde{q}_s) \ge D_\pi(\bar{s}L_o(p), \bar{s}L_o(q)) > l_0$$

and

$$l_0 = D_{\theta(\tilde{q}_{\bar{s}})}(\bar{s}L_o(p), \bar{s}L_o(q)) \ge D_{\theta(\tilde{q}_s)}(\bar{s}L_o(p), \bar{s}L_o(q)) = \mathrm{dist}(\tilde{p}_s, \tilde{q}_s) = F_s(\mathrm{dist}(\tilde{o}, \tilde{q}_s)),$$

there exists a unique t_s such that $F_s(t_s) = l_0$. Set $d_s = \mathrm{dist}(\tilde{p}_s, \tilde{\gamma}_s(t_s))$. Thus the broken geodesic $\sigma^s : [0, l_0] \to \widetilde{M}^+$ consisting of the minimizing geodesic $\sigma_1^s : [0, d_s] \to \widetilde{M}^+$ from $\tilde{p}_s$ to $\tilde{\gamma}_s(t_s)$ followed by the minimizing geodesic $\sigma_2^s : [d_s, l_0] \to \widetilde{M}^+$from $\tilde{\gamma}_s(t_s)$ to $\tilde{q}_s$ has total length l_0 and will be regarded as the side of $\triangle^s$ joining $\tilde{p}_s$ to $\tilde{q}_s$.

Let $\sigma : [0, l_0] \to M$ be the side of the triangle $\triangle opq$ joining p to q. Set $f(t) = \mathrm{dist}(o, \sigma(t))$ and $f^s(t) = \mathrm{dist}(\tilde{o}, \sigma^s(t))$ for $0 \le t \le l_0 = \mathrm{dist}(p, q)$. Then, by construction,

(1) $f^s(0) < f(0)$ and $f^s(l_0) < f(l_0)$ for $0 \le s < 1$,
(2) $f^s(t) < f(t)$ for $d_s \le t \le l_0$ when $0 \le s \le \bar{s}$,
(3) $f^s(t)$ is continuous in s and t, and
(4) $f^0 < f$.

Continuity (3) holds because we have assumed there are no cut points of $\tilde{p}_s$ in $\mathrm{int}(\widetilde{M}^+)$ so that σ^s, σ_1^s, and σ_2^s depend uniquely and continuously on their endpoints.

We must show that $f^1 \le f$. Let $s_0 = \sup\{s \in [0, 1] : f^s < f\}$. If $s_0 = 1$, we are done. If not, by (1), (3) and (4), then (i) $f^{s_0}(t) \le f(t)$ for all $t \in [0, l_0]$, and (ii) there exists a $t_0 \in (0, l_0)$ such that $f^{s_0}(t_0) = f(t_0)$. Moreover, by (2) we even have $t_0 \in (0, d_{s_0})$ if $s_0 \le \bar{s}$. Thus f^{s_0} is smooth at t_0. Using Corollary 2.3, it follows that

$$(f^{s_0})'(t_0) \ge f'_-(t_0) \ge f'_+(t_0) \ge (f^{s_0})'(t_0).$$

Thus $f'_-(t_0) = f'_+(t_0) = (f^{s_0})'(t_0)$. By the hypothesis of weaker radial attraction, there exists an $\epsilon > 0$ such that $f(t) \le f^{s_0}(t)$ for all $|t - t_0| < \epsilon$. Thus by (i), $f(t) = f^{s_0}(t)$ for all $|t - t_0| < \epsilon$. Consequently the set where $f = f^{s_0}$ is both open and closed, as well as being nonempty, and thus equals the entire interval $[0, l_0]$. But this contradicts (1) because $s_0 < 1$. Therefore $s_0 = 1$ and $f^1 \le f$, thereby establishing Alexandrov convexity. □

REMARK 4.5. The proof shows that either $f^1(t) < f(t)$ for all $0 < t < l_0$, or $f^1(t) = f(t)$ for all $t \in [0, l_0]$.

LEMMA 4.6. *Alexandrov convexity implies the angle comparison at the top angles.*

PROOF. Starting with $L_o \circ \sigma(t) \ge L_{\tilde{o}} \circ \tilde{\sigma}(t)$ and $L_o \circ \sigma(0) = L_{\tilde{o}} \circ \tilde{\sigma}(0) \le \ell$ we have

$$\cos(\pi - \alpha) \ge (L_o \circ \sigma)'_+(0) \ge (L_{\tilde{o}} \circ \tilde{\sigma})'_+(0) = \cos(\pi - \tilde{\alpha})$$

where α and $\tilde{\alpha}$ are the angles at the top left corner of the triangles. Thus $\tilde{\alpha} \le \alpha$. Similarly for the upper right vertex, but now using the left-hand derivatives. □

LEMMA 4.7. *If $q \notin A$ and $L_o(q) < \ell$, then* $\operatorname{dist}(p, q) < D_\pi(L_o(p), L_o(q))$. (*cf.* [6, Assertion 25].)

PROOF. Consider the triangle $\triangle opq$ in M. We need only consider the case that for each q' on the geodesic γ joining o to q, then $\operatorname{dist}(p, q') < D_\pi(L_o(p), L_o(q'))$. For each $\gamma(t)$ for $0 < t < \operatorname{dist}(o, q)$, let $\tilde{\gamma}(t)$ be the vertex of the triangle corresponding to $\triangle op\gamma(t)$. One proves that $\theta(\tilde{\gamma}(t))$ is nonincreasing, and consequently that $\theta(\tilde{\gamma}(t)) \le \beta$ where $\beta < \pi$ is the angle at o in $\triangle opq$. Thus $\operatorname{dist}(p, \gamma(t)) \le D_\beta(L_o(p), t)$. Thus letting t approach $\operatorname{dist}(o, q)$, we have by Proposition 3.2

$$\operatorname{dist}(p, q) \le D_\beta(L_o(p), L_o(q)) < D_\pi(L_o(p), L_o(q)) .$$

