POLYHARMONIC FUNCTIONS OF INFINITE ORDER ON ANNULAR REGIONS

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Abstract. Polyharmonic functions f of infinite order and type τ on annular regions are systematically studied. The first main result states that the Fourier-Laplace coefficients $f_{k,l}(r)$ of a polyharmonic function f of infinite order and type 0 can be extended to analytic functions on the complex plane cut along the negative semiaxis. The second main result gives a constructive procedure via Fourier-Laplace series for the analytic extension of a polyharmonic function on annular region $A(r_0, r_1)$ of infinite order and type less than $1/2r_1$ to the kernel of the harmonicity hull of the annular region. The methods of proof depend on an extensive investigation of Taylor series with respect to linear differential operators with constant coefficients.

1. Introduction. Let G be a domain in the euclidean space $\mathbf{R}^d$. A function $f : G \to \mathbf{C}$ is called polyharmonic of order p if f is $2p$ times continuously differentiable and $\Delta^p f(x) = 0$ for all $x \in G$, where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$ is the Laplace operator and Δ^p the p-th iterate of Δ. Polyharmonic functions have been investigated by several authors (see e.g. [1], [10], [11], [12], [13], [16], [20], [21], [28], [33], [34], [35], [38]) and they have recently many applications in approximation theory, radial basis functions and wavelet analysis (see e.g. [6], [22], [23], [24], [29]).

Aronszajn introduced in 1935 the concept of a polyharmonic function of *infinite order* (see [3] and [26]). On the one hand, this class of functions contains the class of classical polyharmonic functions of all finite orders p, and on the other hand it retains many properties of the latter class, e.g. analytic extendibility to the harmonicity hull; the monograph [2] is devoted to this subject and additional information can be found in the research book of Avanissian [4]. Important examples are eigenfunctions of the Laplacian, i.e., functions satisfying the equation $\Delta f(x) = \lambda f(x)$ for some $\lambda \in \mathbf{C}$, or so-called *metaharmonic* functions (e.g. [40]).

Let us recall that a function $f : G \to \mathbf{C}$ is *polyharmonic of infinite order and type* $\tau \geq 0$ if, for any compact set $K \subset G$ and for all $\varepsilon > 0$, there exists a constant $C_{K,\varepsilon} > 0$ such that

$$(1) \qquad \max_{x \in K} |\Delta^p f(x)| \leq C_{K,\varepsilon} (2p)! (\tau + \varepsilon)^{2p}$$

for all natural numbers p. An equivalent way to express this inequality is to require that for any compact subset K of G the inequality

$$(2) \qquad \overline{\lim_{p \to \infty}} \max_{x \in K} \sqrt[2p]{\frac{|\Delta^p f(x)|}{(2p)!}} \leq \tau$$

holds. [4, Theorem 1.4] characterizes real-analyticity in terms of estimates of the Laplacian. Namely, an infinitely differentiable function $f : G \to \mathbf{C}$ is real-analytic if and only if for any compact subset K of G there exists a constant C_K and a constant τ_K such that

$$(3) \qquad \max_{x \in K} |\Delta^p f(x)| \leq C_K (2p)! (\tau_K)^{2p}$$

for all natural numbers p. Thus polyharmonic functions of infinite order and type τ are real-analytic and they allow the explicit control of the constant τ_K in (3).

In the present paper we shall study polyharmonic functions of infinite order on the annular region

$$A(r_0, r_1) := \left\{ x \in \mathbf{R}^d; r_0 < |x| < r_1 \right\} \quad \text{for } 0 \leq r_0 < r_1 \leq \infty.$$

In this case tools from harmonic analysis, like the Fourier-Laplace series, are available. Our first goal is to describe properties of the Fourier-Laplace coefficients $f_{k,l}$ of a polyharmonic function f of infinite order. Let us recall some basic notations: Let

$$Y_{k,l}(x), \quad \text{for } l = 1, \ldots, a_k,$$

be an orthonormal basis of the a_k-dimensional linear space of harmonic homogeneous polynomials of degree $k \geq 0$, which are orthonormal with respect to the scalar product

$$\langle f, g \rangle_{\mathbf{S}^{d-1}} := \int_{\mathbf{S}^{d-1}} f(\theta) \overline{g(\theta)} d\theta,$$

where $\mathbf{S}^{d-1} = \left\{ x \in \mathbf{R}^d; |x| = 1 \right\}$ is the unit sphere (see [5], [22], [32], [39]). Let f be a continuous function on the annular region $A(r_0, r_1)$. Then the *Fourier-Laplace coefficients* $f_{k,l}$ of f are defined by

$$(4) \qquad f_{k,l}(r) = \int_{\mathbf{S}^{d-1}} f(r\theta) \overline{Y_{k,l}(\theta)} d\theta \quad \text{for } r \in (r_0, r_1).$$

The formal series

$$(5) \qquad \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l}(r) Y_{k,l}(\theta)$$

is called the *Fourier-Laplace series* of f. The special case of a harmonic function f defined on $A(r_0, r_1)$ shall serve us as a guiding example: We have

$$(6) \qquad f_{k,l}(r) = \begin{cases} \alpha_k r^k + \beta_k r^{-k-d+2} & \text{for } d > 2, \text{ or } d = 2, \ k \geq 1 \\ \alpha_0 + \beta_0 \log r & \text{for } d = 2, \ k = 0, \end{cases}$$

on the open interval (r_0, r_1), for suitable complex coefficients α_k and β_k. More generally, it is known that if f is polyharmonic of order p and d is odd then there exist polynomials $p_{k,l}$ and $q_{k,l}$ of degree $p - 1$ such that $f_{k,l}(r) = r^k p_{k,l}(r^2) + r^{-k-d+2} q_{k,l}(r^2)$ (see [38], [41] or

[22]). Thus for odd dimension $f_{k,l}(r)$ extends to an analytic function on the punctured plane $\mathbf{C}^* := \{z \in \mathbf{C}; z \neq 0\}$, while for even dimension we can only infer that $f_{k,l}(r)$ is an analytic function on the slit plane $\mathbf{C} \setminus (-\infty, 0]$.

