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TIME PERIODIC SOLUTIONS OF THE NAVIER-STOKES EQUATIONS WITH THE TIME PERIODIC POISEUILLE FLOW IN TWO AND THREE DIMENSIONAL PERTURBED CHANNELS

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Abstract. H. Beirão da Veiga proved that, for a straight channel in $\boldsymbol{R}^n$ (n arbitarily large) and for a given flux with the time periodicity, there exists a unique time periodic Poiseuille flow in a straight channel in $\boldsymbol{R}^n$. Furthermore, the existence of a time periodic solution in a perturbed channel (Leray's problem) is shown for the Stokes problem (arbitary dimension) and for the Navier-Stokes problem ($n \leq 4$). Concerning the Navier-Stokes case, a quatitative condition requaired to show the existence of a time periodic solution depends not just on the flux of the time periodic Poiseuille flow but also on the domain it self. In this paper, by applying the result of H. Beirão da Veiga and C. J. Amick, we succeed in proving the independence of such a condition on the particular domain.

1. The time periodic Poiseuille flow. In this section, for a straight channel in $\boldsymbol{R}^n$ ($n = 2, 3$), which is parallel to the x_1-axis, let us consider a time periodic flow of an incompressible viscous fluid which is also parallel to the x_1-axis.

In the case $n = 2$, for $a > 0$ we suppose $\Sigma := (-a, a)$. In the case $n = 3$, we suppose that Σ is a bounded smooth simply connected domain in $\boldsymbol{R}^2$. We write

$$\omega = \boldsymbol{R} \times \Sigma .$$

Then ω is a straight and infinite channel, which is parallel to the x_1-axis, and Σ is a cross section of the channel ω.

In the straight and infinite channel ω, let us consider the nonstationary Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{0} \quad \text{in} \quad \boldsymbol{R} \times \omega , \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \boldsymbol{R} \times \omega , \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \boldsymbol{R} \times \partial\omega \tag{1.3}$$

with the time periodic condition

$$\boldsymbol{u}(t) = \boldsymbol{u}(t + T) \quad \text{in} \quad \omega \tag{1.4}$$

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and the flux condition

$$\int_{\Sigma} \boldsymbol{u}(t) \cdot \boldsymbol{n} dS = \alpha(t) \quad (t \in \boldsymbol{R}), \tag{1.5}$$

where $\boldsymbol{u} = \boldsymbol{u}(t, x)$ and $p = p(t, x)$ are the unknown velocity and the unknown pressure of the fluid motion in ω, respectively, ν is the given viscosity constant, $T(> 0)$ is a given constant, $\boldsymbol{n}$ is the unit parallel vector to the x_1-axis and $\alpha(t)$ is a given T-periodic real function.

Since we look for a solution pallalel to the x_1-axis, we may assume that

$$\boldsymbol{u}(t, x) = (v(t, x), 0) \qquad (n = 2),$$
$$\boldsymbol{u}(t, x) = (v(t, x), 0, 0) \quad (n = 3).$$

Then it follows that v does not depend on x_1 from (1.2), $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{0}$ and p depends only on t and x_1 from (1.1). Therefore we obtain the problem

$$\frac{\partial v}{\partial t} - \nu \Delta v = -\frac{\partial p}{\partial x_1} \quad \text{in} \quad \boldsymbol{R} \times \Sigma, \tag{1.6}$$

$$v = 0 \quad \text{on} \quad \boldsymbol{R} \times \partial \Sigma \tag{1.7}$$

with the time periodic condition

$$v(t) = v(t + T) \quad \text{in} \quad \Sigma \quad (t \in \boldsymbol{R}) \tag{1.8}$$

and the flux condition

$$\int_{\Sigma} v(t) dS = \alpha(t) \quad (t \in \boldsymbol{R}), \tag{1.9}$$

where $\Delta = \partial^2/\partial x_2^2$ $(n = 2)$, $\Delta = \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ $(n = 3)$. We can consider the corresponding problem also in $\boldsymbol{R}^2$. We recall that, by appealing to the ideas introduced [4], Galdi and Robertson [8] showed that, for $n = 2$, the axial pressure gradient and the flow rate are connected through a simple relation.

It is easy to see that v does not depend on x_1 and p depends only on t and x_1. Therefore it follows from the equation (1.6) that $\partial v/\partial t - \nu \Delta v$ and $\partial p/\partial x_1$ depends only on t. Moreover, we assume that

$$p(t, x_1) = \psi(t) x_1,$$

where $\psi = \psi(t)$ is the unknown function. Integrating (1.6) on Σ, we obtain

$$\psi(t) = -\frac{1}{|\Sigma|}\left(\alpha'(t) - \nu \int_{\Sigma} \Delta v(t) dS\right),$$

where $|\Sigma|$ is the Lebesgue measure of Σ. Therefore there exists a time periodic solution $\boldsymbol{u}$ of the Navier-Stokes equations (1.1)–(1.5) in ω, with the form $\boldsymbol{u} = (v, 0)$ or $\boldsymbol{u} = (v, 0, 0)$, if and only if v is a solution of the problem

$$v' + \nu A v - \frac{\nu}{|\Sigma|}(Av, e)e = \frac{\alpha'}{|\Sigma|} e \tag{1.10}$$

with the time periodic condition

$$v(t) = v(t + T) \quad (t \in \boldsymbol{R}) \tag{1.11}$$

and the flux condition

$$(1.12)\qquad (v(t), e) = \alpha(t) \quad (t \in \boldsymbol{R}),$$

where $e(y) = 1$ $(y \in \Sigma)$, $A = -\Delta$ with the domain $D(A) = H^2(\Sigma) \cap H_0^1(\Sigma)$, $(v, e) = \int_\Sigma ved S$.

Before stating the time periodic result, we introduce the function space. Let X be a Banach space. We set

$$\begin{aligned} H_\pi^1(\boldsymbol{R}) &= \{\varphi \in H_{\mathrm{loc}}^1(\boldsymbol{R}); \varphi(t) = \varphi(t+T) \text{ a.e. } t \in \boldsymbol{R}\}, \\ L_\pi^2(\boldsymbol{R}; X) &= \{\varphi \in L_{\mathrm{loc}}^2(\boldsymbol{R}; X); \varphi(t) = \varphi(t+T) \text{ in } X \text{ for a.e. } t \in \boldsymbol{R}\}, \\ C_\pi(\boldsymbol{R}; X) &= \{\varphi \in C(\boldsymbol{R}; X); \varphi(t) = \varphi(t+T) \text{ in } X \text{ for } t \in \boldsymbol{R}\}. \end{aligned}$$

Beirão da Veiga [4] proved that for $n \geq 2$ if a flux $\alpha \in H_\pi^1(\boldsymbol{R})$ is given, then there exists a unique time periodic solution v^α of this problem (1.10)–(1.12) satisfying

$$\begin{aligned} v^\alpha &\in L_\pi^2(\boldsymbol{R}; H_0^1(\Sigma) \cap H^2(\Sigma)) \cap C_\pi(\boldsymbol{R}; H_0^1(\Sigma)), \\ (v^\alpha)' &\in L_\pi^2(\boldsymbol{R}; L^2(\Sigma)). \end{aligned}$$

In this paper, we consider the case $n = 2$ and $n = 3$.

We apply the regularity result for the heat equations to v^α. Therefore it is easy to prove that

$$\begin{aligned} v^\alpha &\in L_\pi^2(\boldsymbol{R}; H_0^1(\Sigma) \cap H^{n+1}(\Sigma)) \cap C_\pi(\boldsymbol{R}; H_0^1(\Sigma)), \\ (v^\alpha)' &\in L_\pi^2(\boldsymbol{R}; H^{n-1}(\Sigma)). \end{aligned}$$

Set

$$\begin{aligned} \boldsymbol{V}^\alpha(t, x) &= (v^\alpha(t, x), 0) \qquad (n = 2), \\ \boldsymbol{V}^\alpha(t, x) &= (v^\alpha(t, x), 0, 0) \quad (n = 3). \end{aligned}$$

In this paper, let us call $\boldsymbol{V}^\alpha$ "the time periodic Poiseuille flow".

