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EXTREMAL LORENTZIAN SURFACES WITH NULL r-PLANAR GEODESICS IN SPACE FORMS

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Abstract. We show a congruence theorem for oriented Lorentzian surfaces with horizontal reflector lifts in pseudo-Riemannian space forms of neutral signature. As a corollary, a characterization theorem is obtained for the Lorentzian Boruvka spheres, that is, a full real analytic null r-planar geodesic immersion with vanishing mean curvature vector field is locally congruent to the Lorentzian Boruvka sphere in a $2r$-dimensional space form of neutral signature.

1. Introduction.

To study minimal surfaces in a unit sphere, the twistor lift plays an important role. For instance, Calabi [2] proves a rigidity theorem for minimal immersions of surfaces with genus zero in Euclidean spheres using twistor lifts. An application of the rigidity result shows that a minimal isometric immersion of the 2-sphere into a unit sphere is congruent to the d-th standard immersion (also called the Boruvka sphere in the unit $2d$-sphere) for a positive integer d. As a result, Boruvka spheres have horizontal twistor lifts. Chern [4] reinterprets Calabi's work and investigates minimal 2-spheres in a unit sphere by using the higher order osculating spaces and higher fundamental forms. We refer to Bryant [1] also. One of the aims in this paper is to characterize Boruvka spheres in indefinite pseudo-Riemannian geometry, using an indefinite version of twistor lifts.

The Boruvka spheres with Lorentzian metric, a family of isometric immersions of Lorentzian 2-sphere into the pseudo-Riemannian spheres are, via Wick rotations, constructed from the standard immersions of Riemannian 2-sphere in Ding and Wang [5] and Miura [10]. These immersions have vanishing mean curvature vector fields, thus, these are *extremal*. In this paper, we call these immersions the Lorentzian Boruvka spheres (LBSs). We focus on the fact that the target spaces of the LBSs are always *neutral*. Then it is natural that we use reflector lifts instead of twistor lifts. The notion of reflector lifts established on neutral pseudo-Riemannian manifolds is corresponding to that of twistor lifts in Riemannian geometry. See Jensen and Rigoli [7] for details. We see that extremal helical geodesic immersions (HGIs) from Lorentzian surfaces into a space form have horizontal reflector lifts. Note that the LBSs have helical geodesics. As a property of HGIs, we propose a notion of *null r-planar geodesic immersions* (PGIs). For a precise definition, see Definition 4.1. We provide a congruence theorem for oriented Lorentzian surfaces with horizontal reflector lifts. As an application of our

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congruence theorem, we characterize the Lorentzian Boruvka spheres as extremal Lorentzian surfaces with null r-planar geodesic.

The paper is organized as follows. In Section 2, we prepare a general theoretical setting and basic equations for Lorentzian surfaces. Furthermore, we investigate extremal Lorentzian surfaces by using their isotropic higher fundamental forms and furnish several lemmas. In Section 3, we introduce the notion of reflector lifts. A congruence theorem for oriented Lorentzian surfaces with horizontal reflector lifts is proved. In Section 4, the definitions of HGIs and null r-PGIs are clearly stated. Moreover, based on [10], we explain the construction of LBSs briefly. Finally, in Section 5, we investigate extremal null r-planar geodesic immersions from Lorentzian surfaces of constant Gaussian curvature and provide our main theorem.

2. Preliminaries. Throughout this paper, all manifolds and maps are assumed to be smooth unless otherwise mentioned. Let E be a vector bundle over a manifold M and E_p the fiber of E over a point $p \in M$. We write TM (resp. T^*M) for the tangent (resp. cotangent) bundle of M. For vector bundles E, E' over M, we denote the homomorphism bundle whose fiber is the space of linear mappings E_p to E'_p by $\mathrm{Hom}(E, E')$, and set $\mathrm{End}(E) := \mathrm{Hom}(E, E)$. The space of all sections of a vector bundle E is denoted by $\Gamma(E)$. We denote the space of E-valued 1-forms on M by $\wedge^1(E) := \Gamma(T^*M \otimes E)$. Let $\varphi : N \to M$ be a smooth map and E a vector bundle over M. The pull back bundle of E by φ is denoted by $\varphi^\# E$. In this paper, a pair (E, g^E) is a *pseudo-Riemannian* vector bundle if the bundle metric g^E of E is nondegenerate of constant index. The set of all metric connections of E with respect to g^E is denoted by $\mathcal{C}(E, g^E)$.

2.1. Basic definitions and equations. In this subsection, we recall some basic definitions and equations for pseudo-Riemannian manifolds and submanifolds. Let $(\widetilde{M}^n_t, \widetilde{g})$ be an n-dimensional pseudo-Riemannian manifold with nondegenerate metric $\widetilde{g}$ of constant index t. We may denote $(\widetilde{M}^n_t, \widetilde{g})$ by $\widetilde{M}^n_t$ for short. We say that $\widetilde{M}^n_t$ is of *neutral* signature if $n = 2t$, and *Lorentzian* if $n > 1$ and $t = 1$. If there is no confusion, we omit the dimension and index, i.e., $\widetilde{M} = \widetilde{M}^n_t$. A tangent vector X to $\widetilde{M}$ is called *spacelike* if $\widetilde{g}(X, X) > 0$ or $X = 0$, *null* if $\widetilde{g}(X, X) = 0$ and $X \neq 0$, and *timelike* if $\widetilde{g}(X, X) < 0$.

Let $\mathbb{R}^n_t$ be the n-dimensional pseudo-Euclidean space of the index t with the flat standard metric. Let $(x^1, \dots, x^{n+1})$ be the standard coordinate on $\mathbb{R}^{n+1}$. The n-dimensional pseudosphere $S^n_t(r)$ of the index t and the radius $r > 0$ is defined by

$$S^n_t(r) = \left\{ p \in \mathbb{R}^{n+1}_t \,\middle|\, -\sum_{i=1}^{t} (x^i(p))^2 + \sum_{j=t+1}^{n+1} (x^j(p))^2 = r^2 \right\}.$$

Similarly, the n-dimensional pseudohyperbolic space $H^n_t(r)$ of the index t and the radius $r > 0$ is defined by

$$H^n_t(r) = \left\{ p \in \mathbb{R}^{n+1}_{t+1} \,\middle|\, -\sum_{i=1}^{t+1} (x^i(p))^2 + \sum_{j=t+2}^{n+1} (x^j(p))^2 = -r^2 \right\}.$$

The spaces $\mathbb{R}^n_t$, $S^n_t(r)$ and $H^n_t(r)$ are of constant curvature 0, $1/r^2$ and $-1/r^2$ respectively. We denote the space form of constant curvature c by $Q^n_t(c)$ which is one of $\mathbb{R}^n_t$, $S^n_t(r)$ or $H^n_t(r)$.

From now on, we provide the basic equations for isometric immersions in pseudo-Riemannian geometry. For more details, we refer to [15, IV, pp.163–188] in the case of Riemannian geometry. Let (M, g) be a pseudo-Riemannian submanifold in $(\widetilde{M}, \widetilde{g})$ isometrically immersed by f. We denote the Levi-Civita connection of $\widetilde{g}$ (resp. g) by $\widetilde{\nabla}$ (resp. ∇). The mean curvature vector field of M is denoted by H. If $H = 0$, then M is called an *extremal* submanifold ([12, p. 299]). We often omit the symbol "f" for the induced objects of the immersion f if there is no confusion for the simplicity. We define $\widetilde{\nabla}(X_1) := X_1$, $\widetilde{\nabla}(X_1, X_2) := \widetilde{\nabla}_{X_1} X_2$ and inductively for $i \geq 3$

$$\widetilde{\nabla}(X_1, X_2, \ldots, X_i) := \widetilde{\nabla}_{X_1} \widetilde{\nabla}(X_2, \ldots, X_i)\,, \quad \text{and}$$
$$\widetilde{\nabla}(X_1, X_2, \ldots, X_i)(p) := \widetilde{\nabla}_{X_{1p}} \widetilde{\nabla}(X_2, \ldots, X_i)\,,$$

where $X_k \in \Gamma(TM)$ $(1 \leq k \leq i)$ and $p \in M$. We define the i-th osculating space of f by $\mathrm{Osc}^0(f) := M \times \{0\} \subset f^{\#}T\widetilde{M}$ and for any positive integer i,

$$\mathrm{Osc}^i(f) := \bigcup_{p \in M} \mathrm{Osc}^i_p(f)\,, \quad \text{where}$$
$$\mathrm{Osc}^i_p(f) := \mathrm{Span}\{\widetilde{\nabla}(X_1, \ldots, X_k)(p) \mid X_l \in \Gamma(TM), 1 \leq l \leq k \leq i\}\,.$$

Since f is an immersion, $\mathrm{Osc}^1(f) = TM$. Therefore, there is the unique integer $d \geq 1$ such that

- $\mathrm{Osc}^0(f) \subsetneqq \mathrm{Osc}^1(f) \subsetneqq \mathrm{Osc}^2(f) \subsetneqq \cdots \subsetneqq \mathrm{Osc}^d(f)$,
- $\mathrm{Osc}^i(f)$ is a smooth subbundle of $f^{\#}T\widetilde{M}$ and the induced metric is *nondegenerate* of constant index for each $i = 1, 2, \ldots, d$,
- $\mathrm{Osc}^{d+1}(f) = \mathrm{Osc}^d(f)$ or $\mathrm{Osc}^{d+1}(f)$ is not a pseudo-Riemannian subbundle (i.e., the induced symmetric tensor in $\mathrm{Osc}^{d+1}(f)$ from $T\widetilde{M}$ is degenerate or $\mathrm{Osc}^{d+1}(f)$ is not a smooth subbundle of $f^{\#}T\widetilde{M}$).

If f satisfies the three conditions above, we say that f is *nicely curved of order d*. For $i = 0, 1, \ldots, d-1$, we can take the i-th normal space $N^i(f)$ such that

$$\mathrm{Osc}^{i+1}(f) = \mathrm{Osc}^i(f) \oplus N^i(f)\,,$$

where $N^i(f)$ is the orthogonal complement subbundle of $\mathrm{Osc}^i(f)$ in $\mathrm{Osc}^{i+1}(f)$. We denote $N^i(f)$ by N^i for short. Because of $\mathrm{Osc}^0(f) = M \times \{0\}$ and $\mathrm{Osc}^1(f) = TM$, we have $N^0 = TM$. Moreover we put $N^d := \mathrm{Osc}^d(f)^{\perp}$ which is the orthogonal complement subbundle of $\mathrm{Osc}^d(f)$ in $f^{\#}T\widetilde{M}$. Notice that N^d need not be contained in the osculating space $\mathrm{Osc}^k(f)$ for an integer $k > 0$. Therefore we often need separate arguments for objects related to the highest normal bundle. Then we obtain

$$f^{\#}T\widetilde{M} = \bigoplus_{i=0}^{d} N^i\,.$$

We denote the induced symmetric tensor in N^i from $\widetilde{g}$ by g^i. Note that $g = g^0, g^1, \dots, g^d$ are pseudo-Riemannian, since f is nicely curved of order d.

For a vector $\zeta \in f^{\#}T\widetilde{M}$, we denote the N^i-component of ζ by $(\zeta)^{N^i}$. For $i = 0, 1, \dots, d$, we define the $(i+1)$st fundamental form α^{i+1} by

$$\alpha^{i+1}(X_1, \dots, X_{i+1}) := (\widetilde{\nabla}(X_1, \dots, X_{i+1}))^{N^i} .$$

By definitions, we can see that $\alpha^1 = \mathrm{id}_{TM} \in \Gamma(\mathrm{End}\, TM)$, $N^i = \mathrm{Span}\,(\mathrm{Im}\,\alpha^{i+1})$ for $i = 0, 1, \dots, d-1$ and $N^d \supset \mathrm{Span}\,(\mathrm{Im}\,\alpha^{d+1})$. We note that $\alpha(X, Y) = \alpha^2(X, Y)$, where α is the usual second fundamental form and $X, Y \in \Gamma(TM)$. The following lemma is proved in a way similar to that in [15, pp. 171–172].