To show that the function $\hat{\theta}(t) = \theta(\tilde{\gamma}(t))$ is decreasing, it suffices to show its upper right Dini derivate satisfies $D^+\hat{\theta}(t) \le 0$ for all $t \in (0, \operatorname{dist}(o, q))$. If not, suppose that $D^+\hat{\theta}(t_0) > 0$ at some t_0. Let $\widehat{\gamma}$ be the meridian from $\tilde{o}$ running through $\tilde{\gamma}(t_0)$. Then $L_{\tilde{o}}(\widehat{\gamma}(t)) = L_{\tilde{o}}(\tilde{\gamma}(t)) = L_o(\gamma(t)) = t$ and $\theta(\widehat{\gamma}(t)) = \hat{\theta}(t_0)$ for all t. Set $C_0 = \frac{1}{2}D^+\hat{\theta}(t_0) > 0$. Thus there exists a sequence $h_k > 0$ converging to 0 as $k \to \infty$ such that $\hat{\theta}(t_0 + h_k) - \hat{\theta}(t_0) > C_0 h_k$ for all k. Taking a compact neighborhood K of $\tilde{\gamma}(t_0) = \widehat{\gamma}(t_0)$ contained in $\operatorname{int}(\widetilde{M}^+)$ of the form in Corollary 3.4, there exists by that corollary a constant $C_1 > 0$ such that $L_{\tilde{p}}(\tilde{\gamma}(t_0 + h_k)) - L_{\tilde{p}}(\widehat{\gamma}(t_0 + h_k)) \ge C_1(\hat{\theta}(t_0 + h_k) - \hat{\theta}(t_0)) > C_0 C_1 h_k$ for all sufficiently large k. Since

$$\begin{aligned}
&\frac{L_{\tilde{p}}(\tilde{\gamma}(t_0 + h)) - L_{\tilde{p}}(\widehat{\gamma}(t_0 + h))}{h} \\
&= \frac{L_{\tilde{p}}(\tilde{\gamma}(t_0 + h)) - L_{\tilde{p}}(\tilde{\gamma}(t_0))}{h} - \frac{L_{\tilde{p}}(\widehat{\gamma}(t_0 + h)) - L_{\tilde{p}}(\widehat{\gamma}(t_0))}{h} \\
&= \frac{L_p(\gamma(t_0 + h)) - L_p(\gamma(t_0))}{h} - \frac{L_{\tilde{p}}(\widehat{\gamma}(t_0 + h)) - L_{\tilde{p}}(\widehat{\gamma}(t_0))}{h}
\end{aligned}$$

it follows that $(L_p \circ \gamma)'_+(t_0) - (L_{\tilde{p}} \circ \widehat{\gamma})'_+(t_0) \ge C_0 C_1 > 0$. On the other hand, if α is the upper right vertex angle of $\triangle op\gamma(t_0)$ and $\tilde{\alpha}$ that of $\triangle \tilde{o}\tilde{p}\tilde{\gamma}(t_0)$, then by Lemma 4.6 , $\tilde{\alpha} \le \alpha$.

Thus $(L_{\tilde{p}} \circ \widehat{\gamma})'_+(t_0) = \cos\tilde{\alpha} \geq \cos\alpha \geq (L_p \circ \gamma)'_+(t_0)$ which is a contradiction. Thus $\hat{\theta}$ is decreasing.

To finish the proof, we will show that $\hat{\theta}(0^+) = \lim_{t\to 0^+} \hat{\theta}(t) \leq \beta$ where β is the base angle of $\triangle opq$. Pick an arbitrary angle $\widehat{\beta} > \beta$ and let $\widehat{\gamma}$ be the meridian with $\theta(\gamma(t)) = \widehat{\beta}$ in $\widetilde{M}^+$. Then

$$(L_p \circ \gamma)'_+(0) \leq \cos(\pi - \beta) < \cos(\pi - \widehat{\beta}) = (L_{\tilde{p}} \circ \widehat{\gamma})'_+(0)\,.$$

Thus there exists an $\epsilon > 0$ such that $L_{\tilde{p}}(\tilde{\gamma}(t)) = L_p(\gamma(t)) < L_{\tilde{p}}(\widehat{\gamma}(t))$ for all $0 \leq t < \epsilon$. Thus, for all such t, $\hat{\theta}(t) = \theta(\tilde{\gamma}(t)) < \widehat{\beta}$ by Proposition 3.2(1), and hence $\hat{\theta}(0^+) < \widehat{\beta}$. Therefore $\hat{\theta}(0^+) \leq \beta$ because $\widehat{\beta} > \beta$ was arbitrary. □

LEMMA 4.8. *If $q \notin A$ then $L_o(q) < \ell$.*

PROOF. We may assume $\ell < \infty$ as otherwise there is nothing to prove. If $L_o(q) \not< \ell$, there exists a point q' on the minimizing geodesic joining o to q such that $L_o(q') = \ell$. Since $q \notin A$, neither is q'. Choose a sequence of points q_n along the geodesic from o to q' converging to q'. Then $q_n \notin A$ and $L_o(q_n) < \ell$. Hence $\mathrm{dist}(p, q_n) < D_\pi(L_o(p), L_o(q_n))$ by Lemma 4.7. Thus on taking limits as n approaches infinity,

$$\mathrm{dist}(p, q') \leq D_\pi(L_o(p), \ell) = \ell - \mathrm{dist}(o, p)\,.$$

Therefore $\mathrm{dist}(o, p) + \mathrm{dist}(p, q') = \ell = \mathrm{dist}(o, q')$ by the triangle inequality. Thus q' lies on the axis A. This is a contradiction. □

REMARK 4.9. By taking arbitrary points p, every maximal minimizing geodesic emanating from o can be considered to be part of an axis. Thus if there exists a point q with $\mathrm{dist}(o, q) = \ell$, the above argument implies that q is the cut point at distance ℓ along every geodesic emanating from o. We can conclude that M is homeomorphic to a sphere. (See [2, Chapter 5].) This is a version of the Maximal Radius Theorem.

The next statement is an immediate consequence of the previous lemmas.

COROLLARY 4.10. *If $q \notin A$, there exists a corresponding triangle satisfying Alexandrov convexity and the angle comparison. Moreover, $L_o(q) < \ell$ and* $\mathrm{dist}(p, q) < D_\pi(L_o(p), L_o(q))$.