The **first main result** of this paper states the following: The Fourier-Laplace coefficients $f_{k,l}(r)$ of a polyharmonic function $f : A(r_0, r_1) \to \mathbf{C}$ of infinite order and type 0 possess analytic extensions to the slit plane $\mathbf{C} \setminus (-\infty, 0]$ (cf. Theorem 20 below). For odd dimension we can sharpen the result: There exist *entire functions* $p_{k,l}$ and $q_{k,l}$ such that

$$(7) \qquad f_{k,l}(r) = r^k p_{k,l}(r^2) + r^{-k-d+2} q_{k,l}(r^2)$$

for all $r \in (r_0, r_1)$. In particular, it follows that the Fourier-Laplace coefficients $f_{k,l}$ defined on the interval (r_0, r_1) can be analytically extended to the punctured plane $\mathbf{C}^* := \{z \in \mathbf{C}; z \neq 0\}$. We refer to Theorem 23 below.

The **second main result** of the paper addresses the problem of extending analytically a polyharmonic function $f : A(r_0, r_1) \to \mathbf{C}$ of infinite order and type $\tau \geq 0$ to a suitable domain in $\mathbf{C}^d$. It is well known that polyharmonic functions of infinite order and type $\tau = 0$ defined on a domain G in $\mathbf{R}^d$ can be extended analytically to the so-called kernel $\widetilde{G}$ of the harmonicity hull $\widehat{G}$ (see [2], [4], or for a generalization [9]). In the case of the annular region $A(r_0, r_1)$ we can give an explicit formula for the analytic extension via Fourier-Laplace series and, as a by-product, we show that it suffices to assume that the function f is polyharmonic of infinite order and type $\tau < 1/2r_1$ instead of the stronger assumption of type $\tau = 0$. We refer to Theorem 25 and Theorem 26 below.

In order to make the results more precise we recall some basic notations in complex analysis in several variables: For $z = (z_1, \dots, z_d) \in \mathbf{C}^d$ define $|z|^2_{\mathbf{C}^d} = |z_1|^2 + \cdots + |z_d|^2$ and $q(z) := z_1^2 + \cdots + z_d^2$. The upper and lower *Lie-norm* $L_+ : \mathbf{C}^d \to [0, \infty)$ and $L_- : \mathbf{C}^d \to [0, \infty)$ are defined by

$$L_\pm(z) = \sqrt{|z|^2_{\mathbf{C}^d} \pm \sqrt{|z|^4_{\mathbf{C}^d} - |q(z)|^2}}.$$

The Lie-ball of radius $R \in (0, \infty]$ is defined by $\widehat{B_R} := \{z \in \mathbf{C}^d; L_+(z) < R\}$ and it is also called the classical domain of E. Cartan of the type IV, we refer to [2, p. 59], [17] or [31] for further details.

In the above terms our *second main result* says that a polyharmonic function $f : A(r_0, r_1) \to \mathbf{C}$ of infinite order and type $\tau < 1/2r_1$ can be extended to an analytic function on the domain

$$\{z \in \mathbf{C}^d; r_0 < L_-(z) \leq L_+(z) < r_1\} \setminus q^{-1}(-\infty, 0]).$$

The proof depends on a Laurent type decomposition of the function f: for odd dimension $d > 1$ we show that there exists an analytic function f_1 defined on $\{z \in \mathbf{C}^d; L_+(z) < r_1\}$ and an analytic function f_2 defined on $\{z \in \mathbf{C}^d; r_0 < L_-(z) \leq L_+(z) < 1/2\tau\}$ such that the function F defined by

$$F(z) = f_1(z) + \left(z_1^2 + \cdots + z_d^2\right)^{(2-d)/2} f_2(z)$$

is an analytic extension of f. A similar result is formulated in Section 7 for even dimension.

The paper is organized as follows: In the Section 2 we recall some basic facts about the action of the Laplace operator Δ on Fourier-Laplace series. For polyharmonic functions of infinite order we obtain estimates of derivatives of the Fourier-Laplace coefficients with respect to certain linear differential operators depending on the radius r. The results in Section 3 belong to the main technical merits of the paper: They are devoted to an extensive discussion of the so-called fundamental function of a linear differential operator with constant coefficients and the concept of a generalized Taylor series with respect to the corresponding fundamental functions, the climax being Theorem 16 below. These results are crucial for the main goals of the paper and are also of independent interest.

Section 4 contains the first main result about the analytic extendibility of the Fourier-Laplace coefficients for polyharmonic functions of infinite order and type $\tau < 1/2r_0$, and Section 5 discusses the special case of odd dimension. In Section 6 we discuss the analytic extendibility as described above for odd dimension, and in Section 7 the case of even dimension is addressed. The paper concludes with an Appendix concerning estimates of generalized derivatives of odd order by even orders in the framework of linear differential operators with constant coefficients.

Throughout the paper it is assumed that $d \geq 2$. By ω_{d-1} we define the surface area of S^{d-1} with respect to the rotation invariant measure $d\theta$.