LEMMA 2.1. *For a section $\zeta_i \in \Gamma(N^i)$ $(i = 0, 1, \dots, d)$ and $X \in T_pM$,*

$$\begin{aligned}
\widetilde{\nabla}_X \zeta_0 &\in N_p^0 \oplus N_p^1 , \\
\widetilde{\nabla}_X \zeta_i &\in N_p^{i-1} \oplus N_p^i \oplus N_p^{i+1} \quad (i = 1, \dots, d-1) , \\
\widetilde{\nabla}_X \zeta_d &\in N_p^{d-1} \oplus N_p^d .
\end{aligned}$$

By Lemma 2.1, we can define for $\zeta_i \in \Gamma(N^i)$ and $X \in \Gamma(TM)$,

$$\begin{aligned}
S_X^i \zeta_i &:= -(\widetilde{\nabla}_X \zeta_i)^{N^{i-1}} && (i = 1, 2, \dots, d) , \\
\nabla_X^i \zeta_i &:= (\widetilde{\nabla}_X \zeta_i)^{N^i} && (i = 0, 1, \dots, d) , \\
T_X^i \zeta_i &:= (\widetilde{\nabla}_X \zeta_i)^{N^{i+1}} && (i = 0, 1, \dots, d-1) .
\end{aligned}$$

It is easy to check that $\nabla^0 = \nabla$, $T_X^0 Y = \alpha^2(X, Y)$ for any $X, Y \in \Gamma(TM)$ and

$$S^i \in \wedge^1(\mathrm{Hom}(N^i, N^{i-1})) , \quad \nabla^i \in \mathcal{C}(N^i, g^i) , \quad T^{i-1} \in \wedge^1(\mathrm{Hom}(N^{i-1}, N^i))$$

for $i = 1, 2, \dots, d$. Consequently we obtain the *Frenet formulas* of f:

$$\widetilde{\nabla}_X Y = \nabla_X^0 Y + T_X^0 Y , \tag{2.1}$$

$$\widetilde{\nabla}_X \zeta_i = -S_X^i \zeta_i + \nabla_X^i \zeta_i + T_X^i \zeta_i \quad (i = 1, 2, \dots, d-1) , \tag{2.2}$$

$$\widetilde{\nabla}_X \zeta_d = -S_X^d \zeta_d + \nabla_X^d \zeta_d , \tag{2.3}$$

where $X, Y \in \Gamma(TM) = \Gamma(N^0)$ and $\zeta_i \in \Gamma(N^i)$ for $i = 1, 2, \dots, d$. We note that (2.1) is the (usual) Gauss formula, and S^1 is the (usual) shape operator restricted to the first normal space N^1. We denote the normal connection by $\nabla^\perp$. Then we obtain $\nabla_X^\perp \zeta_1 = \nabla_X^1 \zeta_1 + T_X^1 \zeta_1$ for any $X \in \Gamma(TM)$ and $\zeta_1 \in \Gamma(N^1)$. Moreover we have $g^{i-1}(\zeta_{i-1}, S_X^i \zeta_i) = g^i(T_X^{i-1} \zeta_{i-1}, \zeta_i)$ for $i = 1, 2, \dots, d$. We define the differentiation of N^i-valued $(0, k)$-tensor field P by

$$(D_X^i P)(X_1, \dots, X_k) := \nabla_X^i (P(X_1, \dots, X_k)) - \sum_{j=1}^{k} P(X_1, \dots, \nabla_X X_j, \dots, X_k) .$$

By a straightforward calculation, we can see the following lemma.

LEMMA 2.2. *Let $f : M \to Q^n_t(c)$ be an isometric immersion into a space of constant curvature c. If f is nicely curved of order d, then the following equations hold.*

$$R(X,Y)Z = c\,(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + S^1_X T^0_Y Z - S^1_Y T^0_X Z\,, \tag{2.4}$$

$$R^i(X,Y)\zeta_i = T^{i-1}_X S^i_Y \zeta_i - T^{i-1}_Y S^i_X \zeta_i + S^{i+1}_X T^i_Y \zeta_i - S^{i+1}_Y T^i_X \zeta_i \quad (i = 1, 2, \dots, d-1)\,, \tag{2.5}$$

$$R^d(X,Y)\zeta_d = T^{d-1}_X S^d_Y \zeta_d - T^{d-1}_Y S^d_X \zeta_d\,, \tag{2.6}$$

$$(D^{i-1}_X S^i)_Y \zeta_i = (D^{i-1}_Y S^i)_X \zeta_i \qquad (i = 1, 2, \dots, d)\,, \tag{2.7}$$

$$(D^{i+1}_X T^i)_Y \zeta_i = (D^{i+1}_Y T^i)_X \zeta_i \qquad (i = 0, 1, \dots, d-1)\,, \tag{2.8}$$

$$S^{i-1}_X S^i_Y \zeta_i = S^{i-1}_Y S^i_X \zeta_i \qquad (i = 2, 3, \dots, d)\,, \tag{2.9}$$

$$T^{i+1}_X T^i_Y \zeta_i = T^{i+1}_Y T^i_X \zeta_i \qquad (i = 0, 1, \dots, d-2)\,, \tag{2.10}$$

where $X, Y, Z \in \Gamma(TM)$, $\zeta_i \in \Gamma(N^i)$, and $R(= R^0)$ and R^i are the curvature tensor of the Levi-Civita connection $\nabla(= \nabla^0)$ of M and ∇^i, respectively.

Lemma 2.1, (2.10) and the symmetry of T^0 show the following corollary.

COROLLARY 2.3. *Under the same assumption as in Lemma* 2.2, *we have*

$$\alpha^{i+1}(X_1, \dots, X_{i+1}) = T^{i-1}_{X_1}\alpha^i(X_2, \dots, X_{i+1}) = T^{i-1}_{X_1} T^{i-2}_{X_2} \cdots T^0_{X_i} X_{i+1}, \tag{2.11}$$

where $X_j \in \Gamma(TM)$ for $j \le i+1$. Moreover, α^{i+1} is $(i+1)$-symmetric for $i \le d$.

By virtue of Corollary 2.3, for the simplicity, we are allowed to write

$$\alpha^{i+1}(X^k, Y^l) := \alpha^{i+1}(\underbrace{X, \dots, X}_{k}, \underbrace{Y, \dots, Y}_{l})$$

for $X, Y \in \Gamma(TM)$ and $k + l = i + 1$.

The tensor field $T^{d-1} \in \wedge^1(\mathrm{Hom}(N^{d-1}, N^d))$ closely relates to the reduction of the codimension of isometric immersions which are nicely curved of order d.

LEMMA 2.4. *Let $f : M \to Q^n_t(c)$ be an isometric immersion which is nicely curved of order d. If there exists an open piece U of M such that T^{d-1} is vanishing on U, then there exists a totally geodesic pseudo-Riemannian submanifold P of $Q^n_t(c)$ such that $f(U) \subset P$.*

PROOF. By the assumption, the subbundle $\oplus_{i=0}^{d-1} N^i$ of $f^\# TQ$ is parallel with respect to $\widetilde{\nabla}$ on U, where TQ is the tangent bundle of $Q^n_t(c)$. Putting $P := \exp^Q(\oplus_{i=0}^{d-1} N^i|_U)$, we get this lemma, where $\exp^Q$ is the exponential map of $Q^n_t(c)$. □

2.2. Extremal Lorentzian surfaces. Let $(V, \langle\ ,\ \rangle)$ be a Lorentzian vector 2-space and (ξ, η) a null basis of V such that $\langle \xi, \xi\rangle = \langle \eta, \eta\rangle = 0$ and $\langle \xi, \eta\rangle = \mu \ne 0$. In this paper, we say that the signature of the null basis (ξ, η) is *positive* (resp. *negative*), if $\mu > 0$ (resp. $\mu < 0$). For any vector $v \in V$, we have

$$v = \frac{1}{\mu}\langle v, \eta\rangle \xi + \frac{1}{\mu}\langle v, \xi\rangle \eta\,. \tag{2.12}$$

It is easy to show the following lemma.

LEMMA 2.5. *Let (E, g^E) be a Lorentzian plane bundle over a manifold M and $\nabla^E \in \mathcal{C}(E, g^E)$. Let (ξ, η) be a local frame field of E such that $g^E(\xi, \xi) = g^E(\eta, \eta) = 0$ and $g^E(\xi, \eta)$ is nonzero constant. Then there exists a local 1-form ρ^E on M such that*

$$\nabla^E \xi = \rho^E \otimes \xi\,, \qquad R^{\nabla^E}(X, Y)\xi = d\rho^E(X, Y)\xi\,,$$
$$\nabla^E \eta = -\rho^E \otimes \eta\,, \qquad R^{\nabla^E}(X, Y)\eta = -d\rho^E(X, Y)\eta\,.$$

PROOF. Put $\lambda := g^E(\xi, \eta) \in \mathbb{R}$ and $\rho^E(X) := (1/\lambda)g^E(\nabla^E_X \xi, \eta)$. Using (2.12), we have

$$\nabla^E_X \xi = \frac{1}{\lambda} g^E(\nabla^E_X \xi, \eta)\xi + \frac{1}{\lambda} g^E(\nabla^E_X \xi, \xi)\eta = \rho^E(X)\xi\,.$$

It is easy to obtain the other equations. □

An endomorphism $J^E \in \Gamma(\mathrm{End}\,(E))$ is called a *parahermitian structure* of (E, g^E), if J^E satisfies $(J^E)^2 = \mathrm{id}_E$ and $g^E(J^E(\zeta_1), J^E(\zeta_2)) = -g^E(\zeta_1, \zeta_2)$ for any $\zeta_1, \zeta_2 \in \Gamma(E)$.

Local null frames (ξ, η) on U and (ξ', η') on U' have the *same* signature, if $\langle \xi, \eta\rangle\langle \xi', \eta'\rangle > 0$ on $U \cap U' \neq \emptyset$. Moreover we assume that (ξ, η) and (ξ', η') have the same orientation. Then we note that

$$\begin{pmatrix} \xi' & \eta' \end{pmatrix} = \begin{pmatrix} \xi & \eta \end{pmatrix} \begin{bmatrix} \sqrt{\langle \xi', \eta'\rangle / \langle \xi, \eta\rangle}\, e^{\theta} & 0 \\ 0 & \sqrt{\langle \xi', \eta'\rangle / \langle \xi, \eta\rangle}\, e^{-\theta} \end{bmatrix} \quad \text{on } U \cap U'\,,$$

where θ is a smooth function on $U \cap U'$.

LEMMA 2.6. *Let (E, g^E) be a Lorentzian plane bundle over a manifold M and $\nabla^E \in \mathcal{C}(E, g^E)$. If E is orientable, then E admits a ∇^E-parallel parahermitian structure of (E, g^E).*

PROOF. Let (ξ, η) be an oriented local null frame on $U \subset M$ of (E, g^E). We can define $J^E \in \Gamma(\mathrm{End}\,(E))$ by $J^E(\xi) := \xi$, $J^E(\eta) := -\eta$. Let (ξ', η') be another oriented local null frame on U' ($U \cap U' \neq \emptyset$) with the same signature of (ξ, η). Defining the endomorphism $J^{E'}$ of E on U' by $J^{E'}(\xi') := \xi'$, $J^{E'}(\eta') := -\eta'$, we can see that $J^E = J^{E'}$ on $U \cap U'$. Since E is orientable, J^E is well-defined on M. Taking a connection form ρ^E as in Lemma 2.5, we see that J^E is a ∇^E-parallel parahermitian structure of (E, g^E). □

From the proof above, we note that oriented null frames with the same signature define the same parahermitian structure J^E. In particular, when the parahermitian structure J^E is defined by oriented null frames with positive (resp. negative) signature, we call J^E *positive* (resp. *negative*).

Let $(M^2_1(K), g)$ be an oriented 2-dimensional Lorentzian surface of the Gaussian curvature K. Let (e_1, e_2) be an oriented local orthonormal frame of M such that $g(e_i, e_j) = (-1)^i \delta_{ij}$, where δ_{ij} is the Kronecker delta. We put

$$e_+ := \frac{1}{\sqrt{2}}(e_1 + e_2)\,, \quad e_- := \frac{1}{\sqrt{2}}(-e_1 + e_2)\,, \tag{2.13}$$

which satisfy $g(e_\pm, e_\pm) = 0$ and $g(e_+, e_-) = 1$, hence (e_+, e_-) is a local null frame with positive signature and the same orientation to (e_1, e_2). Then, by Lemma 2.5, there exists a local 1-form ρ on M such that

$$\nabla e_\pm = \pm\rho \otimes e_\pm\,, \quad d\rho(e_+, e_-) = K\,, \tag{2.14}$$

where ∇ is the Levi-Civita connection of M. By virtue of Lemma 2.6, we can take a ∇-parallel parahermitian structure $J \in \Gamma(\mathrm{End}(TM))$ such that $J(e_\pm) = \pm e_\pm$. We call this endomorphism J the *canonical paraKähler structure* on $(M_1^2(K), g)$.