THEOREM 4.11. *For every geodesic triangle $\triangle opq$ with* $\mathrm{dist}(o, p) < \ell$, *there exists a corresponding triangle satisfying Alexandrov convexity and the angle comparison. Moreover, $L_o(q) \leq \ell$ and* $\mathrm{dist}(p, q) \leq D_\pi(L_o(p), L_o(q))$.

PROOF. By Corollary 4.10, it suffices to let $q \in A$. Since q is in the closure of $M\backslash A$, it follows from Lemma 4.8 that $L_o(q) \leq \ell$ and $|L_o(p) - L_o(q)| \leq \mathrm{dist}(p, q) \leq D_\pi(L_o(p), L_o(q))$. There are three cases to consider.

Case 1 : $\mathrm{dist}(p, q) = |L_o(p) - L_o(q)|$. (a) If $\mathrm{dist}(p, o) = \mathrm{dist}(p, q) + \mathrm{dist}(o, q)$, $\triangle opq$ is degenerate with $\angle q = \pi$ and $\angle p = 0 = \angle o$. The corresponding geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ exists and is a similarly degenerate triangle in which equality holds in Alexandrov convexity and the corresponding angles are equal. (b) If $\mathrm{dist}(o, p) + \mathrm{dist}(p, q) = \mathrm{dist}(o, q)$, which by

Remark 4.9, occurs when $L_o(q) = \ell$, then $\angle p = \pi$. The corresponding geodesic triangle is a degenerate triangle whose sides are all contained in the meridian through $\tilde{p}$ and satisfies $\angle \tilde{p} = \pi$ and $\angle \tilde{o} = \angle \tilde{q} = 0$. Thus the angle comparison holds and Alexandrov convexity is easily checked.

Case 2: $L_o(q) < \ell$ and $|L_o(p) - L_o(q)| < \operatorname{dist}(p, q) < D_\pi(L_o(p), L_o(q))$. In this case there exists a corresponding geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ with $\tilde{q} \in \operatorname{int}(\widetilde{M}^+)$, and thus the argument of Lemma 4.4 proves Alexandrov convexity, and consequently the angle comparison by Lemmas 4.6 and 4.7.

Note that if there is a minimizing geodesic joining o to q which is not contained in A, then Case 2 occurs, assuming we are not in Case 1. Let γ be a minimizing geodesic joining o to q and not contained in A. Consider the geodesic triangles $\triangle op\gamma(t)$ for $0 < t < \operatorname{dist}(o, q)$. Then $\gamma(t) \notin A$, and the base angles of these triangles are a constant β with $0 < \beta < \pi$. Thus by Lemma 4.7, $\operatorname{dist}(p, \gamma(t)) \leq D_\beta(L_o(p), L_o(\gamma(t)))$. On taking the limit as t approaches $\operatorname{dist}(o, q)$ we obtain $\operatorname{dist}(p, q) \leq D_\beta(L_o(p), L_o(q)) < D_\pi(L_o(p), L_o(q))$.

Case 3: $L_o(q) < \ell$, $\operatorname{dist}(p, q) = D_\pi(L_o(p), L_o(q))$ and the minimizing geodesic joining o to q is unique and contained in A. In this case $\angle o = \pi$. There are two subcases to consider. (a) If $\operatorname{dist}(p, q) = \operatorname{dist}(o, p) + \operatorname{dist}(o, q)$ then there exists a degenerate corresponding triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ with $\angle\tilde{o} = \pi$ and $\angle\tilde{p} = \angle\tilde{q} = 0$. Hence the angle comparison holds and one checks that so does Alexandrov convexity. (Note that $\tilde{q}$ lies in the segment of the meridian opposite to $\tilde{p}$ between the vertex $\tilde{o}$ and the first conjugate point to $\tilde{p}$.) (b) If $\operatorname{dist}(p, q) < \operatorname{dist}(o, p) + \operatorname{dist}(o, q)$, then the minimal geodesic σ from p to q is not contained in A. We may then take a sequence of points q_n along σ which converge to q. Thus $q_n \in M \backslash A$, and the sides and angles of the sequence of the geodesic triangles $\triangle opq_n$ converge to the corresponding sides and angles of $\triangle opq$, since by construction, the sides pq_n converge to the side pq, which is σ, and the sides oq_n converge to oq by uniqueness of the side oq. Since $q_n \notin A$, there exist corresponding Alexandrov triangles $\triangle\tilde{o}\tilde{p}\tilde{q}_n$ which satisfy both the angle comparison and Alexandrov convexity. By taking a subsequence we may assume that the $\triangle\tilde{o}\tilde{p}\tilde{q}_n$ converge to a geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ which therefore is a corresponding triangle for $\triangle opq$. (Note that $\tilde{q}$ lies in the interior of the cut locus of $\tilde{p}$.) □

REMARK 4.12. In the above argument we assumed that $\operatorname{dist}(o, p) < \ell$. If $\operatorname{dist}(o, p) = \ell$, then by Remark 4.9, $\{p\} = C(o)$. Thus in the triangle $\triangle opq$, we would have $\operatorname{dist}(o, q) < \ell$, and could argue as above with q instead of p to complete the proof of Theorem 1.2.

We conclude this section by noting that weaker radial attraction is the optimal hypothesis to prove Toponogov's theorem.

PROPOSITION 4.13. *Let (M, o) be a pointed Riemannian manifold, and let $(\widetilde{M}, \tilde{o})$ be a model surface. Assume that for every geodesic triangle $\triangle opq$ in M there exists a geodesic triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ whose corresponding sides have the same length and smaller angles than those in $\triangle opq$. Then $(\widetilde{M}, \tilde{o})$ has weaker radial attraction than (M, o).*