2. Basic estimates and examples.

By $C^m(G)$ we denote the set of all functions $f : G \to C$ which are continuously differentiable up to the order m. It is well known that the Fourier-Laplace series (5) converges absolutely and compactly in $A(r_0, r_1)$ to f if $f \in C^m(A(r_0, r_1))$ for $m > \frac{1}{2}(d-1)$. We refer to [18] for questions of convergence of Fourier-Laplace series and the cited literature therein.

Let $f \in C^\infty(A(r_0, r_1))$ and let $f_{k,l}, k \in N_0, l = 1, \ldots, a_k$, be the Fourier-Laplace coefficients defined in (4). Recall that a_k is the dimension of the space of all harmonic homogeneous polynomials of degree k.

For $x \in A(r_0, r_1)$ we use spherical coordinates $x = r\theta$ where $\theta = x/|x|$ and $r = |x|$. It is not difficult to establish the formula

$$(8) \qquad \left(\Delta^p f\right)(r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} L_k^p\left(f_{k,l}\right)(r) \cdot Y_{k,l}(\theta) \,,$$

where the series converges absolutely and uniformly on compact subsets of $A(r_0, r_1)$ and L_k^p is the p-th iterate of the differential operator

$$(9) \qquad L_k = \frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} - \frac{k(k+d-2)}{r^2}$$

(see [22]). It follows that the (k, l)-th Fourier-Laplace coefficient of $\Delta^p f$ is equal to $L_k^p f_{k,l}(r)$, i.e., that

$$(10) \qquad L_k^p\left(f_{k,l}\right)(r) = \int_{S^{d-1}} \left(\Delta^p f\right)(r\theta) \, \overline{Y_{k,l}(\theta)} d\theta$$

for any $r \in (r_0, r_1)$. Parseval's formula yields

$$\int_{S^{d-1}} \left| \Delta^p f(r\theta) \right|^2 d\theta = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \left| L_k^p f_{k,l}(r) \right|^2 < \infty$$

for any r with $r_0 < r < r_1$.

THEOREM 1. *Let $f \in A(r_0, r_1) \to C$ be polyharmonic of infinite order and type $\tau \geq 0$. Then for each subinterval $[a, b]$ of (r_0, r_1) and for all $\varepsilon > 0$ there exists a positive number $C_{a,b,\varepsilon}$ such that, for all $k \in N_0, l = 1, \ldots, a_k$, for all $r \in [a, b]$ and $p \in N_0$,*

$$\left| L_k^p \left(f_{k,l} \right) (r) \right| \leq C_{a,b,\varepsilon} \sqrt{\omega_{d-1}} \, (2p)! \, (\tau + \varepsilon)^{2p} \ .$$

PROOF. Let $[a, b] \subset (r_0, r_1)$ and $K(a, b) := \left\{ x \in R^d ; a \leq |x| \leq b \right\}$. Then (10) implies that

(11) $$\left| L_k^p \left(f_{k,l} \right) (r) \right| \leq \max_{x \in K(a,b)} \left| \Delta^p f(x) \right| \cdot \int_{S^{d-1}} \left| Y_{k,l}(\theta) \right| d\theta$$

for all $r \in [a, b]$. The integral on the right-hand side in (11) can be estimated by the Cauchy Schwarz inequality

$$\int_{S^{d-1}} \left| Y_{k,l}(\theta) \right| d\theta \leq \sqrt{\int_{S^{d-1}} 1 d\theta} \sqrt{\int_{S^{d-1}} \left| Y_{k,l}(\theta) \right|^2 d\theta} = \sqrt{\omega_{d-1}} \ .$$

Now the result follows from the definition of a polyharmonic function of infinite order and type $\tau \geq 0$ given in (1). $\square$

The rest of this Section is devoted to an instructive example: For a real number α and a harmonic homogeneous polynomial $Y_k(x)$ of degree $k \in N_0$, define the function

$$H_{\alpha,k}(x) = \left(x_1^2 + \cdots + x_d^2 \right)^{\alpha} \cdot Y_k(x) = |x|^{2\alpha} \cdot Y_k(x) \ .$$

Obviously $H_{\alpha,k}$ can be defined on the annular region $R^d \setminus \{0\}$. In the next result we restrict $H_{\alpha,k}$ to the annular region $A(r_0, r_1)$ with $r_0 > 0$ and we shall show that $H_{\alpha,k}$ is polyharmonic of infinite order and type at most $1/r_0$.

THEOREM 2. *For $\alpha \in N_0$ or $\alpha = 1 - \frac{1}{2}d - k + j$ with $j \in N_0$ the function $H_{\alpha,k}$ is polyharmonic of finite order. If $\alpha \in R$ is different from these numbers, then $H_{\alpha,k}$, as a function on the annular region $A(r_0, r_1)$ with $r_0 > 0$, is polyharmonic of infinite order and type at most $1/r_0$.*

PROOF. A straightforward calculation provides the formula

$$\Delta \left(|x|^{2\alpha} Y_k(x) \right) = 2\alpha \left(2\alpha + d - 2 + 2k \right) \cdot |x|^{2\alpha - 2} Y_k(x) \ .$$

Hence $\Delta^p \left(|x|^{2\alpha} Y_k(x) \right) = c_{\alpha, p} |x|^{2\alpha - 2p} Y_k(x)$ where

$$c_{\alpha, p} = 2\alpha \left(2\alpha - 2 \right) \cdots \left(2\alpha - 2(p-1) \right)$$
$$\cdot \left(2\alpha + d - 2 + 2k \right) \cdots \left(2\alpha + d - 2 + 2k - 2(p-1) \right) \ .$$

Thus $\Delta^p \left(|x|^{2\alpha} Y_k(x) \right) = 0$ if and only if $2\alpha - 2j = 0$ or $2\alpha + d - 2 + 2k - 2j = 0$ for some $j = 0, \ldots, p - 1$. This means that $\alpha = j$ for some $j \in \{0, \ldots, p - 1\}$ or $\alpha = 1 - \frac{1}{2}d - k + j$ for some $j \in \{0, \ldots, p\}$. Hence the first statement is proven.