Let $f : M_1^2(K) \to Q_t^n(c)$ be an isometric immersion. Then the mean curvature vector field H of f is

$$H = \frac{1}{2}\,(-\alpha(e_1, e_1) + \alpha(e_2, e_2)) = \alpha(e_+, e_-)\,,$$

where (e_1, e_2) and (e_+, e_-) are local frames on M in (2.13). Thus, f is extremal if and only if $\alpha(e_+, e_-) = 0$.

LEMMA 2.7. *Let $f : M_1^2(K) \to Q_t^n(c)$ be an extremal isometric immersion. Then,*

$$(D^1_{e_-}\alpha^2)(e_+^2) = 0\,, \quad (D^1_{e_+}\alpha^2)(e_-^2) = 0\,. \tag{2.15}$$

Moreover, if f is nicely curved of order d, then we have

$$\alpha^{i+1}(e_+^k, e_-^l) = 0 \quad (i = 1, 2, \ldots, d,\ k + l = i + 1,\ k, l \geq 1)\,. \tag{2.16}$$

PROOF. By (2.8), we have

$$\begin{aligned}(D^1_{e_-}\alpha^2)(e_+^2) &= \nabla^1_{e_-}\alpha^2(e_+^2) - 2\rho(e_-)\alpha^2(e_+^2) = (D^1_{e_-}T^0)_{e_+}(e_+)\\ &= (D^1_{e_+}T^0)_{e_-}(e_+) = \nabla^1_{e_+}T^0_{e_-}e_+ = \nabla^1_{e_+}\alpha^2(e_+, e_-) = 0\,.\end{aligned}$$

By a similar calculation, we obtain $(D^1_{e_+}\alpha^2)(e_-^2) = 0$. If $k, l \geq 1$,

$$\alpha^{i+1}(e_+^k, e_-^l) = T^{i-1}_{e_+} \cdots T^{i-k+1}_{e_+} T^{i-k}_{e_-} \cdots T^1_{e_-}\alpha^2(e_+, e_-) = 0\,.$$

This completes the proof. □

For an extremal isometric immersion $f : M_1^2(K) \to Q_t^n(c)$ which is nicely curved of order d, using (2.13) and (2.16), by the equation $\alpha^2(e_1, e_1) = \alpha^2(e_2, e_2)$ and arguments similar to that in the proof of Lemma 2.7, we have

$$\begin{cases}\alpha^{i+1}(e_+^{i+1}) = (\sqrt{2})^{i-1}\big(\alpha^{i+1}(e_1, e_2^i) + \alpha^{i+1}(e_2^{i+1})\big)\,,\\ \alpha^{i+1}(e_-^{i+1}) = (\sqrt{2})^{i-1}\big(-\alpha^{i+1}(e_1, e_2^i) + \alpha^{i+1}(e_2^{i+1})\big)\,.\end{cases} \tag{2.17}$$

Noting N^i is nonzero for $i = 1, 2, \ldots, d-1$, by (2.17), we have

$$N^i = \mathrm{Span}\{\alpha^{i+1}(e_1, e_2^i),\ \alpha^{i+1}(e_2^{i+1})\} = \mathrm{Span}\{\alpha^{i+1}(e_+^{i+1}),\ \alpha^{i+1}(e_-^{i+1})\}\,, \tag{2.18}$$

$$\mathrm{rank}\,N^i = 1 \quad \text{or} \quad 2\,. \tag{2.19}$$

2.3. Surfaces with isotropic higher fundamental forms. In this subsection, we study extremal isometric immersions of Lorentzian surfaces with isotropic higher fundamental forms. The property that higher fundamental forms are isotropic is closely related to horizontal reflector lifts mentioned in the next section. We provide several lemmas which are often used in the following sections.

Let V and W be vector spaces with inner products and $\beta : V \times \cdots \times V \to W$ a k-multilinear map into W. We say that β is *spacelike* (resp. *timelike*) *isotropic* if there exists a constant λ such that $\langle \beta(u^k), \beta(u^k)\rangle = \lambda$ for any spacelike (resp. timelike) unit vectors u. For the simplicity, we say that such a map is spacelike (resp. timelike) λ-isotropic. Then we note the following lemma.

LEMMA 2.8. *Under the notation above, a k-multilinear map β is spacelike λ-isotropic if and only if β is timelike $(-1)^k\lambda$-isotropic. Moreover, if β is spacelike isotropic, then $\langle \beta(e^k), \beta(e^k)\rangle = 0$ for all null vector $e \in V$.*

PROOF. If β is spacelike λ-isotropic, then we have

$$\langle \beta((v/\|v\|)^k), \beta((v/\|v\|)^k)\rangle = \lambda \quad \text{for any spacelike vector } 0 \neq v \in V .$$

Hence we obtain the equation $(*)$ $\langle \beta(v^k), \beta(v^k)\rangle = \lambda \, \langle v, v\rangle^k$ on the set of all spacelike vectors of V, which forms nonempty open subset in V. This equation $(*)$ holds on V, since the function $V \ni w \mapsto \langle \beta(w^k), \beta(w^k)\rangle - \lambda \, \langle w, w\rangle^k \in \mathbb{R}$ is real analytic (more precisely, it is a polynomial in n variables $w = (w_1, \dots, w_n)$, where $n = \dim V$). So, we have

$$\langle \beta(v^k), \beta(v^k)\rangle = (-1)^k \lambda \quad \text{for any unit timelike vector } v \in V.$$

We can similarly see the converse and the statement for null vectors. □

By Lemma 2.8, in the case that V is indefinite, we use the term "isotropic" as "spacelike isotropic".

We say that the $(k+1)$st fundamental form α^{k+1} of an isometric immersion $f : M \to \widetilde{M}$ is (*spacelike*) *isotropic* if α_p^{k+1} is $\lambda_{k,p}$-isotropic at each point $p \in M$. The function $\lambda_k : M \to \mathbb{R}$ defined by $\lambda_k(p) := \lambda_{k,p}$ is called the (spacelike) isotropic function. If the isotropic function λ_k is constant on M, then α^{k+1} is called *constant* λ_k-isotropic. We note that α^{k+1} is λ_k-isotropic if and only if

$$g^k(\alpha^{k+1}(e_1, e_2^k), \alpha^{k+1}(e_2^{k+1})) = 0 \tag{2.20}$$

for any orthonormal tangent vectors e_1, e_2 to M (e.g. [8, Lemma 1.1]). Moreover, in the case that $f : M_1^2(K) \to Q_t^n(c)$ is extremal, by (2.17), we obtain

$$g^k(\alpha^{k+1}(e_2^{k+1}), \alpha^{k+1}(e_2^{k+1})) = -g^k(\alpha^{k+1}(e_1, e_2^k), \alpha^{k+1}(e_1, e_2^k)) = \lambda_k , \tag{2.21}$$

where (e_1, e_2) is an orthonormal basis of T_pM such that $g(e_i, e_j) = (-1)^i\delta_{ij}$.

LEMMA 2.9. *Let $f : M_1^2(K) \to Q_t^n(c)$ be an extremal isometric immersion which is nicely curved of order d. We assume that there exists a positive integer $i\,(\leq d)$ such that α^{i+1} is λ_i-isotropic. Then λ_i is everywhere nonzero on M if and only if $i < d$. In the case of $i < d$,*

N^i is a Lorentzian plane bundle over M. Moreover if M is oriented, then N^i is orientable and (N^i, g^i) admits a ∇^i-parallel parahermitian structure J^i.

PROOF. Let (e_1, e_2) be an orthonormal basis of T_pM such that $g(e_i, e_j) = (-1)^i\delta_{ij}$. Noting (2.18), (2.20) and (2.21), we see that λ_i is equal to zero at p if and only if N^i_p is a degenerate plane, a null line or zero at p. Since f is nicely curved of order d, we can see that λ_i is everywhere nonzero if and only if $i < d$. Then N^i is a Lorentzian plane bundle over M, since the normal vectors $\alpha^{i+1}(e_1, e_2^i)$, $\alpha^{i+1}(e_2^{i+1})$ span a Lorentzian plane.

When M is oriented, taking oriented local orthonormal frames (e_1, e_2) and (e_1', e_2') on an open set U of M, we can get a function $\theta \in C^\infty(U)$ such that $e_1' = (\cosh\theta)e_1 + (\sinh\theta)e_2$ and $e_2' = (\sinh\theta)e_1 + (\cosh\theta)e_2$ on U. The following local frames $(\alpha^{i+1}(e_1, e_2^i), \alpha^{i+1}(e_2^{i+1}))$ and $(\alpha^{i+1}(e_1', e_2'^i), \alpha^{i+1}(e_2'^{i+1}))$ of N^i are local orthogonal frames with same orientation of N^i. In fact, we have

$$\begin{aligned}&\left(\alpha^{i+1}(e_1', e_2'^i) \quad \alpha^{i+1}(e_2'^{i+1})\right)\\ &\quad = \left(\alpha^{i+1}(e_1, e_2^i) \quad \alpha^{i+1}(e_2^{i+1})\right)\begin{bmatrix}\cosh((i+1)\theta) & \sinh((i+1)\theta)\\ \sinh((i+1)\theta) & \cosh((i+1)\theta)\end{bmatrix}.\end{aligned}$$

Thus N^i is orientable. From Lemma 2.6, we obtain a ∇^i-parallel parahermitian structure J^i of (N^i, g^i). □

When $f : M_1^2(K) \to Q_t^n(c)$ is extremal and nicely curved of order d, and there exists a positive integer $i(< d)$ such that α^{i+1} is λ_i-isotropic, by Lemma 2.9, we can take the following local orthonormal frame (e_{2i+1}, e_{2i+2}) of N^i defined by

$$e_{2i+1} := \frac{1}{\sqrt{|\lambda_i|}}\,\alpha^{i+1}(e_1, e_2^i)\,, \quad e_{2i+2} := \frac{1}{\sqrt{|\lambda_i|}}\,\alpha^{i+1}(e_2^{i+1})\,,$$

where (e_1, e_2) is a local oriented orthonormal frame on M such that $g(e_i, e_j) = (-1)^i\delta_{ij}$. Moreover, noting (2.17), we put the signature $\varepsilon_i := \lambda_i/|\lambda_i| \in \{1, -1\}$,

$$\xi_i := \frac{1}{\sqrt{2}}(e_{2i+1} + e_{2i+2}) = \frac{1}{(\sqrt{2})^i\sqrt{|\lambda_i|}}\alpha^{i+1}(e_+^{i+1})\,, \tag{2.22}$$

$$\eta_i := \frac{1}{\sqrt{2}}(-e_{2i+1} + e_{2i+2}) = \frac{1}{(\sqrt{2})^i\sqrt{|\lambda_i|}}\alpha^{i+1}(e_-^{i+1})\,, \tag{2.23}$$

which satisfy

$$\begin{aligned}&g^i(e_{2i+1}, e_{2i+1}) = -\varepsilon_i\,, && g^i(e_{2i+1}, e_{2i+2}) = 0\,, && g^i(e_{2i+2}, e_{2i+2}) = \varepsilon_i\,,\\ &g^i(\xi_i, \xi_i) = 0\,, && g^i(\xi_i, \eta_i) = \varepsilon_i\,, && g^i(\eta_i, \eta_i) = 0\,.\end{aligned}$$

LEMMA 2.10. *Let $f : M_1^2(K) \to Q_t^n(c)$ be an extremal isometric immersion which is nicely curved of order d. We assume that there exists a positive integer $i < d$ such that α^{i+1} is λ_i-isotropic. Then we obtain*

$$\nabla_X^i\alpha^{i+1}(e_\pm^{i+1}) = \left(\pm\rho_i(X) + \frac{1}{2}d(\log|\lambda_i|)(X)\right)\alpha^{i+1}(e_\pm^{i+1})\,, \tag{2.24}$$

where (e_+, e_-) is a local null frame on $M_1^2(K)$ and ρ_i is the connection form of ∇^i with respect to (ξ_i, η_i) defined by (2.22) and (2.23).