PROOF. Let σ and $\tilde{\sigma}$ be geodesics in M and $\widetilde{M}$ respectively that satisfy $L_o(\sigma(0)) = L_{\tilde{o}}(\tilde{\sigma}(0)) < \ell$ and $(L_o \circ \sigma)'_+(0) = (L_{\tilde{o}} \circ \tilde{\sigma})'_+(0)$. Pick geodesics γ and $\tilde{\gamma}$ in M and $\widetilde{M}$ respectively joining o to $p = \sigma(0)$ and $\tilde{o}$ to $\tilde{p} = \tilde{\sigma}(0)$ respectively such that $\langle \gamma'(p), \sigma'(o)\rangle = (L_o \circ \sigma)'_+(0)$ and $\langle \tilde{\gamma}'(p), \tilde{\sigma}'(0)\rangle = (L_{\tilde{o}} \circ \tilde{\sigma})'_+(0)$. Thus the angle between γ and σ at p equals the angle between $\tilde{\gamma}$ and $\tilde{\sigma}$ at $\tilde{p}$. Let t be less than the injectivity radius of $\widetilde{M}$ at $\tilde{p}$, and consider the geodesic triangle $\triangle op\sigma(t)$. By hypothesis there exists a geodesic triangle $\triangle \tilde{o}\tilde{p}\tilde{q}$ in $\widetilde{M}$ corresponding to $\triangle op\sigma(t)$ such that $\mathrm{dist}(\tilde{p}, \tilde{q}) = t$, $\mathrm{dist}(\tilde{o}, \tilde{q}) = \mathrm{dist}(o, \sigma(t))$ and $\angle \tilde{p} \leq \angle p$. Since t is less than the injectivity radius at $\tilde{p}$, Proposition 3.2(2) implies

$$L_{\tilde{o}}(\tilde{\sigma}(t)) \geq L_{\tilde{o}}(\tilde{q}) = L_o(\sigma(t))\,.$$

Thus the weaker radial attraction condition holds taking ϵ equal to the injectivity radius at $\tilde{p}$. □

5. Hessian comparison. Again let M be a complete Riemannian manifold with base point o, and let $\widetilde{M}$ be a model surface of revolution with vertex $\tilde{o}$. Recall that the radial curvature of M is bounded from below by $\widetilde{M}$ if for every $p \in M$ and $\tilde{p} \in \widetilde{M}$ with $\mathrm{dist}(o, p) = \mathrm{dist}(\tilde{o}, \tilde{p})$ the sectional curvature of every 2-plane at p containing a tangent vector to a minimizing geodesic from o to p is greater than or equal to the curvature of $\widetilde{M}$ at $\tilde{p}$.

The next proposition is an immediate consequence of the Hessian Comparison Theorem in [5] and serves to motivate Theorem 5.3.

PROPOSITION 5.1. *Assume that the radial curvature of M is bounded below by $\widetilde{M}$, then the Hessian of $L_{\tilde{o}}$ dominates the Hessian of L_o, that is, if $p \in M$ and $\tilde{p} \in \widetilde{M}$ with $L_o(p) = L_{\tilde{o}}(\tilde{p})$ are not cut points of o and $\tilde{o}$ respectively, and $X \in T_pM$, $\widetilde{X} \in T_{\tilde{p}}\widetilde{M}$, satisfy $|X| = |\widetilde{X}|$ and $\langle T, X\rangle = \langle \widetilde{T}, \widetilde{X}\rangle$, where T and $\widetilde{T}$ are the tangents to the minimizing geodesics joining o and $\tilde{o}$ to p and $\tilde{p}$ respectively, then*

$$\nabla^2 L_o(X, X) \leq \nabla^2 L_{\tilde{o}}(\widetilde{X}, \widetilde{X})\,.$$

REMARK 5.2. The converse is false. As a counter example take $\widetilde{M}$ to be the Euclidean plane with metric in polar coordinates $dr^2 + r^2 d\theta^2$ and M to be the surface of revolution with metric $dr^2 + (re^{-r^2})^2 d\theta^2$. Then the Hessian of $L_{\tilde{o}}$ dominates that of L_o because

$$\frac{\tilde{y}'(r)}{\tilde{y}(r)} = \frac{1}{r} \geq \frac{y'(r)}{y(r)} = \frac{1 - 2r^2}{r}\,.$$

However, $\widetilde{M}$ is flat, while the curvature of M satisfies $\kappa(r) = 6 - 4r^2 < 0$ when $r > \sqrt{3/2}$. In fact the negative curvature is unbounded. More examples are constructed in Section 6.

THEOREM 5.3. *The following are equivalent:*

(1) *The Hessian of $L_{\tilde{o}}$ dominates the Hessian of L_o.*
(2) *The principal curvatures of the geodesic spheres about o are bounded from below by the curvature of the geodesic circles about $\tilde{o}$ of the same radius.*
(3) *$\widetilde{M}$ has weaker radial attraction than M.*

PROOF. (1) and (2) are equivalent on account of Proposition 2.7. Clearly (3) implies (1). The proof that (1) implies (3) will be broken down into a sequence of lemmas.

LEMMA 5.4. *Given* $0 < r_1 < r_2 < \ell$, *then for all sufficiently small* $\delta > 0$ *there exists a model surface* $\widetilde{M}_\delta$ *such that the Hessian of* L_{o_δ} *dominates the Hessian of* $L_{\tilde{o}}$ *and moreover satisfies*

$$\nabla_\delta^2 L_{o_\delta} = \nabla^2 L_{\tilde{o}} + \delta$$

on the interval $[r_1, r_2]$. *Furthermore, as* $\delta \to 0^+$, $\widetilde{M}_\delta$ *converges to* $\widetilde{M}$ *in the disk of radius* r_2 *about the vertex.*

PROOF. In a model surface $\widetilde{M}$ with metric $dr^2 + y(r)^2 d\theta^2$, the Hessian of $L_{\tilde{o}}$ takes the form

$$\nabla^2 L_{\tilde{o}}(X, Y) = \frac{y'(r)}{y(r)} \left(\langle X, Y \rangle - \langle X, T \rangle \langle Y, T \rangle \right) .$$

(See [5, Prop. 2.20].) Thus we need to construct a y_δ that defines $\widetilde{M}_\delta$ such that

$$\frac{y'(r)}{y(r)} \le \frac{y_\delta'(r)}{y_\delta(r)} \tag{5.1}$$

for all r and

$$\frac{y'(r)}{y(r)} + \delta = \frac{y_\delta'(r)}{y_\delta(r)} \tag{5.2}$$

for $r_1 \le r \le r_2$. The details are presented in Example 3 in Section 6. □

LEMMA 5.5. *Given* $0 < r_1 < r_2 < \ell$, *let* $\delta > 0$ *be sufficiently small so that* $\widetilde{M}_\delta$ *satisfies Lemma* 5.4. *Let* σ, $\tilde{\sigma}$, *and* σ_δ, *be geodesics in* M, $\widetilde{M}$, *and* $\widetilde{M}_\delta$ *respectively such that*

$$r_1 < L_o \circ \sigma(0) = L_{\tilde{o}} \circ \tilde{\sigma}(0) = L_{o_\delta} \circ \sigma_\delta(0) < r_2$$

and $(L_o \circ \sigma)_+'(0) = (L_{\tilde{o}} \circ \tilde{\sigma})_+'(0) = (L_{o_\delta} \circ \sigma_\delta)_+'(0)$. *Then there exists an* $\epsilon > 0$ *such that* $L_{o_\delta} \circ \sigma_\delta(t) \ge L_o \circ \sigma(t)$ *for all* $0 \le t < \epsilon$. *In other words,* $\widetilde{M}_\delta$ *has weaker radial attraction than* M *in the region* $r_1 < r < r_2$.