2. Time periodic problem in a perturbed channel. In the case $n = 2$, for $a_i > 0$ $(i = 1, 2)$ we set $\Sigma_i := (-a_i, a_i)$. In the case $n = 3$, we define Σ_i as a bounded smooth simply connected domain in $\boldsymbol{R}^2$. We write

$$\omega_i = \boldsymbol{R} \times \Sigma_i \quad (i = 1, 2).$$

In the channel ω_i, if a flux $\alpha \in H_\pi^1(\boldsymbol{R})$ is given, then there exists a unique solution v_i^α of the time periodic problem (1.10)–(1.12) on Σ_i. Set

$$\begin{aligned} \boldsymbol{V}_i^\alpha(t, x) &= (v_i^\alpha(t, x), 0) \qquad (n = 2), \\ \boldsymbol{V}_i^\alpha(t, x) &= (v_i^\alpha(t, x), 0, 0) \quad (n = 3). \end{aligned}$$

For a certain $L > 0$ we set

$$\begin{aligned} \omega_{01} &= \{x \in \omega_1; x_1 \leq -L\}, \\ \omega_{02} &= \{x \in \omega_2; x_1 \geq L\}. \end{aligned}$$

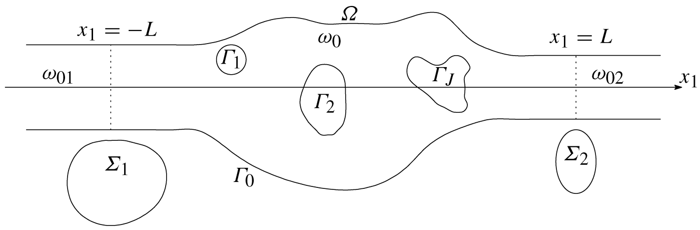

FIGURE 1.

Let Ω a smooth and unbounded domain in $\boldsymbol{R}^n (n = 2, 3)$ and $\partial\Omega$ a boundary of the domain Ω. We suppose

$$\Omega \cap \omega_{01} = \omega_{01} ,$$
$$\Omega \cap \omega_{02} = \omega_{02} .$$

Let ω_0 be $\Omega \backslash (\omega_{01} \cup \omega_{02})$. ω_0 is a perturbed and bounded part, ω_{01} and ω_{02} are channel parts. The boundary $\partial\Omega$ of Ω has connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_J$ of C^∞-surface such that $\Gamma_1, \dots, \Gamma_J$ lie inside of Γ_0 with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, and such that $\partial\Omega = \bigcup_{j=0}^{J} \Gamma_j$. Let us call the domain Ω "a perturbed channel". See the Figure 1.

In this paper, for a perturbed channel Ω we consider the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad (0, T) \times \Omega , \tag{2.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad (0, T) \times \Omega \tag{2.2}$$

with the boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad (0, T) \times \partial\Omega , \tag{2.3}$$

$$\boldsymbol{u} \to \boldsymbol{V}_i^\alpha \quad \text{as} \quad |x| \to \infty \quad \text{in} \quad \omega_{0i} \quad (i = 1, 2) \tag{2.4}$$

and the time periodic condition

$$\boldsymbol{u}(0) = \boldsymbol{u}(T) \quad \text{in} \quad \Omega, \tag{2.5}$$

where $\boldsymbol{u} = \boldsymbol{u}(t, x)$ and $p = p(t, x)$ are the unknown velocity and the unknown pressure of an incompressible viscous fluid in Ω, respectively, while $\nu > 0$ is the kinematic viscosity and $\boldsymbol{f}$ is the prescribed external force.

We will seek a solution of (2.1)–(2.5) of the form

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{V}^\alpha ,$$

where the velocity field $\boldsymbol{V}^\alpha$ is to be such that

$$\begin{aligned} \operatorname{div} \boldsymbol{V}^\alpha &= 0 && \text{in} \quad \Omega , \\ \boldsymbol{V}^\alpha &= \boldsymbol{0} && \text{on} \quad \partial\Omega , \\ \boldsymbol{V}^\alpha &= \boldsymbol{V}_i^\alpha && \text{in} \quad \omega_{0i} \quad (i = 1, 2) , \end{aligned}$$

$$\begin{aligned} &\boldsymbol{V}^\alpha(t) = \boldsymbol{V}_i^\alpha(t+T) \quad \text{in} \quad \Omega \quad (t \in \boldsymbol{R}), \\ &\boldsymbol{V}^\alpha|_{\omega_0} \in H_\pi^1((0,T); \boldsymbol{H}^1(\omega_0)), \end{aligned}$$

where $\boldsymbol{H}^1(\omega_0) = \{H^1(\omega_0)\}^n$ $(n = 2, 3)$. Let us call $\boldsymbol{V}^\alpha$ "the extended time periodic Poiseuille flow".

Beirão da Veiga [4] treated the above time periodic problem. He proved that there exists a time periodic solution with the time periodic Poiseuille flow under the restricted flux condition

$$C(\Omega)\sqrt{\nu + \nu^{-2}}\|\alpha\|_{H^1(0,T)} \le \frac{1}{2}\nu,$$

where the constant $C(\Omega)$ depends on Ω.

Amick [2] obtaind the following stationary result. S is the two or three dimensional unbounded straight cylinder, where in the case $n = 3$ any cross section Σ of S is a circle. V is the classical stationary Poiseuille flow in S, so it does not depend on the time. We set

$$\sigma(V) = \sup_{\varphi \in \boldsymbol{H}_{0,\sigma}^1(S)} \frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, V)_S}{\|\nabla \boldsymbol{\varphi}\|_{2,S}^2}. \tag{2.6}$$

The constant $\sigma(V)$ depends only on the cylinder S and the classical Poiseuille flow V. Amick [2] proved that in a two and three dimensional perturbed cylinderical domain, there exists a stationary solution with the classical Poiseuille flow V under the restricted flux condition

$$\sigma(V) < \nu. \tag{2.7}$$

This result is applied to the time periodic problem under the same inequality (2.7) by Kobayashi [13].

In this paper the above upper bound (2.7) is replaced by a more general, and natural, upper bound, where roughly speaking, in definition (2.6) the classical Poiseuille flow V will be replaced by the time periodic Poiseuille flow $\boldsymbol{V}^\alpha(t)$. Such a constant denotes $\hat{\gamma}^\alpha$. For the details, see Definitions 3.2 and 3.3. If

$$\hat{\gamma}^\alpha < \nu, \tag{2.8}$$

we prove that for the perturbed channel Ω there exists a time periodic weak solution of the Navier-Stokes equations with the time periodic Poiseuille flow.

Before stating our results we introduce some function spaces. $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ is the set of all real smooth vector functions with compact support in Ω such that $\operatorname{div}\boldsymbol{\varphi} = 0$. $\boldsymbol{L}_\sigma^2(\Omega)$ is the closure of $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ in $\boldsymbol{L}^2(\Omega)$. The $\boldsymbol{L}^2$ inner product and norm on Ω are denoted as $(\cdot,\cdot)$ and $\|\cdot\|_2$, respectively. $\boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{H}_{0,\sigma}^1(\Omega)$ are the closures of $\boldsymbol{C}_0^\infty(\Omega)$ and $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ in $\boldsymbol{H}^1(\Omega)$, respectively. $\boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{H}_{0,\sigma}^1(\Omega)$ are the Hilbert spaces with respect to the inner product $((\boldsymbol{u}, \boldsymbol{v})) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$.