PROOF. We obtain (2.24) by a simple calculation. □

LEMMA 2.11. *Let $f : M_1^2(K) \to Q_t^n(c)$ be an extremal isometric immersion which is nicely curved of order d. We assume that there exists a positive integer $i < d$ such that α^{i+1} is λ_i-isotropic and $(D^i \alpha^{i+1})$ is $(i+2)$-symmetric, that is, $(D^i_{e_\mp}\alpha^{i+1})(e_\pm^{i+1}) = 0$. Then we have*

$$(D^{i+1}_{e_-}\alpha^{i+2})(e_+^{i+2}) = 0\,, \quad (D^{i+1}_{e_+}\alpha^{i+2})(e_-^{i+2}) = 0\,.$$

Thus $(D^{i+1}\alpha^{i+2})$ is $(i+3)$-symmetric.

PROOF. We can prove this lemma by a simple calculation. In fact, we have

$$\begin{aligned}
&(D^{i+1}_{e_-}\alpha^{i+2})(e_+^{i+2}) \\
&= \nabla^{i+1}_{e_-}\alpha^{i+2}(e_+^{i+2}) - (i+2)\rho(e_-)\alpha^{i+2}(e_+^{i+2}) \\
&= \nabla^{i+1}_{e_-}T^i_{e_+}\alpha^{i+1}(e_+^{i+1}) - (i+2)\rho(e_-)\alpha^{i+2}(e_+^{i+2}) \\
&= (D^{i+1}_{e_-}T^i)_{e_+}\alpha^{i+1}(e_+^{i+1}) + T^i_{e_+}(D^i_{e_-}\alpha^{i+1})(e_+^{i+1})
\end{aligned}$$

using the Codazzi equation (2.8) for T^i, (2.11), (2.16) and (2.24)

$$\begin{aligned}
&= (D^{i+1}_{e_+}T^i)_{e_-}\alpha^{i+1}(e_+^{i+1}) \\
&= \nabla^{i+1}_{e_+}\alpha^{i+2}(e_-, e_+^{i+1}) + \rho(e_+)\alpha^{i+2}(e_-, e_+^{i+1}) - T^i_{e_-}\nabla^i_{e_+}\alpha^{i+1}(e_+^{i+1}) \\
&= -T^i_{e_-}\left(\rho_i(e_+) + \frac{1}{2}d(\log|\lambda_i|)(e_+)\right)\alpha^{i+1}(e_+^{i+1}) \\
&= -\left(\rho_i(e_+) + \frac{1}{2}d(\log|\lambda_i|)(e_+)\right)\alpha^{i+2}(e_-, e_+^{i+1}) = 0\,.
\end{aligned}$$

In a similar way, we have $(D^{i+1}_{e_+}\alpha^{i+2})(e_-^{i+2}) = 0$. □

LEMMA 2.12. *Under the same assumptions as in Lemma 2.11, if α^{i+1} is constant λ_i-isotropic, then α^{i+1} is ∇^i-parallel, that is, $(D^i\alpha^{i+1}) = 0$.*

PROOF. Since α^{i+1} is isotropic, $g^i(\alpha^{i+1}(e_+^{i+1}), \alpha^{i+1}(e_+^{i+1})) = 0$. Thus we see

$$\begin{aligned}
g^i((D^i_{e_+}\alpha^{i+1})(e_+^{i+1}), \alpha^{i+1}(e_+^{i+1})) &= \frac{1}{2}e_+ g^i(\alpha^{i+1}(e_+^{i+1}), \alpha^{i+1}(e_+^{i+1})) \\
&\quad -(i+1)\rho(e_+)g^i(\alpha^{i+1}(e_+^{i+1}), \alpha^{i+1}(e_+^{i+1})) = 0.
\end{aligned}$$

Using Lemma 2.11, we obtain

$$\begin{aligned}
g^i((D^i_{e_+}\alpha^{i+1})(e_+^{i+1}), \alpha^{i+1}(e_-^{i+1})) &= e_+ g^i(\alpha^{i+1}(e_+^{i+1}), \alpha^{i+1}(e_-^{i+1})) \\
&\quad -g^i(\alpha^{i+1}(e_+^{i+1}), \nabla^i_{e_+}\alpha^{i+1}(e_-^{i+1})) \\
&\quad -(i+1)\rho(e_+)g^i(\alpha^{i+1}(e_+^{i+1}), \alpha^{i+1}(e_-^{i+1}))\,.
\end{aligned}$$

By Lemma 2.10, we have

$$g^i(\alpha^{i+1}(e_+^{i+1}), \nabla^i_{e_+}\alpha^{i+1}(e_-^{i+1})) = -(i+1)\rho(e_+)g^i(\alpha^{i+1}(e_+^{i+1}), \alpha^{i+1}(e_-^{i+1}))\,,$$

and hence, $g^i((D^i_{e_+}\alpha^{i+1})(e_+^{i+1}), \alpha^{i+1}(e_-^{i+1})) = e_+(2^i\lambda_i) = 0$. Because of $i < d$, $(\alpha^{i+1}(e_+^{i+1}), \alpha^{i+1}(e_-^{i+1}))$ is a local frame of N^i. Hence, $(D^i_{e_+}\alpha^{i+1})(e_+^{i+1}) = 0$. In a similar way, we have $(D^i_{e_-}\alpha^{i+1})(e_-^{i+1}) = 0$, thus $(D^i\alpha^{i+1}) = 0$. □

LEMMA 2.13. *Under the same assumptions as in Lemma* 2.11, *we have*

$$\rho_i(X) = \frac{1}{2}(d\log|\lambda_i|)(JX) + (i+1)\rho(X), \tag{2.25}$$

$$(d\rho_i)(e_+, e_-) = \frac{1}{2}\Delta\log|\lambda_i| + (i+1)K, \tag{2.26}$$

where ρ_i (*resp.* ρ) *is the connection form of* ∇^i (*resp.* ∇) *with respect to* (ξ_i, η_i) *defined by* (2.22) *and* (2.23) (*resp.* (e_+, e_-)), *and* Δ *is the Laplace operator of* $M_1^2(K)$.

PROOF. Using Lemma 2.11 and (2.23), we have

$$\begin{aligned}\nabla^i_{e_+}\eta_i &= \sqrt{|\lambda_i|}\left(e_+\left(\frac{1}{\sqrt{|\lambda_i|}}\right) - \frac{1}{\sqrt{|\lambda_i|}}(i+1)\rho(e_+)\right)\eta_i \\ &= -\left(\frac{1}{2}d(\log|\lambda_i|(Je_+)) + (i+1)\rho(e_+)\right)\eta_i\,.\end{aligned}$$

Noting $\nabla^i_X\eta_i = -\rho_i(X)\eta_i$, we have

$$\rho_i(e_+) = \frac{1}{2}d(\log|\lambda_i|)(Je_+) + (i+1)\rho(e_+)\,.$$

In a similar way, we get $\rho_i(e_-) = (1/2)d(\log|\lambda_i|)(Je_-) + (i+1)\rho(e_-)$. These equations show (2.25). Using (2.25) and $(d\rho)(e_+, e_-) = K$, we have (2.26). □

LEMMA 2.14. *Let* $f : M_1^2(K) \to Q_t^n(c)$ *be an extremal isometric immersion which is nicely curved of order* d. *We assume that there exists a positive integer* $i \le d$ *such that* α^i *and* α^{i+1} *are isotropic with isotropic functions* λ_{i-1} *and* λ_i *respectively. In the case of* $i < d$,

$$\begin{cases} T^{i-1}_{e_+}\xi_{i-1} = \sqrt{|2\lambda_i/\lambda_{i-1}|}\xi_i\,, \qquad T^{i-1}_{e_-}\xi_{i-1} = 0\,, \\ T^{i-1}_{e_+}\eta_i = 0\,, \qquad T^{i-1}_{e_-}\eta_{i-1} = \sqrt{|2\lambda_i/\lambda_{i-1}|}\eta_i\,, \end{cases} \tag{2.27}$$

$$\begin{cases} S^i_{e_+}\xi_i = 0\,, \qquad S^i_{e_-}\xi_i = \varepsilon_{i-1}\varepsilon_i\sqrt{|2\lambda_i/\lambda_{i-1}|}\xi_{i-1}\,, \\ S^i_{e_+}\eta_i = \varepsilon_{i-1}\varepsilon_i\sqrt{|2\lambda_i/\lambda_{i-1}|}\eta_{i-1}\,, \qquad S^i_{e_-}\eta_i = 0\,, \end{cases} \tag{2.28}$$

where $\varepsilon_j := \lambda_j/|\lambda_j|$ $(j = i-1, i)$ *and* $\lambda_0 := 1$. *In the case of* $i = d$, *we have*

$$\begin{cases} S^d_{e_+}\alpha^{d+1}(e_+^{d+1}) = 0\,, \qquad S^d_{e_-}\alpha^{d+1}(e_+^{d+1}) = \varepsilon_{d-1}\dfrac{(\sqrt{2})^{d+1}\lambda_d}{\sqrt{|\lambda_{d-1}|}}\xi_{d-1}, \\ S^d_{e_+}\alpha^{d+1}(e_-^{d+1}) = \varepsilon_{d-1}\dfrac{(\sqrt{2})^{d+1}\lambda_d}{\sqrt{|\lambda_{d-1}|}}\eta_{d-1}\,, \qquad S^d_{e_-}\alpha^{d+1}(e_-^{d+1}) = 0\,. \end{cases} \tag{2.29}$$

PROOF. We can simply prove these equations by (2.22) and (2.23). □

For the later use, when α^i and α^{i+1} are isotropic, we rewrite (2.28) as follows.

$$S^i_{e_2}\alpha^{i+1}(e_2^{i+1}) = \frac{\lambda_i}{\lambda_{i-1}}\alpha^i(e_2^i)\,, \tag{2.30}$$

where e_2 is a spacelike unit tangent vector to $M_1^2(K)$.

Since (2.15) in Lemma 2.7 holds, we can repeatedly use Lemma 2.11 under the assumption: $\alpha^2, \alpha^3, \dots, \alpha^{k+1}$ are isotropic. Hence, Lemma 2.11, and equations (2.25) and (2.26) are available for any $i = 1, 2, \dots, k$, and also Lemmas 2.12 and 2.14.

LEMMA 2.15. *Let $f : M_1^2(K) \to Q_t^n(c)$ be an extremal isometric immersion which is nicely curved of order d. If there exists a positive integer $k \le d$ such that α^{i+1} is λ_i-isotropic for any $i = 1, 2, \dots, k$, then we have $\lambda_1 = (c - K)/2$ and, when $k > 1$,*

$$\lambda_{i+1} = \frac{1}{2}\left(c - \binom{i+2}{2}K - \frac{1}{2}\Delta \log|\lambda_1 \cdots \lambda_i|\right)\lambda_i \quad (i = 1, 2, \dots, k-1)\,.$$

Therefore isotropic functions $\lambda_1, \dots, \lambda_k$ depend only on c, K and higher derivatives of K.