PROOF. Set $\sigma(0) = p$, $\sigma'(0) = X$, $\tilde{\sigma}'(0) = \widetilde{X}$ and $\sigma_\delta'(0) = X_\delta$. We may assume $X \ne \pm T$, where $T \in T_p M$ is the tangent vector to the minimizing geodesic γ from o to p such that $\langle X, T \rangle = (L_o \circ \sigma)_+'(0)$, otherwise the result is immediate. Under this assumption, we have

$$\nabla^2 L_o(X, X) \le \nabla^2 L_{\tilde{o}}(\widetilde{X}, \widetilde{X}) < \nabla^2 L_{\tilde{o}}(\widetilde{X}, \widetilde{X}) + \delta(|\widetilde{X}|^2 - \langle \widetilde{X}, \widetilde{T} \rangle^2) = \nabla^2 L_{o_\delta}(X_\delta, X_\delta) , \tag{5.3}$$

by Lemma 5.4, where $\widetilde{T}$ is the unit radial tangent vector at $\tilde{\sigma}(0)$.

First suppose $p \notin C(o)$, then $L_o \circ \sigma$ is smooth in a neighborhood of 0. By equations (2.11) and (5.3),

$$(L_o \circ \sigma)''(0) = \nabla^2 L_o(X, X) < \nabla^2 L_{o_\delta}(X_\delta, X_\delta) = (L_{o_\delta} \circ \sigma_\delta)''(0) .$$

Thus there exists an $\epsilon > 0$ such that $L_o \circ \sigma(t) \le L_{o_\delta} \circ \sigma_\delta(t)$ when $0 \le t < \epsilon$.

On the other hand, if $p \in C(o)$, then it is possible to use Corollary 2.6 to construct an upper support function F_V for $L_o \circ \sigma$ that satisfies

$$F_V''(0) < \nabla^2 L_{o_\delta}(X_\delta, X_\delta) = (L_{o_\delta} \circ \sigma_\delta)''(0)\,, \tag{5.4}$$

and therefore there would exist an $\epsilon > 0$ such that $L_o \circ \sigma(t) \le F_V(t) \le L_{o_\delta} \circ \sigma_\delta(t)$ when $0 \le t \le \epsilon$. To accomplish this, observe that $\eta = \nabla^2 L_{o_\delta}(X_\delta, X_\delta) - \nabla^2 L_o(X, X) > 0$ by equation (5.3). It then suffices to choose t_0 in Corollary 2.6 close enough to $l = \mathrm{dist}(o, p)$ so that both $|\nabla^2 L_o(X, X) - \nabla^2 L_o(P(t_0), P(t_0))| < \eta/2$ and

$$\left| \big(1 - \langle P, T\rangle^2\big) \int_{t_0}^{l} \kappa(P \wedge T)dt \right| < \eta/2$$

where P is the parallel vector field along γ. For then by Corollary 2.6,

$$F_V''(0) = \nabla^2 L_o(P(t_0), P(t_0)) - \big(1 - \langle P, T\rangle^2\big) \int_{t_0}^{l} \kappa(P \wedge T)dt < \nabla^2 L_{o_\delta}(X_\delta, X_\delta)\,. \tag{5.5}$$

□

LEMMA 5.6. *Given* $0 < r_1 < r_2 < \ell$, *let* $\delta > 0$ *be sufficiently small so that* $\widetilde{M}_\delta$ *satisfies Lemma* 5.4. *Let* $\triangle opq$ *be a geodesic triangle in* M *such that* $r_1 < L_o(p) < r_2$, $\mathrm{dist}(o, p) - \mathrm{dist}(p, q) > r_1$, $\mathrm{dist}(o, p) + \mathrm{dist}(p, q) < r_2$, *and* $\mathrm{dist}(p, q)$ *is strictly less than the injectivity radius of* p_δ *in* $\widetilde{M}_\delta$ *whose polar coordinates are* $(L_o(p), 0)$. *Then there exists a corresponding triangle* $\triangle o_\delta p_\delta q_\delta$ *in* $\widetilde{M}_\delta$ *satisfying Alexandrov convexity. The angle comparison holds for the top angles.*

PROOF. The main point here is that the construction of the collapsing family $\triangle^s$ in the proof of Lemma 4.4 works as long as the construction stays away from the cut loci of $\tilde{p} = p_\delta$ and nearby points $\tilde{p}_s$. Under the given assumptions, there exist $\epsilon_1, \epsilon_2 > 0$ such that

$$\mathrm{dist}(o, p) - \mathrm{dist}(p, q) = r_1 + \epsilon_1 \tag{5.6}$$

and

$$\mathrm{dist}(o, p) + \mathrm{dist}(p, q) = r_2 - \epsilon_2\,. \tag{5.7}$$

We may arrange the construction of $\triangle^s$ so that the points $\tilde{p}_s$ satisfy $\mathrm{dist}(\tilde{p}_s, \tilde{p}) < \min(\epsilon_1, \epsilon_2)$ and $\mathrm{dist}(p, q)$ remains strictly less than the injectivity radius of $\tilde{p}_s$ for all s. Since the length of the (broken) geodesic σ^s is $\mathrm{dist}(p, q)$, we have, by the triangle inequality,