Let X be a Banach space. $C([0,T]; X)$, $L^2((0,T); X)$, $L^\infty((0,T); X)$ and $H^1((0,T); X)$ are the usual Banach spaces. $C_\pi([0,T]; X)$ and $H_\pi^1((0,T); X)$ are the sets of all the $C([0,T]; X)$ and $H_\pi^1((0,T); X)$ functions, respectively, satisfying the time periodic condition $\boldsymbol{u}(0) = \boldsymbol{u}(T)$ in X.

3. Results. Our definition of a time periodic weak solution of the Navier-Stokes equations (2.1)–(2.5) is as follows.

DEFINITION 3.1. A measurable function $\boldsymbol{u} = \boldsymbol{u}(t, x)$ in $(0, T) \times \Omega$ is called a time periodic weak solution of the Navier-Stokes equations if $\boldsymbol{u}$ satisfies the following condition.

(1) $\boldsymbol{v} := \boldsymbol{u} - \boldsymbol{V}^{\alpha} \in L^2((0, T); \boldsymbol{H}^1_{0,\sigma}(\Omega)) \cap L^{\infty}((0, T); \boldsymbol{L}^2_{\sigma}(\Omega))$.

(2) $\boldsymbol{u}$ satisfies

$$\int_0^T -(\boldsymbol{u}, \boldsymbol{\varphi})\psi' + \{\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{\varphi})\}\psi dt$$
$$= \int_0^T {}_{(\boldsymbol{H}^1_{0,\sigma})'}\langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}^1_{0,\sigma}} \psi dt \quad (\boldsymbol{\varphi} \in \boldsymbol{H}^1_{0,\sigma}(\Omega), \psi \in C_0^{\infty}(0, T)).$$

(3) $\boldsymbol{v}$ is time periodic in $\boldsymbol{L}^2(\Omega)$, that is to say, $\boldsymbol{v}$ satisfies

$$\boldsymbol{v}(0) = \boldsymbol{v}(T) \quad \text{in} \quad \boldsymbol{L}^2(\Omega).$$

In the straight channel ω_i (not Ω), we define a constant concerning the time periodic Poiseuille flow.

DEFINITION 3.2. We set

$$\gamma_i^{\alpha}(t) := \sup_{\varphi \in \boldsymbol{H}^1_{0,\sigma}(\omega_i)} \frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{V}_i^{\alpha}(t))_{\omega_i}}{\|\nabla \boldsymbol{\varphi}\|^2_{2,\omega_i}} \quad (i = 1, 2,\ t \in [0, T]). \tag{3.1}$$

For the detailed property of $\gamma_i^{\alpha}(t)$, see Proposition 4.1.

DEFINITION 3.3. We set

$$\hat{\gamma}_i^{\alpha} := \sup_{t \in [0,T]} \gamma_i^{\alpha}(t) \quad (i = 1, 2),$$
$$\hat{\gamma}^{\alpha} := \max\{\hat{\gamma}_1^{\alpha}, \hat{\gamma}_2^{\alpha}\}.$$

Our main theorem on the existence of a time periodic weak solution now reads.

THEOREM 3.1. *Suppose that* $\boldsymbol{f} \in L^2((0, T); (\boldsymbol{H}^1_{0,\sigma}(\Omega))')$ *and* $\hat{\gamma}^{\alpha} < \nu$. *Then there exists a time periodic weak solution.*

REMARK 3.1. In this paper, we assume that the semi-infinite channels ω_{01} and ω_{02} are parallel to the x_1-axis. But the inequality $\hat{\gamma}^{\alpha} < \nu$ implies that these semi-infinite channels may not be parallel.

REMARK 3.2. In this paper, the domain Ω has two outlets ω_{01} and ω_{02}. We can solve the J outlets problem applying the similar inequality. We consider a straight channel $\omega_i = \boldsymbol{R} \times \Sigma_i$ $(i = 1, \dots, J, J \geq 3)$, where Σ_i is a cross section as Section 1. For a given flux function $\alpha_i \in H^1_{\pi}(\boldsymbol{R})$ and in ω_i, we have the time periodic Poiseuille flow $\boldsymbol{V}_i^{\alpha_i}$. We assume that α_i satisfies $\sum_{i=1}^{J} \alpha_i(t) = 0$ $(t \in \boldsymbol{R})$. We suppose that Ω has J outlets ω_{0i} $(i = 1, \dots, J)$ where ω_{0i} is a semi-infinite channel with the cross section Σ_i. In the domain Ω, we consider a time periodic problem with the time periodic Poiseuille flow $\boldsymbol{V}_i^{\alpha_i}$. We define constant $\hat{\gamma} =$

$\max_{1\le i\le J}\{\hat{\gamma}_i^{\alpha_i}\}$ as Definitions 3.2 and 3.3. Suppose that $\hat{\gamma} < \nu$. Then there exists a time periodic weak solution in Ω.

4. The detailed property of $\gamma_i^{\alpha}(t)$. In this section we show the property of the $\gamma_i^{\alpha}(t)$.

PROPOSITION 4.1. *Let $\alpha, \beta \in H^1_\pi(\boldsymbol{R})$. Then we have*
(1) $\gamma_i^{\alpha} \in H^1_\pi(\boldsymbol{R})$,
(2) $\gamma_i^{\alpha}(t) > 0$,
(3) $\gamma_i^{\lambda\alpha}(t) = |\lambda|\gamma_i^{\alpha}(t)$,
(4) $\gamma_i^{\alpha+\beta}(t) = \gamma_i^{\alpha}(t) + \gamma_i^{\beta}(t)$.

By this proposition, Definition 3.2 is meaningful.

4.1. Proof of Proposition 4.1. Let us prove (1). It is easy to see that, for any $|h| < 1$ and $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{0,\sigma}(\omega_i)$, the inequality

$$\frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}_i^{\alpha}(t+h))_{\omega_i} - ((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}_i^{\alpha}(t))_{\omega_i}}{\|\nabla\boldsymbol{\varphi}\|^2_{2,\omega_i}} \le C\|v_i^{\alpha}(t+h) - v_i^{\alpha}(t)\|_{H^{n-1}(\Sigma_i)}$$

holds true, where C depends on Sobolev's Imbedding Theorem $H^{n-1}(\Sigma_i) \hookrightarrow L^\infty(\Sigma_i)$ and the Poincaré inequality in ω_i. Therefore it follows that

$$\left|\gamma_i^{\alpha}(t+h) - \gamma_i^{\alpha}(t)\right| \le C\|v_i^{\alpha}(t+h) - v_i^{\alpha}(t)\|_{H^{n-1}(\Sigma_i)}\,. \tag{4.1}$$

Since $v^{\alpha} \in H^1((0,T); H^{n-1}(\Sigma_i))$, for any interval $(a,b) \subset (0,T)$ we have

$$\int_a^b \frac{\|v_i^{\alpha}(t+h) - v_i^{\alpha}(t)\|^2_{H^{n-1}(\Sigma_i)}}{h}dt \le C\int_0^T \|(v_i^{\alpha})'(t)\|^2_{H^{n-1}(\Sigma_i)}dt\,.$$

Hence it follows from (4.1) that $\gamma_i^{\alpha} \in H^1(0,T)$.