PROOF. By the Gauss equation (2.4), (2.27), (2.28), and $\lambda_0 = 1$, we have

$$\begin{aligned} Ke_+ = R(e_+, e_-)e_+ &= c\,(g(e_-, e_+)e_+ - g(e_+, e_+)e_-) + S^1_{e_+}T^0_{e_-}e_+ - S^1_{e_-}T^0_{e_+}e_+ \\ &= ce_+ - \varepsilon_0\varepsilon_1 2|\lambda_1|\xi_0\,, \end{aligned}$$

hence $\lambda_1 = (c - K)/2$, where $\varepsilon_0 = g(e_+, e_-) = 1$ and $\xi_0 = e_+$. For $i = 1, 2, \dots, k-1$ if $k < d$, or $i = 1, 2, \dots, k-2$ if $k = d$, by the equation (2.5) on N^i, (2.27) and (2.28),

$$R^i(e_+, e_-)\xi_i = T^{i-1}_{e_+}S^i_{e_-}\xi_i - T^{i-1}_{e_-}S^i_{e_+}\xi_i + S^{i+1}_{e_+}T^i_{e_-}\xi_i - S^{i+1}_{e_-}T^i_{e_+}\xi_i = 2\left(\frac{\lambda_i}{\lambda_{i-1}} - \frac{\lambda_{i+1}}{\lambda_i}\right)\xi_i\,.$$

In the case of $k = d$, using (2.5) on N^{d-1} and (2.29), we can see that the equation above holds for $i = k - 1$. On the other hand, from (2.26), we have

$$R^i(e_+, e_-)\xi_i = \left(\frac{1}{2}\Delta \log|\lambda_i| + (i+1)K\right)\xi_i \quad (i = 1, 2, \dots, k-1)\,.$$

Since ξ_i is everywhere nonzero for $i = 1, 2, \dots, k < d$,

$$\frac{1}{2}\Delta \log|\lambda_i| + (i+1)K = 2\left(\frac{\lambda_i}{\lambda_{i-1}} - \frac{\lambda_{i+1}}{\lambda_i}\right)\,.$$

From these equations and $\lambda_1 = (c - K)/2$, we obtain for $i = 1, 2, \dots, k-1$

$$\lambda_{i+1} = \frac{1}{2}\left(c - \frac{(i+1)(i+2)}{2}K - \frac{1}{2}\Delta \log|\lambda_1 \cdots \lambda_i|\right)\lambda_i\,.$$

Thus we complete the proof. □

COROLLARY 2.16. *Let $f : M_1^2(K) \to Q_t^n(c)$ be an extremal isometric immersion which is nicely curved of order d. We assume that there exists a positive integer $k (\le d)$ such that α^{i+1} is λ_i-isotropic for any $i = 1, 2, \dots, k$. If K is constant, then we have*

$$\lambda_i = \frac{1}{2}\left(c - \frac{i(i+1)}{2}K\right)\lambda_{i-1} \quad (i = 1, 2, \dots, k \le d)\,.$$

Therefore $\alpha^2, \alpha^3, \dots, \alpha^{k+1}$ *are constant isotropic, moreover,* $n \geq 2k$ *and* $t \geq k$. *In the case of* $k = d$, *we see that* α^{d+1} *is* 0*-isotropic and* $K = 2c/d(d+1)$.

We furnish a nonexistence result on extremal Lorentzian surfaces by virtue of the corollary above.

THEOREM 2.17. *There are no extremal isometric immersions from* $M_1^2(K)$ *into* $Q_t^n(c)$ *of which all higher fundamental forms are isotropic if the constant Gaussian curvature* $K \neq 2c/i(i+1)$ *for any integer* i.

PROOF. For any isometric immersion $f : M_1^2(K) \to Q_t^n(c)$, there exists a positive integer d such that f is nicely curved of order d. Then, by the assumptions and Lemma 2.9, the constant isotropic function λ_{d-1} is nonzero and λ_d is zero. From Corollary 2.16, we obtain $K = 2c/d(d+1)$ and the proof of the theorem is completed. $\square$

REMARK 2.18. Bryant [1] proves the nonexistence of minimal immersions from surfaces of constant positive Gaussian curvature $K \neq 2c/i(i+1)$ $(i \in \mathbb{N})$ into $S_0^n(1/\sqrt{c})$ (without the isometry condition). See Calabi [2] and Wallach [16] also. The theorem above is a pseudo-Riemannian version of [1, Theorem 1.5]. It is natural to ask whether the condition "isotropicity" of higher fundamental forms is needed or not. We can find an extremal surface of $K = 1$ ($c = 1$ and $i = 1$) whose second fundamental form is not isotropic (see the last paragraph in Section 5).

3. Congruence theorem for immersions with horizontal reflector lifts. Let V be a $2m$-dimensional vector space V with inner product $\langle\ ,\ \rangle$ of neutral signature. A *parahermitian structure* on V is an endomorphism $J : V \to V$ such that $J^2 = \mathrm{id}_V$ and $\langle JX, JY\rangle = -\langle X, Y\rangle$ for all X and $Y \in V$. The eigenspaces $V_\pm := \mathrm{Ker}\,(J \mp \mathrm{id}_V)$ of a parahermitian structure J are m-dimensional totally isotropic in V, which satisfy $V = V_+ \oplus V_-$. We denote the space of all parahermitian structures on V by $Z(V)$.

Let $(\widetilde{M}, \widetilde{g})$ be a $2m$-dimensional manifold of neutral signature. The reflector space $\mathcal{Z}(\widetilde{M})$ is defined by

$$\mathcal{Z} := \mathcal{Z}(\widetilde{M}) := \bigcup_{p\in\widetilde{M}} Z(T_p\widetilde{M})\,.$$

Note that the reflector space is a subbundle of $\mathrm{End}(T\widetilde{M})$. The bundle projection $p : \mathcal{Z} \to \widetilde{M}$ and the Levi-Civita connection $\widetilde{\nabla}$ on $\widetilde{M}$ induce the decomposition $T\mathcal{Z} = T^h\mathcal{Z} \oplus T^v\mathcal{Z}$ into the horizontal subbundle $T^h\mathcal{Z}$ and the vertical subbundle $T^v\mathcal{Z}$.

Let $f : (M_1^2, g) \to (\widetilde{M}_m^{2m}, \widetilde{g})$ be an isometric immersion. A section of $\widetilde{J} \in \Gamma(f^\#\mathcal{Z})$ is a *reflector lift* of f (or M), if $\widetilde{J}|_{TM} = J$, where J is the canonical paraKähler structure on M. Then, putting $J^\perp := \widetilde{J}|_{T^\perp M}$, we have a parahermitian structure of the normal bundle $T^\perp M$. An isometric immersion f admits a *horizontal reflector lift* if $\widetilde{J}$ is $\widetilde{\nabla}$-parallel, that is, $\widetilde{\nabla}\widetilde{J} = 0$, where $\widetilde{\nabla}$ is the induced connection from the Levi-Civita connection of $\widetilde{M}$. By a straightforward calculation, we have

LEMMA 3.1. *The reflector lift* $\widetilde{J}$ *is horizontal if and only if the* (*usual*) *second fundamental form* α *satisfies* $\alpha(X, JY) = J^{\perp}\alpha(X, Y)$ *for all* $X, Y \in TM$ *and* $\nabla^{\perp} J^{\perp} = 0$.

In Riemannian geometry, a surface with horizontal twistor lift is called *superminimal*. Indeed, superminimal surfaces are minimal. The following proposition is a corresponding result to neutral geometry.

PROPOSITION 3.2. *An isometric immersion with horizontal reflector lift is extremal.*

PROOF. By Lemma 3.1, $\alpha(e_+, e_-) = \alpha(Je_+, e_-) = \alpha(e_+, Je_-) = -\alpha(e_+, e_-)$, which gives $H = \alpha(e_+, e_-) = 0$. □

In this section, we give a congruence theorem for isometric immersions with horizontal reflector lifts.

LEMMA 3.3. *Let* $f : M_1^2(K) \to Q_m^{2m}(c)$ *be an isometric immersion which is nicely curved of order* d. *If* f *has a horizontal reflector lift* $\widetilde{J}$, *then* $\widetilde{J}$ *is* N^i*-preserving. Moreover* $J^i := \widetilde{J}|_{N^i} \in \mathrm{End}\,(N^i)$ *is* ∇^i*-parallel and* $J^i T_X^{i-1}\zeta_{i-1} = T_{JX}^{i-1}\zeta_{i-1}$ *for any* $\zeta_{i-1} \in \Gamma(N^{i-1})$ *and* $i = 1, 2, \dots, d$. *In particular, we obtain*

$$\alpha^{i+1}(X_1, \dots, X_i, JX_{i+1}) = J^i \alpha^{i+1}(X_1, \dots, X_{i+1}) .$$

PROOF. From Lemma 3.1, $\widetilde{J}$ is N^1-preserving. Putting $J^1 := \widetilde{J}|_{N^1}$, we have

$$(\nabla_X^1 J^1)(\zeta_1) = S_X^1(J^1\zeta_1) - J(S_X^1\zeta_1) - T_X^1(J^1\zeta_1) + J^{\perp}(T_X^1\zeta_1) .$$

Since $J^{\perp}$ is $(N^1)^{\perp}$-preserving, we see that J^1 is ∇^1-parallel, $S_X^1(J^1\zeta_1) = J(S_X^1\zeta_1)$ and $\widetilde{J}$ is N^2-preserving. If there exists a positive integer $k(< d)$ such that J^k is ∇^k-parallel and $\widetilde{J}$ is N^k-preserving, then we can put $J^k := \widetilde{J}|_{N^k} \in \Gamma(\mathrm{End}\,(N^k))$. Moreover,

$$(\nabla_X^k J^k)(\zeta_k) = S_X^k(J^k\zeta_k) - J^{k-1}(S_X^k\zeta_k) - T_X^k(J^k\zeta_k) + J^{\perp}(T_X^k\zeta_k) .$$

Since $J^{\perp}$ is $(N^k)^{\perp}$-preserving, we see that J^k is ∇^k-parallel, $S_X^k(J^k\zeta_k) = J^{k-1}(S_X^k\zeta_k)$ and $\widetilde{J}$ is N^{k+1}-preserving. By the inductive method, we have the lemma. □

LEMMA 3.4. *Let* $f : M_1^2(K) \to Q_m^{2m}(c)$ *be an isometric immersion which is nicely curved of order* d. *If* f *has a horizontal reflector lift* $\widetilde{J}$, *then the* i*-th normal bundle* N^i *is a Lorentzian plane bundle, and the* $(i+1)$*st fundamental form* α^{i+1} *is isotropic and the isotropic function* λ_i *is everywhere nonzero for any* i $(i = 1, 2, \dots, d-1)$, *thus* $\mathrm{rank}\, N^d = 2(m-d)$.

PROOF. Let (e_1, e_2) be a local oriented orthonormal frame on M. Then we have

$$\begin{aligned} g^i(\alpha^{i+1}(e_1, e_2^i), \alpha^{i+1}(e_2^{i+1})) &= g^i(\alpha^{i+1}(Je_2, e_2^i), \alpha^{i+1}(e_2^{i+1})) \\ &= g^i(J^i\alpha^{i+1}(e_2^{i+1}), \alpha^{i+1}(e_2^{i+1})) = 0 . \end{aligned}$$

Hence, α^{i+1} is isotropic. Thus, from Lemma 2.9, we can see that its spacelike isotropic function is everywhere nonzero and N^i is a Lorentzian plane bundle over M. □

Let $f : M_1^2(K) \to Q_m^{2m}(c)$ be an isometric immersion which is nicely curved of order d. By Lemma 3.4, we can consider the local null frame (ξ_i, η_i) of N^i defined by (2.22) and

(2.23) for $i = 1, 2, \dots, d-1$. We see that ξ_i (resp. η_i) is a $(+1)$- (resp. (-1)-) eigenvector of J^i by Lemma 3.3.

We obtain a congruence result on isometric immersions with horizontal reflector lift into a space of constant curvature.

THEOREM 3.5. *Let $f, \overline{f} : M_1^2(K) \to Q_d^{2d}(c)$ be isometric immersions with horizontal reflector lifts from a connected oriented Lorentzian surface. If both immersions f and $\overline{f}$ are nicely curved of order d, then there exists an isometry Φ of $Q_d^{2d}(c)$ such that $\overline{f} = \Phi \circ f$.*

PROOF. The corresponding objects associated with $\overline{f}$ are denoted by the symbol with "–", for example, $\overline{T^\perp M}$ is the normal bundle of $\overline{f}$. Let (e_+, e_-) be a local frame field such that $g(e_\pm, e_\pm) = 0$, $g(e_+, e_-) = 1$ and $Je_\pm = \pm e_\pm$. We take a local frame (ξ_i, η_i) for any $i = 1, 2, \dots, d-1$. We define $\Phi : T^\perp M \to \overline{T^\perp M}$ by $\Phi(\xi_i) = \overline{\xi_i}$ and $\Phi(\eta_i) = \overline{\eta_i}$ for $i = 1, 2, \dots, d-1$. Since the reflector lifts of f and $\overline{f}$ are horizontal, Φ preserves the higher fundamental forms. From Lemma 2.15, $\overline{\lambda_i} = \lambda_i$ for $i = 1, 2, \dots, d-1$. By Lemma 2.10 and (2.25), all coefficients of $\nabla^\perp$ with respect to $(\xi_1, \eta_1, \dots, \xi_{d-1}, \eta_{d-1})$ depend only on c, K (and i). Then we see that

$$\Phi(\nabla^\perp_X \xi_i) = \overline{\nabla^\perp}_X \bar{\xi_i}\,, \quad \Phi(\nabla^\perp_X \eta_i) = \overline{\nabla^\perp}_X \bar{\eta_i}\,,$$

that is, Φ preserves the normal connections. By the congruence theorem for isometric immersions into a space form (see [6], for example), we see that there exists an isometry Φ of $Q_d^{2d}(c)$ such that $\overline{f} = \Phi \circ f$. □

REMARK 3.6. An existence theorem for an extremal isometric immersion from a simply connected Lorentzian surface into $Q_d^{2d}(c)$ can be found in [13]. The integrability condition is described by the functions $\lambda_1, \dots, \lambda_{d-1}$ in Lemma 2.15.