Next consider the power series $f(z) = \sum_{p=1}^{\infty} c_{\alpha,p} z^p / (2p)!$ in the complex variable z. Assume that $\alpha \neq j$ and $\alpha \neq 1 - \frac{1}{2}d - k + j$ for all natural numbers $j \in N_0$. Then the convergence radius R can be computed by the ratio test

$$R = \lim_{p \to \infty} \frac{c_{\alpha,p} / (2p)!}{c_{\alpha,p+1} / (2p+2)!} = \lim_{p \to \infty} \frac{(2p+1)(2p+2)}{(2\alpha - 2p)(2\alpha + d - 2 + 2k - 2p)} = 1.$$

The convergence radius formula yields $R = \overline{\lim}_{p \to \infty} \sqrt[p]{|c_{\alpha,p}| / (2p)!} = 1$. Moreover,

$$(12) \qquad \sqrt[2p]{\frac{\left| \Delta^p \left(|x|^{2\alpha} Y_k(x) \right) \right|}{(2p)!}} = \sqrt[2p]{\left| \frac{c_{\alpha,p}}{(2p)!} \right|} \sqrt[2p]{|Y_k(x)|} \sqrt[2p]{|x|^{2\alpha}} \cdot \frac{1}{|x|}.$$

Let now $K \subset A(r_0, r_1)$ be a compact subset. Since Y_k is continuous it is bounded on K, say by M_k. Clearly $|x| > r_0$ for $x \in K$. Thus we can estimate

$$\sqrt[2p]{|Y_k(x)|} \sqrt[2p]{|x|^{2\alpha}} \cdot \frac{1}{|x|} \leq \sqrt[2p]{M_k} \sqrt[2p]{r_1^{2\alpha}} \cdot \frac{1}{r_0}$$

for all $x \in K$. Using this estimate in (12) and taking the limit $p \to \infty$, we see that $H_{\alpha,k}$ is polyharmonic of infinite order and type at most $1/r_0$. □

3. Linear differential operators with constant coefficients.

In this section we shall review some results about linear differential operators with constant coefficients (see e.g. [15]). Mainly we shall study Taylor-type expansion of a C^∞-function with respect to a linear differential operator with constant coefficients. Some material can be found in [37] but we shall need a deeper analysis of this topic. We shall give a self-contained presentation in order to facilitate the readability of the paper and to fix notations.

Let $\lambda_0, \ldots, \lambda_n$ be complex numbers, and define the linear differential operator with constant coefficients L by

$$(13) \qquad L := L_{\lambda_0, \ldots, \lambda_n} := \left(\frac{d}{dx} - \lambda_0 \right) \cdots \left(\frac{d}{dx} - \lambda_n \right).$$

The space of all solutions of $Lu = 0$ is denoted by

$$(14) \qquad E_{(\lambda_0, \ldots, \lambda_n)} := \left\{ f \in C^\infty(R); Lf = 0 \right\}.$$

Elements in $E_{(\lambda_0, \ldots, \lambda_n)}$ are called *exponential polynomials* or sometimes *L-polynomials,* and $\lambda_0, \ldots, \lambda_n$ are called *exponents* or *frequencies* (see e.g. Chapter 3 in [8]).

In the case of *pairwise different* λ_j, $j = 0, \ldots, n$, the space $E_{(\lambda_0, \ldots, \lambda_n)}$ is the linear span generated by the functions $e^{\lambda_0 x}, e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$. In the case when λ_j occurs m_j times in $\Lambda_n = (\lambda_0, \ldots, \lambda_n)$, a basis of the space $E_{(\lambda_0, \ldots, \lambda_n)}$ is given by the linearly independent functions

$$(15) \qquad x^s e^{\lambda_j x} \quad \text{for } s = 0, 1, \ldots, m_j - 1.$$

In the case that $\lambda_0 = \cdots = \lambda_n = 0$, the space $E_{(\lambda_0,\ldots,\lambda_n)}$ is just the space of all polynomials of degree at most n, and we shall refer to this as the *polynomial case*.

3.1. The fundamental function. It is well known that for $\Lambda_n = (\lambda_0, \ldots, \lambda_n) \in \mathbf{C}^{n+1}$ there exists a unique solution $\Phi_{\Lambda_n} \in E_{(\lambda_0,\ldots,\lambda_n)}$ to the Cauchy problem

$$\Phi_{\Lambda_n}(0) = \cdots = \Phi_{\Lambda_n}^{(n-1)}(0) = 0 \quad \text{and} \quad \Phi_{\Lambda_n}^{(n)}(0) = 1 \,.$$

We shall call Φ_{Λ_n} the *fundamental function* in $E_{(\lambda_0,\ldots,\lambda_n)}$ (see e.g. [30]). An explicit formula for Φ_{Λ_n} is