Let us prove (2). Let $\boldsymbol{\varphi}$ be $\boldsymbol{H}^1_{0,\sigma}(\omega_i)$. Set $\boldsymbol{\xi}(x) = \boldsymbol{\varphi}(-x)$. Then $\boldsymbol{\xi} \in \boldsymbol{H}^1_{0,\sigma}(\omega_i)$. We have by a direct calculation

$$\frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}_i^{\alpha}(t))_{\omega_i}}{\|\nabla\boldsymbol{\varphi}\|^2_{2,\omega_i}} = -\frac{((\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\xi}, \boldsymbol{V}_i^{\alpha}(t))_{\omega_i}}{\|\nabla\boldsymbol{\xi}\|^2_{2,\omega_i}}\,.$$

Therefore we get the inequality

$$-\gamma_i^{\alpha}(t) \le \frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}_i^{\alpha}(t))_{\omega_i}}{\|\nabla\boldsymbol{\varphi}\|^2_{2,\omega_i}} \le \gamma_i^{\alpha}(t)\,. \tag{4.2}$$

This proves that $\gamma_i^{\alpha}(t) \ge 0$. Now we assume that

$$\gamma_i^{\alpha}(t) = 0\,. \tag{4.3}$$

It follows from (4.2) that

$$((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}_i^{\alpha}(t))_{\omega_i} = 0 \quad (\boldsymbol{\varphi} \in \boldsymbol{H}^1_{0,\sigma}(\omega_i))\,.$$

Now for any $\psi \in C_0^\infty(\omega_i)$, we set

$$\boldsymbol{\varphi} = \left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right) \quad \text{in} \quad \omega_i \quad (n=2)\,,$$

$$\boldsymbol{\varphi} = \left(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1}, 0 \right) \quad \text{in} \quad \omega_i \quad (n = 3) .$$

Then $\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_{0,\sigma}(\omega_i)$. It follows that

$$0 = ((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{V}^\alpha_i(t))_{\omega_i} = \int_{\omega_i} \psi \frac{\partial^3 \psi}{\partial x_1 \partial x_2^2} v^\alpha_i \, dx .$$

It implies that

$$\frac{\partial^3 \psi}{\partial x_1 \partial^2 x_2} = 0 \quad \text{in} \quad \omega_i . \tag{4.4}$$

Since $\psi \in C^\infty_0(\omega_i)$ is an arbitrary function, this is a contradiction. This proves (2).

Let us prove (3). We know that λv^α_i and $v^{\lambda\alpha}_i$ are solutions of (1.10)–(1.11) with the flux condition $\lambda\alpha$. Therefore we have $\lambda \boldsymbol{V}^\alpha_i = \boldsymbol{V}^{\lambda\alpha}_i$. It implies that $\gamma^{\lambda\alpha}_i(t) = \lambda \gamma^\alpha_i(t)$ for any $\lambda > 0$. Let $\boldsymbol{\varphi}$ be $\boldsymbol{H}^1_{0,\sigma}(\omega_i)$. Set $\boldsymbol{\xi}(x) = \boldsymbol{\varphi}(-x)$. Then $\boldsymbol{\xi} \in \boldsymbol{H}^1_{0,\sigma}(\omega_i)$. We obtain

$$\frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{V}^{\lambda\alpha}_i(t))_{\omega_i}}{\|\nabla \boldsymbol{\varphi}\|^2_{2,\omega_i}} = -\lambda \frac{((\boldsymbol{\xi} \cdot \nabla)\boldsymbol{\xi}, \boldsymbol{V}^\alpha_i(t))_{\omega_i}}{\|\nabla \boldsymbol{\xi}\|^2_{2,\omega_i}}$$

by a direct calculation. It implies that $\gamma^{\lambda\alpha}_i(t) = -\lambda \gamma^\alpha_i(t)$ for any $\lambda < 0$.

Let us prove (4). We know that $v^{\alpha+\beta}_i$ and $v^\alpha_i + v^\beta_i$ are solutions of (1.10)–(1.11) with the flux condition $\alpha + \beta$. Therefore we have $\boldsymbol{V}^{\alpha+\beta}_i = \boldsymbol{V}^\alpha_i + \boldsymbol{V}^\beta_i$. It implies the equality $\gamma^{\alpha+\beta}_i(t) = \gamma^\alpha_i(t) + \gamma^\beta_i(t)$.

5. Preliminary.

5.1. Lemma. In this subsection we show some useful inequlity for the proof of the existence of a time periodic weak solution.

LEMMA 5.1. *Let $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$. Then*

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^4(\Omega)} \le 2^{1/2} \|\boldsymbol{u}\|_2 \|\nabla \boldsymbol{u}\|_2 \quad (n = 2) ,$$
$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^4(\Omega)} \le 2 \|\boldsymbol{u}\|^{1/2}_2 \|\nabla \boldsymbol{u}\|^{3/2}_2 \quad (n = 3) .$$

LEMMA 5.2. *Let $\boldsymbol{u} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ and $\boldsymbol{v},\ \boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$. Then*

$$((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w}) = -((\boldsymbol{u} \cdot \nabla)\boldsymbol{w}, \boldsymbol{v}) ,$$
$$((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{v}) = 0 .$$

Moreover, let $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega)$. Then

$$|((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w})| \le C \|\nabla \boldsymbol{u}\|_2 \|\nabla \boldsymbol{v}\|_2 \|\nabla \boldsymbol{w}\|_2 .$$

LEMMA 5.3. *For any $\varepsilon > 0$ and $\boldsymbol{w} \in C([0,T]; \boldsymbol{L}^n(\Omega))$ $(n = 2, 3)$, there exist an integer N and functions $\boldsymbol{\psi}_j \in \boldsymbol{L}^2(\Omega)$ $(j = 1, \dots, N)$ such that*

$$\int_0^T |((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w})| dt \le \varepsilon \int_0^T (\|\nabla \boldsymbol{u}\|^2_2 + \|\nabla \boldsymbol{v}\|^2_2 + \|\boldsymbol{u}\|_2 \|\nabla \boldsymbol{v}\|_2) dt + \sum_{j=1}^N \int_0^T |(\boldsymbol{u}, \boldsymbol{\psi}_j)|^2 dt$$

for any $\boldsymbol{u}, \boldsymbol{v} \in L^2((0,T); \boldsymbol{H}^1_{0,\sigma}(\Omega))$.

For the proof of this lemma, see Masuda [17, Lemma 2.5].

5.2. Auxiliary proposition for the extended time periodic flow. The following propositions are improved for the time periodic Poiseuille flow. For the original propositions, see Amick [2].

For a fixed $t \in [0, T]$ we define a functional $\boldsymbol{r}(t)$ as

$$\boldsymbol{\varphi} \in \boldsymbol{H}^1_{0,\sigma}(\Omega) \mapsto ((\boldsymbol{V}^\alpha)'(t), \boldsymbol{\varphi}) + \nu(\nabla \boldsymbol{V}^\alpha(t), \nabla\boldsymbol{\varphi}) + ((\boldsymbol{V}^\alpha(t)\cdot\nabla)\boldsymbol{V}^\alpha(t), \boldsymbol{\varphi}) . \tag{5.1}$$

Then we have the following proposition.

PROPOSITION 5.1. *The map* $\boldsymbol{r}(t)$ *is a linear and continuous functional on* $\boldsymbol{H}^1_{0,\sigma}(\Omega)$. *Furthermore we have* ${}_{(\boldsymbol{H}^1_{0,\sigma})'}\langle \boldsymbol{r}, \boldsymbol{\varphi}\rangle_{\boldsymbol{H}^1_{0,\sigma}} \in L^2(0,T)$ *for any* $\boldsymbol{\varphi} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$.

For the proof of this proposition, see Amick [2, Lemma 3.4].

Suppose that $\theta \in C^\infty(\boldsymbol{R})$ satisfies

$$\begin{aligned} 0 \le \theta(s) \le 1 &\quad (s \in \boldsymbol{R}), \\ \theta(s) = 1 &\quad (s \le -1), \\ \theta(s) = 0 &\quad (s \ge 0). \end{aligned}$$

For any $\delta > 0$, we set

$$\theta_\delta(x) = \begin{cases} \theta(\delta x_1) & (x \in \omega_{01}), \\ \theta(-\delta x_1) & (x \in \omega_{02}), \\ 0 & \text{otherwise}. \end{cases}$$

Then we have the following proposition.