4. Lorentzian Boruvka spheres. Hereafter, we provide examples of isometric immersions with horizontal reflector lifts. First of all, we recall a notion of helical geodesic immersions in pseudo-Riemannian geometry.

Let c be a unit speed spacelike curve of a pseudo-Riemannian manifold N. For a positive integer d and positive constants $\kappa_1, \dots, \kappa_{d-1}$, the curve c is a *helix* of type $\Lambda = (d; \kappa_1, \dots, \kappa_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$, if c satisfies the Frenet-Serre formula:

$$\nabla_{c'} c_i = -\varepsilon_{i-1}\varepsilon_i \kappa_{i-1} c_{i-1} + \kappa_i c_{i+1} \quad (i = 1, 2, \dots, d)\,,$$

where ∇ is the Levi-Civita connection of N, $c_1, \dots, c_d$ is an orthonomal frame field along c, $\varepsilon_i = \langle c_i, c_i \rangle \in \{1, -1\}$, $\varepsilon_0 = \kappa_0 = \kappa_d = 0$ and $c_0 = c_{d+1} = 0$. We call the integer d the order of c. A helix of order one is a spacelike (resp. timelike) geodesic of N, if $\varepsilon_1 = +1$ (resp. $\varepsilon_1 = -1$).

Let $f : M \to N$ be an isometric immersion between pseudo-Riemannian manifolds. The immersion f is a *spacelike* (resp. *timelike*) *helical geodesic immersion* (HGI) of type Λ, if f maps arbitrary unit speed spacelike (resp. timelike) geodesic γ of M into a helix of type Λ which is independent of γ. This notion is a generalization in pseudo-Riemannian geometry of that in Sakamoto [14]. In [9], the second author proves the following conditions are equivalent in the case that the domain of f is indefinite.

(1) f is a spacelike HGI of type $\Lambda = (d; \kappa_1, \dots, \kappa_{d-1}; \varepsilon_1, \dots, \varepsilon_d)$.
(2) f is a timelike HGI of type $\overline{\Lambda} = (d; \kappa_1, \dots, \kappa_{d-1}; (-1)^1\varepsilon_1, \dots, (-1)^d\varepsilon_d)$.

Hence we call such immersions HGIs for short. We shall introduce the following notions.

DEFINITION 4.1. An submanifold L of a space form $(Q_t^n(c), \widetilde{g})$ is said to be *totally geodesic* if the Levi-Civita connection $\widetilde{\nabla}$ of $Q_t^n(c)$ naturally induces an affine connection on L, that is, $\widetilde{\nabla}_X Y \in \Gamma(TL)$ for any $X, Y \in \Gamma(TL)$. Furthermore, if the pullback $j^*\widetilde{g}$ is identically vanishing on L, it makes L a *null r-plane* of $Q_t^n(c)$, where $r = \dim L$ and $j : L \hookrightarrow Q_t^n(c)$ is the inclusion map. A null curve c on $Q_t^n(c)$ is *null r-planar*, if there is a null r-plane L of $Q_t^n(c)$ such that $\mathrm{Im}\,(c) \subset L$. An isometric immersion $f : M \to Q_t^n(c)$ between indefinite pseudo-Riemannian manifolds is a *null r-planar geodesic immersion* (PGI), if there exists a positive integer r such that, for each null geodesic γ of M, the null curve $f \circ \gamma$ is null r-planar in $Q_t^n(c)$. If a null r-PGI f is not a null q-PGI for any $q < r$, we say that f is a null *proper* r-PGI.

Notice that a null r-plane L in $Q_t^n(c)$ is contained a null r_0-plane $\overline{L}$ ($r_0 := \min\{n-t, t\}$). Therefore any null r-PGI is null r_0-planar geodesic. For example, any HGIs $f : M \to Q_t^n(c)$ are null r_0-planar geodesic ([9, Theorem D]). In general, the converse is not held. See [11] for details. Simpler examples are totally umbilic isometric immersions, which are null proper 1-PGIs, since the immersions map null geodesics of submanifolds to null geodesics in the ambient space. We deal with this notion in the last section.

In Riemannian geometry, typical examples of HGIs are the standard minimal immersion of compact rank one symmetric spaces. In the case of the n-dimensional sphere, associated with each positive integer d, there exists an isometric minimal immersion $\psi_{n,d} : S^n(r(d)) \to S^{m(d)}$, where $S^n(r(d)) := S_0^n(r(d))$, $S^{m(d)} := S^{m(d)}(1)$, and the radius $r(d)$ and the dimension $m(d)$ are given as follows.

$$r(d) = \sqrt{\frac{d(d+n-1)}{n}}, \quad m(d) = (2d+n-1)\frac{(d+n-2)!}{d!(n-1)!} - 1\,.$$

The immersion $\psi_{n,d}$ is called the d-th standard minimal immersion of $S^n(r(d))$ and (spacelike) HGI of type $\Lambda_{n,d} := (d; \kappa_1, \dots, \kappa_{d-1}; +1, \dots, +1)$, where $\kappa_1, \dots, \kappa_{d-1}$ are certain positive constants. In the case of $n = 2$, these immersions are called the *Boruvka spheres*.

In [10], the second author constructs, associated with each the d-th standard minimal immersion of $S^n(r(d))$, an extremal isometric immersion $\psi_{n,d,t}$ of $S_t^n(r(d))$ into $S_{l(d)}^{m(d)} := S_{l(d)}^{m(d)}(1)$ for arbitrary $t = 1, \dots, n$, where the index $l(d)$ is a certain integer (see [10] for details). In the case of $(n, t) = (2, 1)$, the integer $l(d)$ is equal to d. The constructed immersion $\psi_{n,d,t}$ $(t = 1, \dots, n-1)$ is a spacelike HGI of type $\Lambda_{n,d}$ and $\psi_{n,d,n}$ is a timelike HGI of type $\overline{\Lambda}_{n,d}$ ([10]). See also [5] for a construction of harmonic maps of S_1^2 into a space of constant sectional curvature one.

We recall the d-th standard minimal immersions of $S^2(r(d))$ into the unit sphere. Let Δ_{S^2} be the Laplacian on S^2. It is well-known that all eigenvalues are given by $\mu_d = d(d+1)$ for any nonnegative integer d and the dimension of the eigenspace V_d of Δ_{S^2} corresponding to the eigenvalue μ_d is $2d+1$. Taking an orthonormal basis $f_1, \ldots, f_{2d+1}$ of V_d with respect to the L^2-inner product:

$$g(f_1, f_2) := \int_{S^2} f_1\, f_2\, d\nu_{S^2}\,, \quad f_1,\ f_2 \in V_d\,,$$

where $d\nu_{S^2}$ is proportional to the volume element of S^2 and normalized in such a way that $\int_{S^2} d\nu_{S^2} = \dim V_d = 2d+1$, we can see $(f_1)^2 + \cdots + (f_{2d+1})^2 = 1$ on S^2, and identify $V_d \cong \mathbb{R}^{2d+1}$. Then the d-th standard minimal immersion $\psi_{2,d} : S^2(r(d)) \to S^{2d} \subset \mathbb{R}^{2d+1}$ is given by

$$\psi_{2,d} := (f_1, \ldots, f_{2d+1}) \circ \chi_{1/r(d)}\,,$$

where χ_k is the homothetic transformation in $\mathbb{R}^n$ defined by $\chi_k(v) := kv$ for $v \in \mathbb{R}^n$. We remark that the d-th eigenspace V_d of $S^2 \subset \mathbb{R}^3$ is given by

$$V_d = \{P|_{S^2} \mid P \in H_d(\mathbb{R}^3)\}\,,$$

where $H_d(\mathbb{R}^3)$ is the space of homogeneous harmonic polynomials of degree d on $\mathbb{R}^3$ and $P|_{S^2}$ is the restriction of P to $S^2 \subset \mathbb{R}^3$.

We summarize the construction of extremal immersions obtained in [10] as follows. Let $\mathbb{F}[x] := \mathbb{F}[x_1, x_2, x_3]$ be the polynomial algebra in variables x_1, x_2, x_3, where $\mathbb{F}$ is the set of all complex numbers $\mathbb{C}$ or real numbers $\mathbb{R}$, $\mathbb{F}_d[x]$ the space of homogeneous polynomials of degree d, and $\Delta_{\mathbb{R}^3_1} := -\partial_1^2 + \partial_2^2 + \partial_3^2$ the Laplacian on $\mathbb{R}^3_1$. Putting

$$H_d(\mathbb{R}^3_1) := \{P \in \mathbb{R}_d[x] \mid \Delta_{\mathbb{R}^3_1} P = 0\}\,,$$

we see $\dim H_d(\mathbb{R}^3_1) = 2d+1$. Moreover, we can see that $\Delta_{S^2_1}(P|_{S^2_1}) = d(d+1)P|_{S^2_1}$ for $P \in H_d(\mathbb{R}^3_1)$, where $S^2_1 \subset \mathbb{R}^3_1$ is the unit Lorentzian 2-sphere, and $\Delta_{S^2_1}$ is the Laplacian of S^2_1. Let ρ_1 be the ring endomorphism on $\mathbb{C}[x]$ defined by $\rho_1(1) := 1$, $\rho_1(x_1) := \sqrt{-1}\,x_1$, $\rho_1(x_i) := x_i$ $(i = 2, 3)$. We call ρ_1 a (1-)*Wick rotation*, which satisfies $\rho_1(P) \in H_d(\mathbb{R}^3_1) \oplus \sqrt{-1}H_d(\mathbb{R}^3_1)$ for any $P \in H_d(\mathbb{R}^3)$. We can take a basis $P_{-d}, \ldots, P_d$ of $H_d(\mathbb{R}^3)$ such that

$$(P_{-d})^2 + \cdots + (P_d)^2 = (x_1^2 + x_2^2 + x_3^2)^d\,,$$

$\rho_1(P_i) \in \sqrt{-1}H_d(\mathbb{R}^3_1)$ $(i < 0)$ and $\rho_1(P_i) \in H_d(\mathbb{R}^3_1)$ $(i \geq 0)$. We note that $(\rho_1(P_{-d}))^2 + \cdots + (\rho_1(P_d))^2 = (-x_1^2 + x_2^2 + x_3^2)^d$. Putting $Q_i := -\sqrt{-1}\rho_1(P_i)$ $(i < 0)$ and $\rho_1(P_i)$ $(i \geq 0)$, we have a basis $Q_{-d}, \ldots, Q_d$ of $H_d(\mathbb{R}^3_1)$ such that

$$-(Q_{-d})^2 - \cdots - (Q_{-1})^2 + (Q_0)^2 + (Q_1)^2 + \cdots + (Q_d)^2 = (-x_1^2 + x_2^2 + x_3^2)^d\,.$$

We define the indefinite scalar product g_1 on $H_d(\mathbb{R}^3_1)$ by

$$g_1(Q_i, Q_j) := \varepsilon_i \delta_{ij}\,,$$

where $\varepsilon_i = -1$ ($d < 0$), $\varepsilon_i = +1$ ($d \geq 0$) and identify $H_d(\mathbb{R}^3_1) \cong \mathbb{R}^{2d+1}_d$. Then we can obtain the extremal isometric immersion into the unit neutral $2d$-sphere:

$$\psi_{2,d,1} := (Q_{-d}|_{S^2_1}, \ldots, Q_d|_{S^2_1}) \circ \chi_{1/r(d)} : S^2_1(r(d)) \to S^{2d}_d \subset \mathbb{R}^{2d+1}_d .$$

The immersion $\psi_{2,d,1}$ is called the *Lorentzian Boruvka sphere* (LBS) in S^{2d}_d throughout this paper. We can see explicite representations for $\psi_{2,d,1}$ ($d = 2, 3$) in [9, Examples 3.5, 3.6].

Composing homotheties and anti-isometries of $S^2_1(r(d))$ and S^{2d}_d, we can obtain extremal immersions from $Q^2_1(K_d)$ to $Q^{2d}_d(c)$, where $K_d := 2c/d(d+1)$ and $c \neq 0$. We denote this immersion by $\phi_{d,c}$. This immersion also is referred to as LBS in $Q^{2d}_d(c)$.