(16)
$$\Phi_{\Lambda_n}(x) := \frac{1}{2\pi i} \int_{\Gamma_r} \frac{e^{xz}}{(z - \lambda_0) \cdots (z - \lambda_n)} dz \,,$$

where Γ_r is the path in the complex plane defined by $\Gamma_r(t) = re^{it}$, $t \in [0, 2\pi]$, surrounding all the complex numbers $\lambda_0, \ldots, \lambda_n$ (see Proposition 4 below). Note that (16) implies the useful formula

(17)
$$\left(\frac{d}{dx} - \lambda_{n+1} \right) \Phi_{(\lambda_0,\ldots,\lambda_{n+1})}(x) = \Phi_{(\lambda_0,\ldots,\lambda_n)}(x) \,.$$

The fundamental function can be seen as analogue of the power function x^n in the space $E_{(\lambda_0,\ldots,\lambda_n)}$: In the polynomial case, i.e., $\lambda_0 = \cdots = \lambda_n = 0$, the fundamental function is

(18)
$$\Phi_{\mathrm{pol},n}(x) = \frac{1}{n!} x^n \,.$$

In general, explicit formulae for the fundamental function are complicated. However, in the case of equidistant exponents one can compute the fundamental function in a very simple way (see [27]):

EXAMPLE 3. Assume that $\lambda_k = \alpha + \omega k$ for $k = 0, \ldots, n$, and complex numbers $\omega \neq 0$ and α. Then

(19)
$$\Phi_{\mathrm{equi},n}(x) = \frac{1}{n!} \frac{1}{\omega^n} e^{\alpha x} \left(e^{\omega x} - 1 \right)^n = \frac{1}{n!} \frac{1}{\omega^n} \sum_{k=0}^{n} \binom{n}{k} e^{(\alpha+k\omega)x} (-1)^{n-k} \,.$$

Indeed, it is easy to see that $\Phi_{\mathrm{equi},n}(0) = \cdots = \Phi_{\mathrm{equi},n}^{(n-1)}(0) = 0$ and $\Phi_{\mathrm{equi},n}^{(n)}(0) = 1$, and the right-hand side of (19) shows that $\Phi_{\mathrm{equi},n} \in E_{(\lambda_0,\ldots,\lambda_n)}$.

In the following we shall give estimates of the fundamental function which seem to be new. Our estimates are based on the Taylor expansion of the fundamental function Φ_{Λ_n} which will be described as follows:

PROPOSITION 4. *The function* $\Phi_{(\lambda_0,\ldots,\lambda_n)}$ *defined in* (16) *satisfies* $\Phi_{(\lambda_0,\ldots,\lambda_n)}^{(k)}(0) = 0$ *for* $k = 0, \ldots, n-1$. *For* $k \geq n$ *the formula*

(20)
$$\Phi_{(\lambda_0,\ldots,\lambda_n)}^{(k)}(0) = \sum_{\substack{(s_0,\ldots,s_n) \in \mathbf{N}_0^{n+1} \\ s_0 + \cdots + s_n + n = k,}} \lambda_0^{s_0} \cdots \lambda_n^{s_n}$$

holds. In particular, $\Phi^{(n)}_{(\lambda_0,\ldots,\lambda_n)}(0) = 1$ *and* $\Phi^{(n+1)}_{(\lambda_0,\ldots,\lambda_n)}(0) = \lambda_0 + \cdots + \lambda_n.$

PROOF. For $z \in \mathbf{C}$ with $|z| > |\lambda_j|$, the geometric series

$$\frac{1}{z - \lambda_j} = \frac{1}{z} \cdot \frac{1}{1 - \lambda_j/z} = \sum_{s=0}^{\infty} \lambda_j^s \left(\frac{1}{z}\right)^{s+1}$$

converges. Thus we obtain from (16) that

$$\Phi_{(\lambda_0,\ldots,\lambda_n)}(x) = \sum_{s_0=0}^{\infty} \cdots \sum_{s_n=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_r} \lambda_0^{s_0} \cdots \lambda_n^{s_n} \frac{e^{xz}}{z^{s_0 + \cdots + s_n + n + 1}} dz.$$

By differentiating one obtains

$$\Phi^{(k)}_{(\lambda_0,\ldots,\lambda_n)}(x) = \sum_{s_0=0}^{\infty} \cdots \sum_{s_n=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_r} \lambda_0^{s_0} \cdots \lambda_n^{s_n} \frac{z^k e^{xz}}{z^{s_0 + \cdots + s_n + n + 1}} dz.$$

For $x = 0$ the integral is easy to evaluate and the result is proven. □

In the following proposition we give the first estimate for the fundamental function:

PROPOSITION 5. *Let* $\lambda_0,\ldots,\lambda_n$ *be complex numbers and* $M_n := \max\{|\lambda_j|; j = 0,\ldots,n\}$. *Then the inequality*

$$(21) \qquad \left|\Phi_{(\lambda_0,\ldots,\lambda_n)}(z)\right| \le \Phi_{(|\lambda_0|,\ldots,|\lambda_n|)}(|z|) \le \frac{1}{n!} |z|^n e^{M_n|z|}$$

holds for all $z \in \mathbf{C}$.