PROPOSITION 5.2. *For any* $\varepsilon > 0$, *there exists an* $\boldsymbol{s} \in H^1_\pi((0,T); \boldsymbol{H}^1_{0,\sigma}(\Omega))$ *with compact support such that the inequality*

$$((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{V}^\alpha) \le ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{s}) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{V}^\alpha\theta_\delta^2) + (\varepsilon + c_0\delta)\|\nabla\boldsymbol{v}\|_2^2 \quad (\boldsymbol{v} \in \boldsymbol{H}^1_{0,\sigma}(\Omega))$$

holds true, where the constant c_0 *does not depend on* ε *and* δ.

For the proof of this proposition, see Amick [2, p.495–p.496].

We set

$$\Gamma_\delta(t) := \sup_{v \in \boldsymbol{H}^1_{0,\sigma}(\Omega)} \frac{((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{V}^\alpha(t)\theta_\delta^2)}{\|\nabla\boldsymbol{v}\|^2_{2,\Omega}} . \tag{5.2}$$

Then we have the following proposition.

PROPOSITION 5.3. *We have*

$$\lim_{\delta\to+0} \Gamma_\delta(t) = \max\{\gamma_1(t), \gamma_2(t)\} .$$

For the proof of this proposition, see Amick [2, Theorem 4.3].

6. Proof of Theorem 3.1.

6.1. Time periodic weak solution in a bounded domain. In this section, we prove Theorem 3.1 for a three dimensional domain. The proof for a two dimensional domain is similar.

We consider that a sequence of bounded domains Ω^n ($n \in \boldsymbol{N}$) such that $\omega_0 \subset \Omega^1$, $\Omega^n \subset \Omega^{n+1}$, $\bigcup_{n\in \boldsymbol{N}} \Omega^n = \Omega$ and $\partial\Omega^n$ is of class C^∞. Let Γ_0^n be the outer boundary of Ω^n.

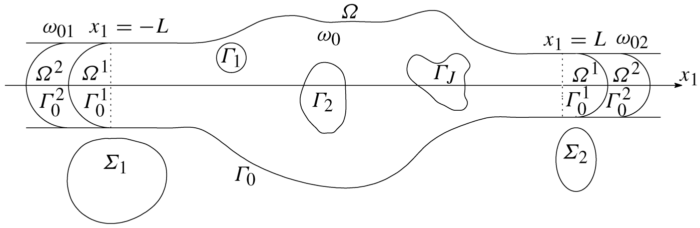

FIGURE 2.

In the bounded domain Ω^n, we consider the time periodic problems of the Navier-Stokes equations with the Dirichlet boundary condition

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad &\text{in} \quad (0,T)\times\Omega^n\,, \\ \operatorname{div}\boldsymbol{u} = 0 \quad &\text{in} \quad (0,T)\times\Omega^n\,, \\ \boldsymbol{u} = \boldsymbol{V}^\alpha \quad &\text{on} \quad (0,T)\times\partial\Omega^n\,, \\ \boldsymbol{u}(0) = \boldsymbol{u}(T) \quad &\text{in} \quad \Omega^n\,. \end{aligned}$$

Precisely, the boundary value $\boldsymbol{V}^\alpha|_{\partial\Omega^n}$ satisfies

$$\begin{aligned} \boldsymbol{V}^\alpha|_{\partial\Omega^n} &\in H^1_\pi((0,T);\boldsymbol{H}^{\frac12}(\partial\Omega^n))\,, \\ \int_{\Gamma_0^n} \boldsymbol{V}^\alpha\cdot\boldsymbol{n}\,dS &= 0\,, \\ \boldsymbol{V}^\alpha = \boldsymbol{0} \quad &\text{on} \quad \Gamma_j \quad (j=1,\dots,J)\,. \end{aligned}$$

Therefore there exists a solenoidal $\boldsymbol{b}^n_\varepsilon \in H^1_\pi((0,T);\boldsymbol{H}^1(\Omega^n))$ such that

$$\boldsymbol{b}^n_\varepsilon = \boldsymbol{V}^\alpha \quad \text{on} \quad (0,T)\times\partial\Omega^n\,,$$

$$|((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{b}^n_\varepsilon(t))_{\Omega^n}| < \varepsilon\|\nabla\boldsymbol{\varphi}\|^2_{\Omega^n} \quad (\boldsymbol{\varphi}\in \boldsymbol{H}^1_{0,\sigma}(\Omega^n), t\in[0,T])\,. \tag{6.1}$$

Here, we obtained the inequality (6.1) for three dimensional bounded domains. Corresponding results in dimension two are given in Kobayashi [14], Lemma 2.6. From (6.1), there exists a solenoidal $\boldsymbol{u}_n \in L^2((0,T);\boldsymbol{H}^1(\Omega^n)) \cap L^\infty((0,T);\boldsymbol{L}^2_\sigma(\Omega^n))$ such that

$$\frac{d}{dt}(\boldsymbol{u}_n,\boldsymbol{\varphi})_n + \nu(\nabla\boldsymbol{u}_n\,,\nabla\boldsymbol{\varphi})_n + ((\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n,\boldsymbol{\varphi})_n = \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle \quad (\boldsymbol{\varphi}\in\boldsymbol{V}(\Omega^n))\,, \tag{6.2}$$

$$\boldsymbol{u}_n = \boldsymbol{V}^\alpha \quad \text{on} \quad (0,T)\times\partial\Omega^n \quad \text{(in the trace sense)}\,, \tag{6.3}$$

$$u_n(0) = u_n(T) \quad \text{in} \quad L^2(\Omega^n)\,, \tag{6.4}$$

where $(\cdot, \cdot)_n$ denotes the L^2 inner product on Ω^n. For the proof, see Yudovič [26].

6.2. The estimate of an initial value. In the pervious subsection, for the bounded domain Ω^n we obtain the time periodic solution u_n. Set

$$v_n = \begin{cases} u_n - V^\alpha & \text{in} \quad \Omega^n\,, \\ \mathbf{0} & \text{in} \quad \Omega \backslash \Omega^n\,. \end{cases}$$

In this subsection, we will prove that $\|v_n(0)\|_{2,\Omega}$ is bounded with respect to n.

It follows that

$$\begin{aligned} & v_n \in L^2((0,T); H^1_{0,\sigma}(\Omega)) \cap L^\infty((0,T); L^2_\sigma(\Omega))\,, \\ & (v_n)' \in L^1((0,T); (H^1_{0,\sigma}(\Omega))')\,, \\ & \frac{d}{dt}(v_n, \varphi) + \nu((v_n, \varphi)) + ((v_n \cdot \nabla)v_n, \varphi) + ((v_n \cdot \nabla)V^\alpha, \varphi) + ((V^\alpha \cdot \nabla)v, \varphi) \\ & = \langle G, \varphi \rangle\,, \end{aligned} \tag{6.5}$$

$$v_n(0) = v_n(T) \quad \text{in} \quad L^2(\Omega)\,, \tag{6.6}$$

where φ is extended as a $\mathbf{0}$ function on the outside Ω^n,

$$\langle G, \varphi \rangle = \langle f, \varphi \rangle - ((V^\alpha)', \varphi) - \nu(\nabla V^\alpha, \nabla \varphi) - ((V^\alpha \cdot \nabla)V^\alpha, \varphi) \quad (\varphi \in H^1_{0,\sigma}(\Omega))\,.$$

Setting $\varphi = v_n$ in (6.5), we obtain

$$\frac{1}{2}\frac{d}{dt}\|v_n\|_2^2 + \nu\|\nabla v_n\|_2^2 = ((v_n \cdot \nabla)v_n, V^\alpha) + \langle G, v_n \rangle\,. \tag{6.7}$$

We apply Proposition 5.2 to $((v_n \cdot \nabla)v_n, V^\alpha)$. Then for any $\varepsilon > 0$ and $\delta > 0$, there exists an $s \in H^1_\pi((0,T); H^1_{0,\sigma}(\Omega))$ with compact support such that

$$((v_n \cdot \nabla)v_n, V^\alpha) \le ((v_n \cdot \nabla)v_n, s) + ((v_n \cdot \nabla)v_n, V^\alpha \theta_\delta^2) + (\varepsilon + c_0\delta)\|\nabla v_n\|_2^2\,. \tag{6.8}$$