An isometric immersion $f : M \to Q^n_t(c)$ is said to be *full* in $Q^n_t(c)$, if there exist no totally geodesic submanifolds N of $Q^n_t(c)$ such that $f(M) \subset N$ and $\dim N < n$. We can give the following.

PROPOSITION 4.2. *Let $f : M^2_1(K) \to Q^n_t(c)$ be a HGI of order d of an oriented Lorentzian surface. If f is extremal, then f is nicely curved of order d and there exists a totally geodesic submanifold P of $Q^n_t(c)$ such that P is isometric to $Q^{2d}_d(c)$ and $f(M)$ is full in P. Let f' be the isometric immersion such that $f = \iota \circ f'$, where ι is the inclusion $P \hookrightarrow Q^n_t(c)$. Then, $f' : M^2_1(K) \to P$ admits a horizontal reflector lift, and K is constant. Moreover α^{i+1} is nonzero constant isotropic for $i = 1, \ldots, d-1$, and α^{d+1} is identically vanishing.*

PROOF. Let $f : M^2_1(K) \to Q^n_t(c)$ be a HGI of type $(d; \kappa_1, \ldots, \kappa_{d-1}; \varepsilon_1, \ldots, \varepsilon_d)$ and nicely curved of order m. In the case of $d = 1$, since f is totally geodesic, the assertion follows. Thus we may assume that $d \geq 2$.

Let $u \in T_pM$ be a spacelike unit tangent vector to M at $p \in M$ and γ a geodesic such that $\gamma(0) = p$ and $\gamma'(0) = u$. From the Frenet-Serre formula of $c := f \circ \gamma$, $\widetilde{\nabla}_U c_1 = \kappa_1 c_2$, where $U := c'$. From the Frenet formula of f,

$$\widetilde{\nabla}_U c_1 = \nabla_U c_1 + T^0_U c_1 = \alpha^2(U^2) .$$

Thus we have $\kappa_1 c_2 = \alpha^2(U^2)$. So α^2 is nonzero $(\varepsilon_2\kappa_1^2)$-constant isotropic by the arbitrarity of spacelike unit tangent vector u. From Lemma 2.9, we see that N^1 is a Lorentzian plane bundle over M, therefore, $m \geq 2$.

We assume that there exists an integer k ($2 \leq k \leq \min\{d, m\}$) such that $\kappa_1 \cdots \kappa_{i-1} c_i = \alpha^i(U^i)$ for any i ($2 \leq i \leq k$). Then α^i is nonzero $(\varepsilon_i \kappa_1^2 \cdots \kappa_{i-1}^2)$-constant isotropic by the arbitrarity of spacelike unit tangent vector u and $c_i \in N^{i-1}$. Thus, by virtue of Lemma 2.9, N^{i-1} is a Lorentzian plane bundles on M and admit a parahermitian structure J^{i-1}. Furthermore, from the Frenet-Serre formula of c,

$$\widetilde{\nabla}_U c_i = -\varepsilon_{i-1}\varepsilon_i \kappa_{i-1} c_{i-1} + \kappa_i c_{i+1} ,$$

and, from the Frenet formula of f,

$$\widetilde{\nabla}_U c_i = -S^{i-1}_U c_i + \nabla^{i-1}_U c_i + T^{i-1}_U c_i .$$

Since α^i is constant isotropic for $i = 2, \ldots, k$, using Lemma 2.12, we obtain $\nabla_U^{i-1} c_i = 0$. Moreover, the equation (2.30) gives $S_U^{i-1} c_i = \varepsilon_{i-1}\varepsilon_i\kappa_{i-1}c_{i-1}$, thus we have $T_U^{i-1} c_i = \kappa_i c_{i+1}$, hence $\alpha^{i+1}(U^{i+1}) = \kappa_1 \cdots \kappa_i c_{i+1}$.

In the case of $k = \min\{d, m\} = d$, by definition, $\kappa_d = 0$. Thus, we can see $\alpha^{d+1} = 0$, that is, $T^d = 0$, which implies that $m = d$ and, using Lemma 2.4, there exists the totally geodesic submanifold P such that P is isometric to $Q_d^{2d}(c)$ and $f(M)$ is full in P. On the other hand, if $k = \min\{d, m\} = m$, then κ_m has zeros by Lemma 2.9, hence $\kappa_m = 0$. By a similar way, we obtain the same conclusion. Since P is totally geodesic in $Q_t^n(c)$, the ∇^i-parallel parahermitian structure J^i on N^i is also one on the i-th normal bundle $N^{i'}$ of f' for any i $(i = 1, 2, \ldots, d-1)$. We can define $J^\perp \in \Gamma(\mathrm{End}(T^\perp M))$ by

$$J^\perp := \bigoplus_{i=1}^{d-1} J^i \,.$$

Then we can check $\alpha(X, JY) = J^\perp\alpha(X, Y)$ and $\nabla^\perp J^\perp = 0$. By Lemma 3.1, f' admits horizontal reflector lift $\widetilde{J} := J \oplus J^\perp$.

Since the second fundamental form α^2 is $(\varepsilon_2\kappa_1^2)$-constant isotropic, using Lemma 2.15, we see that M is of constant Gaussian curvature. □

In Riemannian case, the Boruvka spheres $\psi_{2,d} : S^2(r(d)) \to S^{2d}$ are superminimal, that is, these have horizontal twistor lifts. Note that the LBS $\phi_{d,c} : Q_1^2(K_d) \to Q_d^{2d}(c)$ is an extremal HGI of order d ([10, Proposition 3.8.]) and nicely curved of order d from Proposition 4.2. Therefore, Proposition 4.2 for the LBSs corresponds to the result above for Boruvka spheres. We summarize as follows.

COROLLARY 4.3. *The Lorentzian Boruvka sphere $\phi_{d,c} : Q_1^2(K_d) \to Q_d^{2d}(c)$ is nicely curved of order d and has a horizontal reflector lift.*

5. Extremal surfaces with null r-planar geodesics. In this section, we prove our main result in this paper:

THEOREM 5.1. *Let $f : M_1^2(K) \to Q_t^n(c)$ be an extremal null r-planar geodesic immersion from an oriented connected Lorentzian surface of constant Gaussian curvature K and c nonzero. If f is real analytic and full, then f is locally congruent to the Lorentzian Boruvka sphere $\phi_{r,c}$ with $K = 2c/r(r+1)$, $n = 2r$ and $t = r$. Moreover the order r is proper.*

The proof requires a few technical steps which show that the highest normal bundle is zero. Notice that, in general, the highest normal bundle need not be contained in the i-th osculating space of an isometric immersion for a positive integer i.

Let $f : M_1^2(K) \to Q_t^n(c)$ be an extremal isometric immersion which is nicely curved of order d. Thus, we obtain the decomposition $f^\# TQ = \oplus_{i=0}^d N^i$, where $N^0 = TM$ and N^i is the i-th normal bundle for $i > 0$. Put, for $i = 0, 1, \ldots, d$,

$$\hat{\xi}_i := \alpha^{i+1}(e_+^{i+1}) \,, \quad \hat{\eta}_i := \alpha^{i+1}(e_-^{i+1}) \,,$$

where $e_\pm = (\pm e_1 + e_2)/\sqrt{2}$ and (e_1, e_2) is a local oriented orthonormal frame such that $g(e_i, e_j) = (-1)^i \delta_{ij}$. Then we see $N^i = \mathrm{Span}\,\{\hat{\xi}_i,\ \hat{\eta}_i\}$ for $i = 1, \dots, d-1$ from (2.18) and note that $\mathrm{Span}\,\{\hat{\xi}_d,\ \hat{\eta}_d\} \subset N^d = (\mathrm{Osc}^d(f))^\perp$. We inductively define higher derivatives of α^{d+1} with respect to ∇^d by

$$(D^{d(1)}\alpha^{d+1})(X_1, X_2, \dots, X_{d+2}) := (D^d_{X_1}\alpha^{d+1})(X_2, \dots, X_{d+2}),$$
$$(D^{d(k)}\alpha^{d+1})(X_1, X_2, \dots, X_{d+k+1}) := (D^d_{X_1}(D^{d(k-1)}\alpha^{d+1}))(X_2, \dots, X_{d+k+1})$$

for any positive integer k.

Set $c := f \circ \gamma$ for a null geodesic γ of M. Then we get

$$\widetilde{\nabla}(\dot{c}^1) = \dot{c}\,, \quad \widetilde{\nabla}(\dot{c}^2) = \alpha^2(\dot{c}^2)\,, \quad \widetilde{\nabla}(\dot{c}^3) = f^3_{c,2}\widetilde{\nabla}(\dot{c}^2) + \alpha^3(\dot{c}^3)\,,$$

where $\dot{c}$ is the tangent vector field of c and $f^3_{c,2}$ is a function along c. We inductively obtain

$$\widetilde{\nabla}(\dot{c}^{k+1}) = \sum_{i=2}^{k} f^{k+1}_{c,i}\widetilde{\nabla}(\dot{c}^i) + \alpha^{k+1}(\dot{c}^{k+1}) \quad (k = 0, 1, \dots, d)\,, \tag{5.1}$$

where $f^{k+1}_{c,i}$ is a function along c. By a simple calculation, we have

LEMMA 5.2. *For any nonnegative integer k,*

$$\widetilde{\nabla}(\dot{c}^{d+k+1}) = \sum_{i=2}^{d+k} f^{d+k+1}_{c,i}\widetilde{\nabla}(\dot{c}^i) + (D^{d(k)}\alpha^{d+1})(\dot{c}^{d+k+1})\,,$$

where $f^{d+k+1}_{c,i}$ is a function along a null curve $c = f \circ \gamma$ as above.

Hereafter, when not specified otherwise, we work under the assumption that $f : M^2_1(K) \to Q^n_t(c)$ is an extremal null r-PGI which is nicely curved of order d.

LEMMA 5.3. *For any nonnegative integers i, j,*

$$\widetilde{g}(\hat{\xi}_i, \hat{\xi}_j) = 0, \quad \widetilde{g}(\hat{\eta}_i, \hat{\eta}_j) = 0\,.$$

PROOF. By Lemma 5.2, (5.1) and the definition of null r-PGI, this lemma holds. □

PROPOSITION 5.4. *If $M^2_1(K)$ is oriented, then N^i is an orientable Lorentzian plane bundle, thus N^i admits a ∇^i-parallel parahermitian structure J^i of (N^i, g^i) for $i = 1, \dots, d-1$. Moreover, the $(i+1)$st fundamental form α^{i+1} is λ_i-isotropic for $i = 1, \dots, d$, where $\lambda_i := g^i(\hat{\xi}_i, \hat{\eta}_i)/2^i$. In particular, $\lambda_1, \dots, \lambda_{d-1}$ are non-vanishing and λ_d has zeros on M.*

PROOF. From (2.18) and (2.19), we can see that rank $N^i = 1$ or 2 for $i = 1, 2, \dots, d-1$. Thus, by virtue of Lemma 5.3, $\mu_i := g^i(\hat{\xi}_i, \hat{\eta}_i)$ must be nonvanishing on M for $i = 1, \dots, d-1$. Therefore we can see that N^i is an orientable Lorentzian plane bundle and, using Lemma 2.6, admits a ∇^i-parallel parahermitian structure J^i of (N^i, g^i) for $i = 1, \dots, d-1$. Note that $\mu_d := g^d(\hat{\xi}_d, \hat{\eta}_d)$ has zeros on M. We can see that α^{i+1} is $(\mu_i/2^i)$-isotropic for $i = 1, \dots, d$. In fact, from

$$\hat{\xi}_i = (\sqrt{2})^{i-1}\big(\alpha^{i+1}(e_1, e^i_2) + \alpha^{i+1}(e^{i+1}_2)\big)\,,$$

$$\hat{\eta}_i = (\sqrt{2})^{i-1}\big(-\alpha^{i+1}(e_1, e_2^i) + \alpha^{i+1}(e_2^{i+1})\big)$$

and Lemma 5.3 again, we obtain $g^i(\alpha^{i+1}(e_1, e_2^i), \alpha^{i+1}(e_2^{i+1})) = 0$. By the arbitrarity of a local oriented orthonormal frame (e_1, e_2), α^{i+1} is spacelike $(\mu_i/2^i)$-isotropic for $i = 1, \dots, d$. □