$$\mathrm{dist}(\tilde{p}, \sigma^s(t)) \le \mathrm{dist}(\tilde{p}, \tilde{p}_s) + \mathrm{dist}(\tilde{p}_s, \sigma^s(t)) < \min(\epsilon_1, \epsilon_2) + \mathrm{dist}(p, q) \tag{5.8}$$

for every point $\sigma^s(t)$ on σ^s. Therefore, by the triangle inequality and by (5.8) and (5.6),

$$\begin{aligned}\mathrm{dist}(\tilde{o}, \sigma^s(t)) &\ge \mathrm{dist}(\tilde{o}, \tilde{p}) - \mathrm{dist}(\tilde{p}, \sigma^s(t))\\ &> \mathrm{dist}(o, p) - \mathrm{dist}(p, q) - \min(\epsilon_1, \epsilon_2)\\ &= r_1 + \epsilon_1 - \min(\epsilon_1, \epsilon_2)\\ &\ge r_1\,,\end{aligned}$$

and similarly, by (5.8) and (5.7),

$$\begin{aligned}\operatorname{dist}(\tilde{o}, \sigma^s(t)) &\leq \operatorname{dist}(\tilde{o}, \tilde{p}) + \operatorname{dist}(\tilde{p}, \sigma^s(t)) \\ &< \operatorname{dist}(o, p) + \operatorname{dist}(p, q) + \min(\epsilon_1, \epsilon_2) \\ &= r_2 - \epsilon_2 + \min(\epsilon_1, \epsilon_2) \\ &\leq r_2 .\end{aligned}$$

This proves that σ^s is contained in the region $r_1 < r < r_2$ in $\widetilde{M}_\delta$. Thus on account of Lemma 5.5, the argument in Lemma 4.4 goes through completely to prove Alexandrov convexity holds. The angle comparison for the top angles follows from Lemma 4.5. □

LEMMA 5.7. *$\widetilde{M}$ has weaker radial attraction than M.*

PROOF. Let σ and $\tilde{\sigma}$ be geodesics satisfying $L_o \circ \sigma(0) = L_{\tilde{o}} \circ \tilde{\sigma}(0) < \ell$ and $(L_o \circ \sigma)'_+(0) = (L_{\tilde{o}} \circ \tilde{\sigma})'_+(0)$.

If $p = \sigma(o)$ and $\tilde{p} = \tilde{\sigma}(0)$, choose $\epsilon > 0$ to be less than the injectivity radius of $\tilde{p}$ in $\widetilde{M}$. Because the $\widetilde{M}_\delta$ converge to $\widetilde{M}$, for all sufficiently small δ, ϵ will be smaller than the injectivity radius of the corresponding points p_δ in $\widetilde{M}_\delta$. In accordance with Lemma 2.1, let $\gamma \in Geod(o, p)$ be chosen so that $(L_o \circ \sigma)'(0) = \langle \gamma'(L_o(p)), \sigma'(0) \rangle$. For all sufficiently small δ, let σ_δ be the geodesic starting at p_δ making the same angle with the meridian through p_δ as σ makes with γ. Fix $0 < s < \epsilon$ and set $q = \sigma(s)$. For each $\delta > 0$, find the geodesic triangle in $\widetilde{M}_\delta$ corresponding to $\triangle opq$ according to Lemma 5.6. Its angle at p_δ is smaller than the angle at p. By the monotonicity for small hinges, Proposition 3.2(2), $L_o \circ \sigma(s) \leq L_{\tilde{o}} \circ \sigma_\delta(s)$. Letting $\delta \to 0$, σ_δ approaches $\tilde{\sigma}$. It follows that $L_o \circ \sigma(s) \leq L_{\tilde{o}} \circ \tilde{\sigma}(s)$. □

This completes the proof that (1) implies (3). □

6. Examples. In order to construct examples of smooth model surfaces whose metric in polar coordinates is given by $ds^2 = dr^2 + y(r)^2 d\theta^2$, it suffices to produce smooth functions $y : \mathbb{R} \to \mathbb{R}$ that satisfy all the conditions enunciated in Section 3.

Given two surfaces of revolution M with metric $dr^2 + y(r)^2 d\theta^2$, $0 < r < \ell$ and $\overline{M}$ with metric $dr^2 + \bar{y}(r)^2 d\theta^2$, $0 < r < \bar{\ell}$, it follows from [5, Prop. 2.20] and Theorem 5.3 that $\overline{M}$ has weaker radial attraction than M provided

$$\frac{y'(r)}{y(r)} \leq \frac{\bar{y}'(r)}{\bar{y}(r)} \tag{6.1}$$

for $0 < r < \bar{\ell}$. Note that $\ell \leq \bar{\ell} \leq \infty$. If we set $y(r) = m(r)\bar{y}(r)$, then $y'(r) = \bar{y}'(r)m(r) + \bar{y}(r)m'(r)$ so that

$$\frac{y'(r)}{y(r)} = \frac{\bar{y}'(r)}{\bar{y}(r)} + \frac{m'(r)}{m(r)} . \tag{6.2}$$

Thus $\overline{M}$ has weaker radial attraction than M if $m'(r) \leq 0$. Because $\bar{y}(0) = y(0) = 0$ and $\bar{y}'(0) = y'(0) = 1$ it follows that $m(0) = 1$. In case ℓ is finite, since $y(\ell) = 0$ and $y'(\ell) = -1$, we also have that $m(\ell) = 0$ and $m'(\ell) = \frac{-1}{\bar{y}(\ell)}$. Because y and $\bar{y}$ are odd functions, m is even. Moreover, if ℓ is finite, $m(r)\bar{y}(r)$ must be antisymmetric about ℓ. By choosing m with these properties we can construct examples of manifolds with stronger radial attraction than the given $\overline{M}$.

LEMMA 6.1. *Suppose f_1 and f_2 are smooth functions defined on the interval $[a, b]$. Let $\varphi : [a, b] \to [0, 1]$ be a smooth decreasing function that equals 1 in a neighborhood of a and equals 0 in a neighborhood of b. Consider the function*

$$f = \varphi f_1 + (1 - \varphi) f_2 .$$

Clearly f equals f_1 in a neighborhood of a and equals f_2 in a neighborhood of b.