PROOF. Using (20) we can estimate the Taylor coefficient

$$\left|\Phi^{(k)}_{(\lambda_0,\ldots,\lambda_n)}(0)\right| \le \sum_{s_0 + \cdots + s_n + n = k} \left|\lambda_0^{s_0} \cdots \lambda_n^{s_n}\right| = \sum_{s_0 + \cdots + s_n + n = k} |\lambda_0|^{s_0} \cdots |\lambda_n|^{s_n}$$

which is obviously equal to $\Phi^{(k)}_{(|\lambda_0|,\ldots,|\lambda_n|)}(0)$. Since $\Phi_{\Lambda_n}(z) = \sum_{k=n}^{\infty} \Phi^{(k)}_{(\lambda_0,\ldots,\lambda_n)}(0) z^k / k!$ we can estimate

$$\left|\Phi_{(\lambda_0,\ldots,\lambda_n)}(z)\right| \le \sum_{k=n}^{\infty} \frac{1}{k!} \left|\Phi^{(k)}_{(\lambda_0,\ldots,\lambda_n)}(0)\right| \cdot |z|^k = \Phi_{(|\lambda_0|,\ldots,|\lambda_n|)}(|z|).$$

Since $|\lambda_j| \le M_n$ for all $j = 0,\ldots,n$, we can estimate $|\lambda_0|^{s_0} \cdots |\lambda_n|^{s_n} \le M_n^{s_0 + \cdots + s_n}$. Using (20) for $|\lambda_0|,\ldots,|\lambda_n|$ we obtain for $k \ge n$

$$\Phi^{(k)}_{(|\lambda_0|,\ldots,|\lambda_n|)}(0) = \sum_{s_0 + \cdots + s_n = k - n} |\lambda_0|^{s_0} \cdots |\lambda_n|^{s_n} \le \binom{k - n + n}{k - n} M_n^{k-n}.$$

We conclude that

$$\Phi_{(|\lambda_0|,\ldots,|\lambda_n|)}(|z|) \le \sum_{k=n}^{\infty} \frac{|z|^k}{k!} \binom{k}{k-n} M_n^{k-n} = \sum_{k=0}^{\infty} \frac{|z|^{k+n}}{(k+n)!} \binom{k+n}{k} M_n^k$$

$$= \frac{1}{n!} |z|^n \sum_{k=0}^{\infty} \frac{1}{k!} |M_n z|^k = \frac{1}{n!} |z|^n e^{M_n|z|} .$$

The proof is accomplished. □

Suppose now that $\lambda_0, \lambda_1, \ldots,$ is a *bounded* sequence of complex numbers. Then (21) implies that

(22) $$\overline{\lim_{n\to\infty}} \sqrt[n]{n! |\Phi_{\Lambda_n}(z)|} \leq |z| .$$

In the special case that the exponents λ_n are equal to 0, a stronger conclusion is valid. Namely, by using the explicit formula (18), we know that the limit in (22) exists and

$$\lim_{n\to\infty} \sqrt[n]{n! |\Phi_{\text{pol},n}(z)|} = \lim_{n\to\infty} \sqrt[n]{|z^n|} = |z| .$$

Next suppose that the estimate $|\lambda_n| \leq \beta n$ holds for all natural numbers and some $\beta > 0$. Then (21) yields the estimate

(23) $$\overline{\lim_{n\to\infty}} \sqrt[n]{n! |\Phi_{\Lambda_n}(z)|} \leq |z| e^{\beta|z|} .$$

The estimate (23) seems to be satisfactory. However, the example of the equidistant points $\lambda_n = n\omega$ shows that this is not the optimal estimate. Namely, by using (19) we infer that

$$\lim_{n\to\infty} \sqrt[n]{n! |\Phi_{\text{equi},n}(z)|} = \frac{1}{|\omega|} |e^{\omega z} - 1| \leq |z| e^{|\omega||z|} .$$

Next we shall provide a similar estimate for general exponents λ_n obeying an estimate of the form $|\lambda_n| \leq \beta n$.

PROPOSITION 6. *Let λ_j and μ_j be real numbers satisfying $0 \leq \lambda_j \leq \mu_j$ for $j = 0, \ldots, n$. Then $\Phi_{\Lambda_n}(x)$ is real for all $x \in \mathbf{R}$ and $\Phi_{\Lambda_n}(x) > 0$ for all $x > 0$. Moreover*

$$\left| \Phi_{(\lambda_0,\ldots,\lambda_n)}(z) \right| \leq \Phi_{(\mu_0,\ldots,\mu_n)}(|z|)$$

for all complex numbers z.

PROOF. By (20) the Taylor coefficients of $\Phi_{(\lambda_0,\ldots,\lambda_n)}(x)$ are real, so $\Phi_{(\lambda_0,\ldots,\lambda_n)}(x)$ is a real number for real x. Clearly $0 \leq \lambda_j \leq \mu_j$ implies that $0 \leq \lambda_j^{s_j} \leq \mu_j^{s_j}$ for any natural number s_j, $j = 0, \ldots, n$. Thus $0 \leq \lambda_0^{s_0} \cdots \lambda_n^{s_n} \leq \mu_0^{s_0} \cdots \mu_n^{s_n}$ for any $(s_0, \ldots, s_n) \in \mathbf{N}_0^{n+1}$. By formula (20) we have

$$0 \leq \Phi_{(\lambda_0,\ldots,\lambda_n)}^{(k)}(0) \leq \Phi_{(\mu_0,\ldots,\mu_n)}^{(k)}(0)$$

for all $k \in \mathbf{N}_0$. It follows that $0 \leq \Phi_{(\lambda_0,\ldots,\lambda_n)}(|z|) \leq \Phi_{(\mu_0,\ldots,\mu_n)}(|z|)$. The proof is finished by combing the last inequality with (21). □

THEOREM 7. *Let $\lambda_n, n \in \mathbf{N}_0$, be complex numbers such that $\overline{\lim}_{n\to\infty} |\lambda_n|/n \leq \beta$. Then for any $\varepsilon > 0$ there exists a number $\alpha > 0$ such that*