For a vector bundle E with a bundle connection ∇^E over $M_1^2(K)$ and an E-valued $(0, l)$-tensor field Q, we note the Ricci identity

$$\begin{aligned}(D^E(D^E Q))&(X, Y, X_1, \dots, X_l) - (D^E(D^E Q))(Y, X, X_1, \dots, X_l)\\ &= R^E(X, Y)Q(X_1, \dots, X_l) - \sum_{i=1}^{l} Q(X_1, \dots, R(X, Y)X_i, \dots, X_l)\,,\end{aligned}$$

where an E-valued $(0, l+1)$-tensor field $(D^E Q)$ is defined by

$$(D^E Q)(X, X_1 \dots, X_l) := \nabla_X^E(Q(X_1, \dots, X_l)) - \sum_{i=1}^{l} Q(X_1, \dots, \nabla_X X_i, \dots, X_l)\,.$$

We put for any positive integer k

$$\hat{\xi}_{d+k} := (D^{d(k)}\alpha^{d+1})(e_+^{d+k+1})\,, \quad \hat{\eta}_{d+k} := (D^{d(k)}\alpha^{d+1})(e_-^{d+k+1})\,,$$

which are in $\mathrm{Osc}^{d+k+1}(f) \cap N^d$. Then we have

LEMMA 5.5. *For any positive integer k,*

$$(D^{d(k)}\alpha^{d+1})(e_\mp, e_\pm^{d+k}) \in \mathrm{Osc}^{d+k-1}(f)\,.$$

PROOF. In the case of $k = 1$, we see, in Lemma 2.11,

$$(D^{d(1)}\alpha^{d+1})(e_\mp, e_\pm^{d+1}) = (D_{e_\mp}^d\alpha^{d+1})(e_\pm^{d+1}) = 0 \in \mathrm{Osc}^d(f)\,.$$

For $k \geq 2$, we assume that

$$(D^{d(k-1)}\alpha^{d+1})(e_\mp, e_\pm^{d+k-1}) \in \mathrm{Osc}^{d+k-2}(f)\,.$$

By the Ricci identity,

$$\begin{aligned}(D^{d(k)}\alpha^{d+1})(e_-, e_+^{d+k}) = (D^{d(k)}&\alpha^{d+1})(e_+, e_-, e_+^{d+k-1})\\ &+R^d(e_-, e_+)(D^{d(k-2)}\alpha^{d+1})(e_+^{d+k-1})\\ &-(d+k-1)(d\rho)(e_-, e_+)(D^{d(k-2)}\alpha^{d+1})(e_+^{d+k-1})\,.\end{aligned}$$

On the R.H.S. in the equation above, we get, using (2.6) for the 2nd term,

$$\begin{aligned}\text{(the 1st term)} &= \nabla_{e_+}^d(D^{d(k-1)}\alpha^{d-1})(e_-, e_+^{d+k-1})\\ &\quad -(d+k-2)\rho(e_+)(D^{d(k-1)}\alpha^{d-1})(e_-, e_+^{d+k-1}) \in \mathrm{Osc}^{d+k-1}(f)\,,\\ \text{(the 2nd term)} &= T_{e_-}^{d-1}S_{e_+}^d\hat{\xi}_{d+k-2} - T_{e_+}^{d-1}S_{e_-}^d\hat{\xi}_{d+k-2} \in \mathrm{Osc}^d(f)\,,\\ \text{(the 3rd term)} &= (d+k-1)K\xi_{d+k-2} \in \mathrm{Osc}^{d+k-1}(f)\,.\end{aligned}$$

So we have $(D^{d(k)}\alpha^{d+1})(e_-, e_+^{d+k}) \in \mathrm{Osc}^{d+k-1}(f)$. By a similar calculation, we obtain $(D^{d(k)}\alpha^{d+1})(e_+, e_-^{d+k}) \in \mathrm{Osc}^{d+k-1}(f)$. We finish the proof of this lemma. □

From the lemma above, we have

LEMMA 5.6. *For any nonnegative integer* k,

$$\mathrm{Osc}^k(f) = \mathrm{Span}\,\{\hat{\xi}_l, \hat{\eta}_l \mid l = 0, 1, \ldots, k\}\,.$$

Differentiating sections $\hat{\xi}_{d+k}$, $\hat{\eta}_{d+k}$ of N^d, we obtain some lemmas on N^d.

LEMMA 5.7. *For any nonnegative integer* k,

$$\nabla^d_{e_+}\hat{\xi}_{d+k} = \hat{\xi}_{d+k+1} + (d+k+1)\rho(e_+)\hat{\xi}_{d+k}\,,$$
$$\nabla^d_{e_-}\hat{\eta}_{d+k} = \hat{\eta}_{d+k+1} - (d+k+1)\rho(e_-)\hat{\eta}_{d+k}\,.$$

PROOF. By definition and a simple calculation, we can prove this lemma. □

LEMMA 5.8. *If* $\tilde{g}(\zeta_d, \hat{\xi}_d) = \tilde{g}(\zeta_d, \hat{\eta}_d) = 0$ *for a vector* $\zeta_d \in N^d$, *then* $S^d_X\zeta_d = 0$ *for any* $X \in TM$.

PROOF. We note that $S^d \in \wedge^1\mathrm{Hom}\,(N^d, N^{d-1})$ and $(\hat{\xi}_{d-1}, \hat{\eta}_{d-1})$ is a local null frame of N^{d-1} with $g^{d-1}(\hat{\xi}_{d-1}, \hat{\eta}_{d-1}) = \mu_{d-1} \neq 0$. Using $\alpha^{d+1}(e_\pm, e^d_\mp) = 0$, we have

$$\begin{aligned} S^d_{e_+}\zeta_d &= \mu^{-1}_{d-1}\big(g^{d-1}(S^d_{e_+}\zeta_d, \hat{\eta}_{d-1})\hat{\xi}_{d-1} + g^{d-1}(S^d_{e_+}\zeta_d, \hat{\xi}_{d-1})\hat{\eta}_{d-1}\big) \\ &= \mu^{-1}_{d-1}g^{d-1}(\zeta_d, \hat{\xi}_d)\hat{\eta}_{d-1}\,. \end{aligned}$$

We similarly get $S^d_{e_-}\zeta_d = \mu^{-1}_{d-1}g^{d-1}(\zeta_d, \hat{\eta}_d)\hat{\xi}_{d-1}$ which shows $S^d_X\zeta_d = 0$ for any $X \in TM$. □

LEMMA 5.9. *If* $M^2_1(K)$ *is of constant curvature* K, *then we obtain for any nonnegative integer* k,

$$g^d(\hat{\xi}_d, \hat{\eta}_{d+k}) = g^d(\hat{\eta}_d, \hat{\xi}_{d+k}) = 0\,.$$

PROOF. Since K is constant, we obtain $\lambda_d = 0$ by Corollary 2.16. Thus we see $\tilde{g}(\hat{\xi}_d, \hat{\eta}_d) = 0$. For a positive integer k, we assume that

$$g^d(\hat{\xi}_d, \hat{\eta}_{d+k-1}) = g^d(\hat{\eta}_d, \hat{\xi}_{d+k-1}) = 0.$$

Then we have

$$\begin{aligned} g^d(\hat{\xi}_d, \hat{\eta}_{d+k}) &= g^d(\hat{\xi}_d, \nabla^d_{e_-}\hat{\eta}_{d+k-1} + (d+k)\rho(e_-)\hat{\eta}_{d+k-1}) \\ &= e_-g^d(\hat{\xi}_d, \hat{\eta}_{d+k-1}) - g^d(\nabla^d_{e_-}\hat{\xi}_d, \hat{\eta}_{d+k-1}) \\ &= -g^d(\nabla^d_{e_-}\hat{\xi}_d, \hat{\eta}_{d+k-1})\,, \end{aligned}$$

noting that $\nabla^d_{e_-}\hat{\xi}_d = (d+1)\rho(e_-)\hat{\xi}_d$ from Lemma 2.11,

$$= -(d+1)\rho(e_-)g^d(\hat{\xi}_d, \hat{\eta}_{d+k-1}) = 0\,.$$

Hence we prove $g^d(\hat{\xi}_d, \hat{\eta}_{d+k}) = 0$ for any $k \geq 0$. In a similar way, we get $g^d(\hat{\eta}_d, \hat{\xi}_{d+k}) = 0$ for any $k \geq 0$. □

Put, for any point p of $M^2_1(K)$,

$$W_p := \mathrm{Span}\,\{(\hat{\xi}_{d+k})_p, (\hat{\eta}_{d+k})_p \mid k \text{ is any nonnegative integer}\} \subset N^d_p\,.$$

From Lemmas 5.3, 5.8 and 5.9, we have

LEMMA 5.10. *If K is constant, then $S^d_X \hat{\xi}_{d+k} = S^d_X \hat{\eta}_{d+k} = 0$ for any nonnegative integer k and $X \in T_pM$ $(p \in M)$.*

For an isometric immersion $f : M \to Q^n_t(c)$ and $p \in M$, we put

$$\mathrm{Osc}^{\infty}_p(f) := \bigcup_{i=0}^{\infty} \mathrm{Osc}^i_p(f) \subset (f^\# TQ)_p\,.$$

PROOF OF THEOREM 5.1. Let $f : M^2_1(K) \to Q^n_t(c)$ be nicely curved of order d. From Proposition 5.4, we see that $N^1, \ldots, N^{d-1}$ are orientable Lorentzian subbundles of $f^\# TQ$.

Since f is full and real analytic, there exists a point $p \in M$ such that $\mathrm{Osc}^{\infty}_p(f) = (f^\# TQ)_p$. Thus, we can take a open subset U around p in M such that

$$\mathrm{Osc}^{\infty}(f)|_U = f^\# TQ|_U\,.$$

On U, from Lemma 5.6, we can see $N^d = \mathrm{Span}\,\{\xi_{d+k}, \eta_{d+k} \mid k \geq 0\}$. Using Lemma 5.10, we have $S^d = 0$ on U, if and only if $\alpha^{d+1} = 0$ on U. Thus $\oplus_{i=0}^{d-1} N^i = \mathrm{Osc}^d(f) = \mathrm{Osc}^{\infty}(f) = TQ$ on U. So, $\mathrm{rank}\, N^d = n - 2d = 0$, that is, $n = 2d$. Then, since all vector bundle $N^0 (= TM), N^1, \ldots, N^{d-1}$ are Lorentzian plane bundles, we also get $t = d$. Hence we have $r \leq \min\{n - t, t\} = d$.

Taking a null geodesic γ of M such that $\gamma(0) = p \in M$ and $\dot{\gamma}(0) = e_{+,p} = (\xi_0)_p$ and putting $c_+ := f \circ \gamma$, we obtain by (5.1) and $\alpha^{d+1} = 0$

$$\mathrm{Span}\{\nabla(\dot{c_+}^i)_p \mid i \geq 1\} = \mathrm{Span}\{(\hat{\xi}_0)_p, (\hat{\xi}_1)_p, \ldots, (\hat{\xi}_{d-1})_p\}$$

and its dimension is equal to d. It implies that c_+ is proper d-planar, hence $r \geq d$. So we have $r = d$. Since we can similarly see that any null geodesic of M is proper r-planar in $Q^n_t(c)$, f is null proper r-PG.

By virtue of Corollary 2.16, $\alpha^2, \ldots, \alpha^r$ are (nonzero) isotropic and $K = 2c/r(r+1)$. From Proposition 5.4, we can put the reflector lift $\widetilde{J} := \oplus_{i=0}^{r-1} J^i$ of f. Using (2.1)–(2.3), (2.27) and (2.28), we get $(\widetilde{\nabla}_{e_\pm} \widetilde{J})(\xi_i) = (\widetilde{\nabla}_{e_\pm} \widetilde{J})(\eta_i) = 0$ for $i = 0, 1, \ldots, r-1$, that is, $\widetilde{J}$ is horizontal. By virtue of Theorem 3.5 and Corollary 4.3 , we complete the proof of Theorem 5.1. $\square$

In Riemannian geometry, Calabi shows that a full minimal isometric immersion $f : M^2(K) \to S^n$ satisfies $n = 2d$ and is congruent to the Boruvka sphere $\psi_{2,d}$, in the case that M is of genus zero and of constant Gaussian curvature. As we see in Theorem 5.1, to obtain a corresponding result in pseudo-Riemannian geometry, we need the additional assumption "null r-PG". There exist extremal isometric immersions which are not null r-PG for any $r > 0$. For example, see [3, Theorem 5.1.(b)].

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