Furthermore, it follows that

$$((v_n \cdot \nabla)v_n, V^\alpha \theta_\delta^2) = \frac{((v_n \cdot \nabla)v_n, V^\alpha \theta_\delta^2)}{\|\nabla v_n\|_2^2}\|\nabla v_n\|_2^2 \le \Gamma_\delta(t)\|\nabla v_n\|_2^2\,.$$

We choose $\varepsilon > 0$ such that

$$\nu - \hat{\gamma}^\alpha - 4\varepsilon > 0\,.$$

Applying Proposition 5.3 to $\Gamma_\delta(t)$, we can choose $\delta > 0$ such that

$$\Gamma_\delta(t) \le \hat{\gamma}^\alpha + \varepsilon\,.$$

Moreover, we choose $\delta > 0$ such that

$$\nu - \hat{\gamma}^\alpha - 4\varepsilon - c_0\delta > 0\,.$$

It holds that

$$\frac{1}{2}\frac{d}{dt}\|v_n\|_2^2 + (\nu - \hat{\gamma}^\alpha - 3\varepsilon - c_0\delta)\|\nabla v_n\|_2^2 \le ((v_n \cdot \nabla)v_n, s) + C\|G\|^2_{(H^1_{0,\sigma})'}\,,$$

where the constant C depends only on ε. There exists an $N \in \boldsymbol{N}$ such that $\mathrm{supp}(\boldsymbol{\varphi}) \subset \Omega^n$ for all $n \geq N$. We may set $\boldsymbol{\varphi} = \boldsymbol{s}$ in (6.5). Since

$$\frac{d}{dt}(\boldsymbol{v}_n, \boldsymbol{s}) = \langle \boldsymbol{v}_n', \boldsymbol{s} \rangle + (\boldsymbol{v}_n, \boldsymbol{s}'),$$

we have

$$\begin{aligned}((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{s}) &= -\frac{d}{dt}(\boldsymbol{v}_n, \boldsymbol{s}) - (\boldsymbol{v}_n, \boldsymbol{s}') - \nu((\boldsymbol{v}_n, \boldsymbol{s})) \\ &\quad - ((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{V}^\alpha, \boldsymbol{s}) - ((\boldsymbol{V}^\alpha \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{s}) + \langle \boldsymbol{G}, \boldsymbol{s} \rangle \\ &\leq -\frac{d}{dt}(\boldsymbol{v}_n, \boldsymbol{s}) + \varepsilon \|\nabla \boldsymbol{v}_n\|_2^2 + C(\|\nabla \boldsymbol{s}\|_2^2 + \|\boldsymbol{s}'\|_2^2 + \|\boldsymbol{G}\|^2_{(\boldsymbol{H}^1_{0,\sigma})'}),\end{aligned}$$

where the constant C depnds on $\boldsymbol{V}^\alpha$, ε and the Poincaré inequality. We define $K_1(t)$ by

$$K_1(t) = C(\|\nabla \boldsymbol{s}(t)\|_2^2 + \|\boldsymbol{s}'(t)\|_2^2 + \|\boldsymbol{G}(t)\|^2_{(\boldsymbol{H}^1_{0,\sigma})'}).$$

Then it follows that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + (\nu - \hat{\gamma}^\alpha - 4\varepsilon - c_0\delta)\|\nabla \boldsymbol{v}_n\|_2^2 \leq -\frac{d}{dt}(\boldsymbol{v}_n, \boldsymbol{s}) + K_1. \tag{6.9}$$

Applying the Poincaré inequality to the second term of the left-hand side of (6.9), we have

$$\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + \mu\|\boldsymbol{v}_n\|_2^2 \leq -2\frac{d}{dt}(\boldsymbol{v}_n, \boldsymbol{s}) + 2K_1, \tag{6.10}$$

where

$$\mu = \frac{2(\nu - \hat{\gamma}^\alpha - 4\varepsilon - c_0\delta)}{C(\Omega)^2}.$$

For a certain $\xi > 0$ (smaller than μ), multiplying (6.10) by $e^{(\mu-\xi)t}$, we obtain

$$\begin{aligned}&e^{(\mu-\xi)t}\frac{d}{dt}\|\boldsymbol{v}_n(t)\|_2^2 + \mu e^{(\mu-\xi)t}\|\boldsymbol{v}_n(t)\|_2^2 \\ &\quad \leq -2e^{(\mu-\xi)t}\frac{d}{dt}(\boldsymbol{v}_n(t), \boldsymbol{s}(t)) + 2K_1(t)e^{(\mu-\xi)t} \\ &\quad = -2\frac{d}{dt}\{(\boldsymbol{v}_n(t), \boldsymbol{s}(t))e^{(\mu-\xi)t}\} + 2(\mu-\xi)e^{(\mu-\xi)t}(\boldsymbol{v}_n(t), \boldsymbol{s}(t)) + 2K_1(t)e^{(\mu-\xi)t} \\ &\quad \leq -2\frac{d}{dt}\{(\boldsymbol{v}_n(t), \boldsymbol{s}(t))e^{(\mu-\xi)t}\} + \xi e^{(\mu-\xi)t}\|\boldsymbol{v}_n(t)\|_2^2 + (C\|\boldsymbol{s}(t)\|_2^2 + 2K_1(t))e^{(\mu-\xi)t},\end{aligned} \tag{6.11}$$

where the constant C depends only on μ and ξ. Indeed, it follows from (6.11) that

$$\frac{d}{dt}(e^{(\mu-\xi)t}\|\boldsymbol{v}_n(t)\|_2^2) \leq -2\frac{d}{dt}\{(\boldsymbol{v}_n(t), \boldsymbol{s}(t))e^{(\mu-\xi)t}\} + K_2(t), \tag{6.12}$$

where

$$K_2(t) = (C\|\boldsymbol{s}(t)\|_2^2 + 2K_1(t))e^{(\mu-\xi)t}.$$

Integrating (6.12) on $[0, T]$, we have

$$\|\boldsymbol{v}_n(T)\|_2^2 e^{(\mu-\xi)T} \leq \|\boldsymbol{v}_n(0)\|_2^2 - 2(\boldsymbol{v}_n(T), \boldsymbol{s}(T))e^{(\mu-\xi)T} + 2(\boldsymbol{v}_n(0), \boldsymbol{s}(0)) + K,$$

where

$$K = \int_0^T K_2(t)dt\,.$$

Since $\boldsymbol{v}_n$ and $\boldsymbol{s}$ are time periodic in $\boldsymbol{L}^2(\Omega)$, for any $\lambda > 0$ we have

$$\begin{aligned}\|\boldsymbol{v}_n(0)\|_2^2 e^{(\mu-\xi)T} &\le \|\boldsymbol{v}_n(0)\|_2^2 + (\lambda\|\boldsymbol{v}_n(0)\|_2^2 + C\|\boldsymbol{s}(0)\|_2^2)e^{(\mu-\xi)T}\\ &\quad + \lambda\|\boldsymbol{v}_n(0)\|_2^2 + C\|\boldsymbol{s}(0)\|_2^2 + K\,,\end{aligned}$$

where the constant C depends only on λ. We set

$$\begin{aligned}H &= Ke^{-(\mu-\xi)T} + C\|\boldsymbol{s}(0)\|_2^2(e^{-(\mu-\xi)T}+1)\,,\\ \theta &= 1-\lambda-(1+\lambda)e^{-(\mu-\xi)T}\,.\end{aligned}$$

We choose $\lambda > 0$ such that θ is greater than 0. Then we have

$$\|\boldsymbol{v}_n(0)\|_2^2 \le \frac{H}{\theta} =: M_0\,.$$

Consequetly, the sequence $\|\boldsymbol{v}_n(0)\|_2$ is bounded with respect to n.