(24) $$n! \left| \Phi_{\Lambda_n}(z) \right| \leq e^{\alpha|z|} \left(\frac{e^{(1+\varepsilon)\beta|z|} - 1}{(1+\varepsilon)\beta} \right)^n$$

for all natural numbers n and for all complex numbers z. In other words,

$$(25) \qquad \overline{\lim_{n \to \infty}} \sqrt[n]{n! |\Phi_{\Lambda_n}(z)|} \leq \frac{e^{\beta |z|} - 1}{\beta} \, .$$

PROOF. Let $\varepsilon > 0$. Then there exist n_0 such that $|\lambda_n| \leq (1 + \varepsilon) \beta n$ for all $n \geq n_0$. Take $\alpha > 0$ large enough so that $|\lambda_n| \leq (1 + \varepsilon) \beta n + \alpha$ for all natural numbers n. Define $\mu_n := \alpha + (1 + \varepsilon) \beta n$ for all n, so $|\lambda_n| \leq \mu_n$ for all $n \in N_0$. Propositions 5 and 6 and Example 3 show that

$$\left| \Phi_{(\lambda_0, \ldots, \lambda_n)}(z) \right| \leq \Phi_{(|\lambda_0|, \ldots, |\lambda_n|)}(|z|) \leq \Phi_{(\mu_0, \ldots, \mu_n)}(|z|) = \frac{1}{n!} e^{\alpha |z|} \left(\frac{e^{(1+\varepsilon)\beta |z|} - 1}{(1 + \varepsilon) \beta} \right)^n .$$

This shows (24) and clearly (25) is a simple consequence of (24).   □

The next theorem is our main result in this subsection and it will be used in later sections:

THEOREM 8. *Let $\beta > 0$ and $\lambda_n, n \in N_0$, be complex numbers such that $\overline{\lim}_{n \to \infty} |\lambda_n| / n \leq \beta$. Let a_n be complex numbers for $n \in N_0$ and define R^* through*

$$(26) \qquad \frac{1}{R^*} = \overline{\lim_{n \to \infty}} \sqrt[n]{\left| \frac{a_n}{n!} \right|} .$$

If $R^* > 0$, then the series $\sum_{n=0}^{\infty} a_n \Phi_{\Lambda_n}(z - x_0)$ converges compactly and absolutely in the ball with center x_0 and radius

$$\frac{1}{\beta} \ln(1 + \beta R^*) .$$

PROOF. Let $\rho < (1/\beta) \ln (1 + \beta R^*)$. Then $\left(e^{\beta \rho} - 1 \right) / R^* < \beta$. Take now $\varepsilon > 0$ small enough so that

$$(27) \qquad \frac{e^{(1+\varepsilon)\beta \rho} - 1}{(1 + \varepsilon) (R^* - \varepsilon)} < \beta \, .$$

Since $1/R^* < 1/ (R^* - \varepsilon)$, formula (26) shows that there exists a natural number n_0 such that

$$\left| \frac{a_n}{n!} \right| \leq \left(\frac{1}{R^* - \varepsilon} \right)^n$$

for all $n \geq n_0$. By Theorem 7 there exists a natural number $\alpha > 0$ such that

$$n! \left| \Phi_{\Lambda_n}(z) \right| \leq e^{\alpha |z|} \left(\frac{e^{(1+\varepsilon)\beta |z|} - 1}{(1 + \varepsilon) \beta} \right)^n$$

for all complex numbers z and for all natural numbers n. The last two inequalities lead to

$$\sum_{n=n_0}^{\infty} \left| a_n \Phi_{\Lambda_n}(z - x_0) \right| \leq e^{\alpha \rho} \sum_{n=n_0}^{\infty} \left(\frac{e^{(1+\varepsilon)\beta \rho} - 1}{(1 + \varepsilon) \beta (R^* - \varepsilon)} \right)^n$$

valid for all z with $|z - x_0| \leq \rho$. This series converges in view of the estimate (27).   □

EXAMPLE. Let $\lambda_n = n + 1$ for $n \in \mathbf{N}_0$, and consider the constant function $f(x) = 1$. Then

$$a_n := \left(\frac{d}{dx} - \lambda_0\right) \cdots \left(\frac{d}{dx} - \lambda_{n-1}\right) f(x_0) = (-1)^n \lambda_0 \cdots \lambda_{n-1} = (-1)^n n!.$$

Thus $\lim_{n \to \infty} \sqrt[n]{|a_n|/n!} = 1$. Further $\Phi_{\Lambda_n}(x) = e^x (e^x - 1)^n /n!$. According to Theorem 8 (with $x_0 = 0$ and $\beta = 1$) the series

$$(28) \qquad \sum_{n=0}^{\infty} a_n \Phi_{\Lambda_n}(x) = \sum_{n=0}^{\infty} (-1)^n e^x (e^x - 1)^n = e^x \frac{1}{1 - (1 - e^x)} = 1$$

converges for all complex numbers z with $|z| < \ln 2$. Of course, this can be seen directly for real x. Namely, if $e^x - 1 < 1$ (which means that $e^x < 2$, so $x < \ln 2$), the series obviously converges. On the other hand, for $e^x - 1 \geq 1$ we do not have convergence. Theorem 16 below provides the following interpretation: The constant function 1 has the Taylor series expansion (28) on $(-\ln 2, \ln 2)$ with respect to the differential operators $(d/dx - \lambda_0) \cdots (d/dx - \lambda_{n-1})$.