(1) *If $f_1 \geq f_2$ and both f_1 and f_2 are decreasing, then f is decreasing.*
(2) *If $f_1 \leq f_2$ and both f_1 and f_2 are increasing, then f is increasing.*

PROOF. Consider

$$f' = \varphi'(f_1 - f_2) + \varphi f_1' + (1 - \varphi) f_2' .$$

In the case (1), $f' \leq 0$ because $\varphi' \leq 0$, $f_1 - f_2 \geq 0$, $\varphi \geq 0$, $f_1' \leq 0$, $1 - \varphi \geq 0$, and $f_2' \leq 0$. Similarly, in the case (2), $f' \geq 0$. □

EXAMPLE 1. Given any $\overline{M}$, here is how to construct an M with stronger radial attraction than $\overline{M}$ whose radial curvature is not bounded from below by $\overline{M}$. First note that since $\bar{y}(0) = 0$ and $\bar{y}'(0) = 1$ there exists a $b > 0$ such that $\bar{y}(r) > 0$ and $\bar{y}'(r) > 0$ for all $0 < r < b$. Moreover, there exists a $r_0 \in (0, b)$ such that

$$\left(\frac{\bar{y}'}{\bar{y}}\right)'(r_0) \neq 0 . \tag{6.3}$$

For if not, $\frac{\bar{y}'}{\bar{y}} = k$ on $(0, b)$ for some constant k. The solution of this differential equation takes the form $\bar{y}(r) = Ce^{kr}$ on $(0, b)$ for some constant C. But this is incompatible with $\bar{y}(0) = 0$ and $\bar{y}'(0) = 1$. Set $A = 2\frac{\bar{y}'(r_0)}{\bar{y}(r_0)} > 0$.

Now pick ℓ between r_0 and $\bar{\ell}$. Then

$$e^{-A\ell} > 0 = \frac{\ell - \ell}{\bar{y}(\ell)} \quad \text{and} \quad \frac{d}{dr}\left(\frac{\ell - r}{\bar{y}(r)}\right)\bigg|_{r=\ell} = \frac{-1}{\bar{y}(\ell)} < 0 .$$

Thus by continuity there exists an $r_1 \in (r_0, \ell)$ such that

$$e^{-Ar} \geq \frac{\ell - r}{\bar{y}(r)} \quad \text{and} \quad \frac{d}{dr}\left(\frac{\ell - r}{\bar{y}(r)}\right) < 0 \quad \text{for all} \quad r \in [r_1, \ell] . \tag{6.4}$$

Let φ_1 be a smooth decreasing function on $[0, r_0]$ that equals 1 in a neighborhood of 0 and equals 0 in a neighborhood of r_0, and let φ_2 be a smooth decreasing function on $[r_1, \ell]$ that equals 1 in a neighborhood of r_1 and equals 0 in a neighborhood of ℓ. By construction, the function $m : [0, \ell] \to \mathbb{R}$ defined by

$$m(r) = \begin{cases} \varphi_1(r) + (1 - \varphi_1(r))e^{-Ar} & r \in [0, r_0] \\ e^{-Ar} & r \in [r_0, r_1] \\ \varphi_2(r)e^{-Ar} + (1 - \varphi_2(r))\frac{\ell - r}{\bar{y}(r)} & r \in [r_1, \ell] \end{cases} \tag{6.5}$$

equals 1 in a neighborhood of 0, e^{-Ar} in a neighborhood of r_0, and $\frac{\ell-r}{\bar{y}(r)}$ in a neighborhood of ℓ. The hypotheses of Lemma 6.1(1) are clearly satisfied on $[0, r_0]$ and on $[r_1, \ell]$ by (6.4). We conclude that m is a smooth decreasing function on $[0, \ell]$.

Consequently $y(r) = m(r)\bar{y}(r)$ equals $\bar{y}(r)$ in a neighborhood of 0, $e^{-Ar}\bar{y}(r)$ in a neighborhood of r_0, and $\ell - r$ in a neighborhood of ℓ. Thus y extends smoothly to an odd function on $\mathbb{R}$ with period 2ℓ which is antisymmetric about ℓ. The curvature function near r_0 satisfies

$$\kappa_M(r) = \kappa_{\overline{M}}(r) + A\left(2\frac{\bar{y}'(r)}{\bar{y}(r)} - A\right)$$

which shows that the two curvatures are equal at r_0 but that their difference has different signs on different sides of r_0.

EXAMPLE 2. This example shows that it is possible that the top angles in the corresponding triangle can be equal without the base angles being equal so that rigidity fails. Let $\widetilde{M}$ be the standard sphere of radius 1. Thus we have $\bar{y}(r) = \sin(r)$. Let ℓ be between $\frac{\pi}{2}$ and π. Because

$$\frac{1}{2} > 0 = \frac{\ell - \ell}{\sin(\ell)} \quad \text{and} \quad \frac{d}{dr}\left(\frac{\ell - r}{\sin(r)}\right)\bigg|_{r=\ell} = \frac{-1}{\sin(\ell)} < 0\,,$$

by continuity there exists an r_1 between $\frac{\pi}{2}$ and ℓ such that

$$\text{(6.6)} \qquad \frac{1}{2} \geq \frac{\ell - r}{\sin(r)} \quad \text{and} \quad \frac{d}{dr}\left(\frac{\ell - r}{\sin(r)}\right) < 0 \quad \text{for all} \quad r \in [r_1, \ell]\,.$$