6.3. A priori estimate and weak limit. In this subsection, we prove that $\boldsymbol{v}_n$ is a bounded sequence in $L^2((0,T);\boldsymbol{H}^1_{0,\sigma}(\Omega)) \cap L^\infty((0,T);\boldsymbol{L}^2_\sigma(\Omega))$.

It follows from the equation (6.7) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + \nu\|\nabla\boldsymbol{v}_n\|_2^2 &= ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n, \boldsymbol{V}^\alpha) + \langle \boldsymbol{G}, \boldsymbol{v}_n\rangle\\ &\le C(\boldsymbol{V}^\alpha)\|\boldsymbol{v}_n\|_2\|\nabla\boldsymbol{v}_n\|_2 + \|\boldsymbol{G}\|_{(\boldsymbol{H}^1_{0,\sigma})'}\|\nabla\boldsymbol{v}_n\|_2\\ &\le \frac{C_1}{2}\|\boldsymbol{v}_n\|_2^2 + \frac{\nu}{2}\|\nabla\boldsymbol{v}_n\|_2^2 + \frac{C_2}{2}\|\boldsymbol{G}\|^2_{(\boldsymbol{H}^1_{0,\sigma})'}\,,\end{aligned}$$

where the constant C_1 depends only on $\boldsymbol{V}^\alpha$ and ν, the constant C_2 depends only on ν. So it follows

$$\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + \nu\|\nabla\boldsymbol{v}_n\|_2^2 \le C_1\|\boldsymbol{v}_n\|_2 + C_2\|\boldsymbol{G}\|_{(\boldsymbol{H}^1_{0,\sigma})'}\,. \tag{6.13}$$

Applying the Gronwall inequality to (6.13) and integrating from 0 to t $(\le T)$, we obtain

$$\|\boldsymbol{v}_n(t)\|_2^2 \le M_0e^{C_1T} + C_2\int_0^T e^{C_1t}\|\boldsymbol{G}\|^2_{(\boldsymbol{H}^1_{0,\sigma})'}dt =: M_1\,.$$

Integrating (6.13) on $[0,T]$, we deduce the inequality

$$\int_0^T \|\nabla\boldsymbol{v}_n\|_2^2dt \le \frac{1}{\nu}(C_1TM_1 + C_2\int_0^T \|\boldsymbol{G}\|^2_{(\boldsymbol{H}^1_{0,\sigma})'}dt) =: M_2\,.$$

For each $\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_{0,\sigma}(\Omega)$, we choose $J \in \boldsymbol{N}$ such that $\mathrm{supp}(\boldsymbol{\varphi}) \subset \Omega_n$ for any $n \ge J$. Then we have

$$|(\boldsymbol{v}_n(t), \boldsymbol{\varphi})| \le \|\boldsymbol{v}_n(t)\|_2\|\boldsymbol{\varphi}\|_2 \le M_1\|\boldsymbol{\varphi}\|_2$$

and

$$|(\boldsymbol{v}_n(t), \boldsymbol{\varphi}) - (\boldsymbol{v}_n(s), \boldsymbol{\varphi})|$$

$$= \left| \int_s^t \frac{d}{d\tau}(\boldsymbol{v}_n(\tau), \boldsymbol{\varphi}) d\tau \right|$$
$$\leq \int_s^t \nu |((\boldsymbol{v}_n, \boldsymbol{\varphi}))| + |((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{\varphi})| + |((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{V}^\alpha, \boldsymbol{\varphi})| + |((\boldsymbol{V}^\alpha \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{\varphi})| + |\langle \boldsymbol{G}, \boldsymbol{\varphi}\rangle| d\tau$$
$$\leq \int_s^t (\nu \|\nabla \boldsymbol{v}_n\|_2 + 2\|\boldsymbol{v}_n\|_2^{1/2} \|\nabla \boldsymbol{v}_n\|_2^{3/2} + 2C(\boldsymbol{V}^\alpha)\|\nabla \boldsymbol{v}_n\|_2 + \|\boldsymbol{G}\|_{(\boldsymbol{H}^1_{0,\sigma})'})\|\nabla \boldsymbol{\varphi}\|_2 d\tau$$
$$\leq (M_3|t-s|^{1/2} + M_4|t-s|^{1/4})\|\nabla \boldsymbol{\varphi}\|_2 ,$$

where the constant M_3 and M_4 do not depend on n. Therefore $\{(\boldsymbol{v}_n(t), \boldsymbol{\varphi})\}_{n \geq J}$ is uniformly bounded and equicontinuous on $[0, T]$.

Since $\{\boldsymbol{v}_n\}$ is a bounded sequence in $L^2((0,T); \boldsymbol{H}^1_{0,\sigma}(\Omega)) \cap L^\infty((0,T); \boldsymbol{L}^2_\sigma(\Omega))$, there exists a subsequence $\{\boldsymbol{v}_{n,k}\}_k$ of $\{\boldsymbol{v}_n\}$ and $\boldsymbol{v} \in L^2((0,T); \boldsymbol{H}^1_{0,\sigma}(\Omega)) \cap L^\infty((0,T); \boldsymbol{L}^2_\sigma(\Omega))$ such that

$$\boldsymbol{v}_{n,k} \to \boldsymbol{v} \quad (k \to \infty) \quad \text{in} \quad \begin{cases} L^\infty((0,T); \boldsymbol{L}^2_\sigma(\Omega)) & \text{weak star}, \\ L^2((0,T); \boldsymbol{H}^1_{0,\sigma}(\Omega)) & \text{weakly}. \end{cases} \tag{6.14}$$

For any $\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_{0,\sigma}(\Omega)$, there exists a subsequence $\{\boldsymbol{v}_{n,k,i}\}$ of $\{\boldsymbol{v}_{n,k}\}$ such that

$$(\boldsymbol{v}_{n,k,i}, \boldsymbol{\varphi}) = (\boldsymbol{v}, \boldsymbol{\varphi}) \quad \text{uniformly on} \quad [0,T] \quad (i \to \infty) \tag{6.15}$$

holds by the Ascoli-Arzelá Theorem.

We will prove that the convergence (6.15) holds for any $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$. We have the orthogonal decomposition

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_\sigma + \boldsymbol{\varphi}_p \quad (\boldsymbol{\varphi}_\sigma \in \boldsymbol{L}^2_\sigma(\Omega), \boldsymbol{\varphi}_p \in (\boldsymbol{L}^2_\sigma(\Omega))^\perp).$$

Since $\boldsymbol{C}^\infty_{0,\sigma}(\Omega)$ is dence in $\boldsymbol{L}^2_\sigma(\Omega)$, for any $\eta > 0$ there exists a $\boldsymbol{\varphi}^\eta_\sigma \in \boldsymbol{C}^\infty_{0,\sigma}(\Omega)$ such that

$$\|\boldsymbol{\varphi}^\eta_\sigma - \boldsymbol{\varphi}_\sigma\|_2 < \eta .$$

We have

$$\begin{aligned} |(\boldsymbol{v} - \boldsymbol{v}_n, \boldsymbol{\varphi})| &\leq |(\boldsymbol{v} - \boldsymbol{v}_n, \boldsymbol{\varphi}_\sigma - \boldsymbol{\varphi}^\eta_\sigma)| + |(\boldsymbol{v} - \boldsymbol{w}_n, \boldsymbol{\varphi}^\eta_\sigma)| \\ &\leq 2M_1\eta + |(\boldsymbol{v} - \boldsymbol{w}_n, \boldsymbol{\varphi}^\eta_\sigma)| \end{aligned} \tag{6.16}$$

because $\boldsymbol{v}_n$ is bounded in $L^\infty((0,T); \boldsymbol{L}^2_\sigma(\Omega))$. We can choose a subsequence $\{\boldsymbol{v}_{n,k}\}_k$ of $\{\boldsymbol{v}_n\}_n$ such that the second term of the right-hand side of (6.16) goes to 0. Therefore, for any $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$ there exists a subsequence $\{\boldsymbol{v}_{n,k,i}\}$ such that $(\boldsymbol{v}_{n,k,i}, \boldsymbol{\varphi})$ converges to $(\boldsymbol{v}, \boldsymbol{\varphi})$ uniformly on $[0, T]$.

6.4. Time periodic weak solution. In this subsection we prove that $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{V}^\alpha$ is a time periodic weak solution.

For any $\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_{0,\sigma}(\Omega)$, we choose $J \in \boldsymbol{N}$ such that $\mathrm{supp}(\boldsymbol{\varphi}) \subset \Omega^n$ for any $n \geq J$. We mulitiply (6.5) by $\psi \in C^\infty_0(0,T)$ and integrate on $[0,T]$. Then we have

$$\begin{aligned} \int_0^T &-(\boldsymbol{v}_n, \boldsymbol{\varphi})\psi' + \{\nu((\boldsymbol{v}_n, \boldsymbol{\varphi})) + ((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{\varphi}) \\ &+ ((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{V}^\alpha), \boldsymbol{\varphi}) + ((\boldsymbol{V}^\alpha \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{\varphi})\}\psi dt \end{aligned} \tag{6.17}$$

$$= \int_0^T \langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle \psi \, dt \, .$$

It follows from (6.14) that there exists a subsequence $\{\boldsymbol{v}_{n,k}\}_{k \in \boldsymbol{N}}$ such that the left-hand side of (6.17) except the nonliner term converges to

$$\int_0^T -(\boldsymbol{v}, \boldsymbol{\varphi})\psi' + \{\nu((\boldsymbol{v}, \boldsymbol{\varphi})) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{V}^\alpha), \boldsymbol{\varphi}) + ((\boldsymbol{V}^\alpha \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi})\}\psi \, dt \, .$$

We prove that there exists a subsequence $\{\boldsymbol{v}_{n,k,i}\}$ such that

$$\int_0^T ((\boldsymbol{v}_{n,k,i} \cdot \nabla)\boldsymbol{v}_{n,k,i}, \boldsymbol{\varphi})\psi \, dt \to \int_0^T ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi})\psi \, dt \quad (i \to \infty) \, . \tag{6.18}$$

We have

$$\begin{aligned} &\int_0^T ((\boldsymbol{v}_{n,k} \cdot \nabla)\boldsymbol{v}_{n,k}, \boldsymbol{\varphi})\psi \, dt - \int_0^T ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi})\psi \, dt \\ &\quad = \int_0^T ((\boldsymbol{v}_{n,k} - \boldsymbol{v}) \cdot \nabla \boldsymbol{v}_{n,k}, \boldsymbol{\varphi})\psi \, dt - \int_0^T (\psi \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{v}_{n,k} - \boldsymbol{v}) dt \quad (=: I_1 + I_2) \, . \end{aligned} \tag{6.19}$$

Now let us consider I_1. Applying Lemma 5.3 to I_1, we see that for any $\eta > 0$ there exists an integer N_1 and $\boldsymbol{\psi}_l \in \boldsymbol{L}^2(\Omega)$ $(l = 1, \ldots, N_1)$ such that

$$|I_1| \le M_5 \eta + \sum_{l=1}^{N_1} \int_0^T |(\boldsymbol{v}_{n,k} - \boldsymbol{v}, \boldsymbol{\psi}_l)|^2 dt, \tag{6.20}$$

where the constant M_5 does not depend on n.

Let us consider I_2. Since we know that $\psi(t)\boldsymbol{v}(t) \cdot \nabla \boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$ for a.e. $t \in (0, T)$, it follows that

$$\psi(t)\boldsymbol{v}(t) \cdot \nabla \boldsymbol{\varphi} = \boldsymbol{\Phi}_\sigma(t) + \boldsymbol{\Phi}_p(t) \quad (\boldsymbol{\Phi}_\sigma(t) \in \boldsymbol{L}^2_\sigma(\Omega), \boldsymbol{\Phi}_p(t) \in (\boldsymbol{L}^2_\sigma(\Omega))^\perp) \, .$$

Since $\boldsymbol{v}_{n,k}(t) - \boldsymbol{v}(t) \in \boldsymbol{L}^2_\sigma(\Omega)$ for a.e. $t \in (0, T)$, we have

$$\int_0^T (\psi \boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}, \boldsymbol{v}_{n,k} - \boldsymbol{v}) dt = \int_0^T (\boldsymbol{\Phi}_\sigma, \boldsymbol{v}_{n,k} - \boldsymbol{v}) dt \, . \tag{6.21}$$

Consequently, it follows from (6.20) and (6.21) that

$$|I_1| + |I_2| \le M_5 \eta + \sum_{l=1}^{N_1} \int_0^T |(\boldsymbol{v}_{n,k} - \boldsymbol{v}, \boldsymbol{\psi}_l)|^2 dt + \left| \int_0^T (\boldsymbol{\Phi}_\sigma, \boldsymbol{v}_{n,k} - \boldsymbol{v}) dt \right| \, . \tag{6.22}$$

We can choose a subsequence $\{\boldsymbol{v}_{n,k,i}\}_{i \in \boldsymbol{N}}$ of $\{\boldsymbol{v}_{n,k}\}_{k \in \boldsymbol{N}}$ such that the second and third terms of the right-hand side of (6.22) converge to 0 from (6.15). This proves the convergence of (6.18).

The above convergence implies that $\boldsymbol{v}$ satisfies

$$\begin{aligned} &\int_0^T -(\boldsymbol{v}, \boldsymbol{\varphi})\psi' + \{\nu((\boldsymbol{v}, \boldsymbol{\varphi})) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi}) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{V}^\alpha, \boldsymbol{\varphi}) + ((\boldsymbol{V}^\alpha \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi})\}\psi \, dt \\ &\quad = \int_0^T \langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle \psi \, dt \, . \end{aligned}$$

Since $\boldsymbol{C}^{\infty}_{0,\sigma}(\Omega) \subset \boldsymbol{H}^1_{0,\sigma}(\Omega)$ is dense, we have

$$\int_0^T -(\boldsymbol{v}, \boldsymbol{\varphi})\psi' + \{\nu((\boldsymbol{v}, \boldsymbol{\varphi})) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi}) + ((\boldsymbol{v} \cdot \nabla)\boldsymbol{V}^{\alpha}, \boldsymbol{\varphi}) + ((\boldsymbol{V}^{\alpha} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi})\}\psi dt$$
$$= \int_0^T \langle \boldsymbol{G}, \boldsymbol{\varphi}\rangle \psi dt \quad (\boldsymbol{\varphi} \in \boldsymbol{H}^1_{0,\sigma}(\Omega), \psi \in C^{\infty}_0(0, T)) .$$

Lastly, we prove that $\boldsymbol{v}$ is time periodic in $\boldsymbol{L}^2(\Omega)$. For any $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\Omega)$, there exists a subsequence $\{\boldsymbol{v}_{n,k}\}$ such that the limit (6.15) holds true. Therefore, it follows that

$$(\boldsymbol{v}(0) - \boldsymbol{v}(T), \boldsymbol{\varphi}) = (\boldsymbol{v}(0) - \boldsymbol{v}_{n,k}(0), \boldsymbol{\varphi}) + (\boldsymbol{v}_{n,k}(T) - \boldsymbol{v}(T), \boldsymbol{\varphi}) \to 0 \quad (k \to \infty) ,$$

that is to say, $\boldsymbol{v}$ is time periodic in $\boldsymbol{L}^2(\Omega)$. We set

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{V}^{\alpha} .$$

Then $\boldsymbol{u}$ is a time periodic weak solution.

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