Let φ_1 be a smooth decreasing function on $[0, \frac{\pi}{2}]$ that equals 1 in a neighborhood of 0 and equals 0 in a neighborhood of $\frac{\pi}{2}$, and let φ_2 be a smooth decreasing function on $[r_1, \ell]$ that equals 1 in a neighborhood of r_1 and equals 0 in a neighborhood of ℓ. Lemma 6.1(1) can be applied to show that the function $m : [0, \ell] \to \mathbb{R}$ defined by

$$\text{(6.7)} \qquad m(r) = \begin{cases} \varphi_1(r) + (1 - \varphi_1(r))\frac{1}{2} & r \in [0, \frac{\pi}{2}] \\ \frac{1}{2} & r \in [\frac{\pi}{2}, r_1] \\ \varphi_2(r)\frac{1}{2} + (1 - \varphi_2(r))\frac{\ell - r}{\sin(r)} & r \in [r_1, \ell] \end{cases}$$

is a smooth decreasing function that equals 1 in a neighborhood of 0, $\frac{1}{2}$ in a neighborhood of $\frac{\pi}{2}$, and $\frac{\ell-r}{\sin(r)}$ in a neighborhood of ℓ. Thus $y(r) = m(r)\bar{y}(r)$ defines a smooth rotationally symmetric surface on the sphere M with stronger radial attraction than $\widetilde{M}$. Consider the triangle $\triangle opq$ where $p = (\frac{\pi}{2}, 0)$ and $q = (\frac{\pi}{2}, \theta)$ in polar coordinates on M. Then the corresponding triangle $\triangle\tilde{o}\tilde{p}\tilde{q}$ has $\tilde{p} = (\frac{\pi}{2}, 0)$ and $\tilde{q} = (\frac{\pi}{2}, \theta/2)$. The top angles in $\triangle\tilde{o}\tilde{p}\tilde{q}$ are right angles equal to those of the original triangle $\triangle opq$ in M but the base angle is half of that of the original triangle. Clearly, the interior of $\triangle\tilde{o}\tilde{p}\tilde{q}$ does not embed isometrically into M.

If on the other hand we start with a surface of revolution M with metric $ds^2 = dr^2 + y(r)^2 d\theta^2$ for $0 < r < \ell$ and wish to construct a surface of revolution $\overline{M}$ with metric $d\bar{s}^2 = dr^2 + \bar{y}(r)^2 d\theta$ for $0 < r < \bar{\ell}$ (with $\ell < \bar{\ell}$) with weaker radial attraction than M, then if we write $\bar{y}(r) = \bar{m}(r)y(r)$ for $0 < r < \ell$, by a calculation like that above we need $\bar{m}(0) = 1$,

$\bar{m}(r)$ to be increasing, and in case ℓ is finite, near ℓ, $\bar{m}(r)$ is asymptotic to $\frac{k}{\ell-r}$ for some constant k. More precisely we need $\bar{m}(r)y(r)$ to be smooth near ℓ.

EXAMPLE 3. In this example we explain how to construct the surface $\widetilde{M}_\delta$ needed in Lemma 5.4. We are given the model surface $\widetilde{M}$ with metric $ds^2 = dr^2 + y(r)^2 d\theta^2$ and $0 < r_1 < r_2 < \ell$. Let $\delta > 0$.

First consider the case $\ell = \infty$. Since $1 \le e^{\delta r}$ and both the constant function 1 and $e^{\delta r}$ are increasing for $r \ge 0$, by using Lemma 6.1(2) we can paste 1 and $e^{\delta r}$ together to obtain an increasing function $\bar{m}_\delta : [0, \infty) \to \mathbb{R}$ which equals 1 in a neighborhood of 0 and equals $e^{\delta r}$ on $[r_1, r_2]$. Indeed, let φ be a smooth desreasing function on $[0, r_1]$ which equals 1 in a neighborhood of 0 and equals 0 in a neighborhood of r_1, and define

$$\bar{m}_\delta(r) = \begin{cases} \varphi(r) + (1 - \varphi(r))e^{\delta r} & 0 \le r \le r_1 \\ e^{\delta r} & r_1 \le r\,. \end{cases} \tag{6.8}$$

Then on setting $y_\delta(r) = \bar{m}_\delta(r)y(r)$ we have that $y_\delta = y$ in a neighborhood of 0 and

$$\frac{y'(r)}{y(r)} + \delta = \frac{y'_\delta(r)}{y_\delta(r)} \quad \text{for} \quad r_1 \le r \le r_2\,. \tag{6.9}$$

Next consider the case $\ell < \infty$. Since $y(\ell) = 0$ and $y'(\ell) = -1$, $\lim_{r\to\ell^-} \frac{1}{y(r)} = \infty$. Hence there exists a r_3 with $r_2 < r_3 < \ell$ such that $\frac{1}{y(r_3)} > 1$ and $\frac{1}{y(r)}$ is increasing on $[r_3, \ell)$. Thus if $\delta > 0$ is small enough so that $e^{\delta\ell} < \frac{1}{y(r_3)}$, then $e^{\delta r} \le e^{\delta \ell} < \frac{1}{y(r_3)} \le \frac{1}{y(r)}$ on $[r_3, \ell]$. Now define

$$\bar{m}_\delta(r) = \begin{cases} \varphi_1(r) + (1 - \varphi_1(r))e^{\delta r} & 0 \le r \le r_1 \\ e^{\delta r} & r_1 \le r \le r_3 \\ \varphi_2(r)e^{\delta r} + (1 - \varphi_2(r))\frac{1}{y(r)} & r_3 \le r < \ell \end{cases} \tag{6.10}$$

where φ_1 a smooth decreasing function on $[0, r_1]$ which equals 1 in a neighborhood of 0 and equals 0 in a neighborhood of r_1, and φ_2 a smooth decreasing function on $[r_3, \ell]$ which equals 1 in a neighborhood of r_3 and equals 0 in a neighborhood of ℓ. Then by Lemma 6.1(2) $\bar{m}_\delta(r)$ is an increasing function which by construction equals 1 in a neighborhood of 0, equals $e^{\delta r}$ on $[r_1, r_2]$ and equals $\frac{1}{y(r)}$ in a neighborhood of ℓ. On setting $y_\delta = \bar{m}_\delta(r)y(r)$ we have that $y_\delta = y$ in a neighborhood of 0,

$$\frac{y'(r)}{y(r)} + \delta = \frac{y'_\delta(r)}{y_\delta(r)} \quad \text{for} \quad r_1 \le r \le r_2\,, \tag{6.11}$$

and equals 1 in a neighborhood of ℓ. Thus we may extend y_δ smoothly as we like beyond ℓ to produce a complete model surface. Indeed, setting $y_\delta(r) = 1$ for $r > \ell$ would work.

In either case the surface $\widetilde{M}_\delta$ with metric $ds^2 = dr^2 + y_\delta(r)^2 d\theta^2$ has the properties needed in Lemma 5.4. Clearly by construction y_δ and its derivatives converge to y and its derivatives as $\delta \to 0^+$ uniformly on $[0, r_2]$.

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