An analogue of Theorem 8 can be proved for a *bounded* sequence of exponents λ_n (see Theorem 9 below), either by using Proposition 5 or by applying Theorem 8 for $\beta > 0$ arbitrary. In the latter case, the rule of L'Hospital

$$\lim_{\beta \to 0} \frac{\ln(1 + \beta R^*)}{\beta} = \lim_{\beta \to 0} \frac{R^*}{1 + \beta R^*} = R^*$$

can be used for computing the correct radius of convergence:

THEOREM 9. *Let $\lambda_n, n \in \mathbf{N}_0$, be a bounded sequence of complex numbers. Let a_n be complex numbers for $n \in \mathbf{N}_0$ and define R^* as in (26). If $R^* > 0$ then the series $\sum_{n=0}^{\infty} a_n \Phi_{\Lambda_n}(z - x_0)$ converges compactly and absolutely in the ball with center x_0 and radius R^*.*

It is a natural and interesting question whether in (25) the *limit* exists. We mention two results addressing this problem but we omit the proofs since we shall not need them in the following.

THEOREM 10. *Suppose that $\lambda_n, n \in \mathbf{N}_0$, is a bounded sequence of real numbers and $\Lambda_n = (\lambda_0, \ldots, \lambda_n)$. Then the following limit exists for all $x \geq 0$:*

$$\lim_{n \to \infty} \sqrt[n]{n! \Phi_{\Lambda_n}(x)} = x.$$

THEOREM 11. *Let $\lambda_n, n \in \mathbf{N}_0$ be a sequence of real numbers such that the limit $\lim_{n \to \infty} \lambda_n/n = \beta$ exists. Then the following limit exists for all $x \geq 0$:*

$$\lim_{n \to \infty} \sqrt[n]{n! \Phi_{\Lambda_n}(x)} = \frac{e^{\beta x} - 1}{\beta}.$$

3.2. Taylor series for linear differential operators with constant coefficients. Let $\lambda_0, \dots, \lambda_n$ be complex numbers. As analogue of the n-th derivative $f^{(n)}(t)$ of a function $f(t)$, we define in the setting of linear differential operators

$$D^{(n)} f(t) := \left(\frac{d}{dt} - \lambda_0 \right) \cdots \left(\frac{d}{dt} - \lambda_{n-1} \right) f(t) .$$

To avoid overburdened indexes, we dropped the dependence on λ_j in the above notation $D^{(n)} f(t)$. For $n = 0$ we define $D^{(0)} f(t) = f(t)$. We shall also use the notation

$$D_\lambda f(t) := \frac{d}{dt} f(t) - \lambda f(t)$$

which should be distinguished from the notation $D^{(n)} f$.

The main result of this subsection is Theorem 16 providing a Taylor type expansion of a smooth function according to the fundamental functions $\Phi_n(x - x_0)$. As a preparation we need the following well-known result whose proof is included for convenience of the reader (cf. e.g. [37]).

THEOREM 12. *Let $\lambda_0, \lambda_1, \dots, \lambda_n$ be complex numbers and define $\Lambda_k = (\lambda_0, \dots, \lambda_k)$ for $k \in \{0, \dots, n\}$. Assume that $f : [x_0, x_0 + \gamma] \to C$ is C^{n+1} for some $\gamma > 0$. Then for any $m \le n$ and $x \in [x_0, x_0 + \gamma]$*

$$(29) \qquad f(x) = \sum_{k=0}^{m} D^{(k)} f(x_0) \Phi_{\Lambda_k}(x - x_0) + \int_{x_0}^{x} D^{(m+1)} f(t) \cdot \Phi_{\Lambda_m}(x - t) \, dt .$$

PROOF. We shall prove the statement by induction over m. For $m = 0$ this means that

$$f(x) = f(x_0) \Phi_{(\lambda_0)}(x - x_0) + \int_{x_0}^{x} \left(\frac{d}{dt} - \lambda_0 \right) f(t) \cdot \Phi_{(\lambda_0)}(x - t) \, dt .$$

Since $\Phi_{(\lambda_0)}(x) = e^{\lambda_0 x}$, this is equivalent to

$$f(x) - e^{\lambda_0(x - x_0)} f(x_0) = \int_{x_0}^{x} \left(\frac{d}{dt} - \lambda_0 \right) f(t) \cdot e^{\lambda_0(x - t)} dt = e^{\lambda_0 x} \int_{x_0}^{x} \frac{d}{dt} \left(e^{-\lambda_0 t} f(t) \right) dt ,$$

which is obviously true since

$$\frac{d}{dt} \left(e^{-\lambda_0 t} f(t) \right) = e^{-\lambda_0 t} \left(\frac{d}{dt} - \lambda_0 \right) f(t) .$$

Suppose now that the statement is true for $m < n$ and we want to prove it for $m + 1 \le n$. It suffices to prove that

$$A_m := \int_{x_0}^{x} D^{(m+1)} f(t) \cdot \Phi_{\Lambda_m}(x - t) \, dt$$

is equal to

$$B_m := D^{(m+1)} f(x_0) \Phi_{\Lambda_{m+1}}(x - x_0) + \int_{x_0}^{x} D^{(m+2)} f(t) \cdot \Phi_{\Lambda_{m+1}}(x - t) \, dt .$$

Using the recursion $\Phi'_{\Lambda_{m+1}}(t) = \Phi_{\Lambda_m}(t) + \lambda_{m+1}\Phi_{\Lambda_{m+1}}(t)$ in (17), one obtains

$$\frac{d}{dt}\left(\Phi_{\Lambda_{m+1}}(x-t)\right) = -\Phi'_{\Lambda_{m+1}}(x-t) = -\Phi_{\Lambda_m}(x-t) - \lambda_{m+1}\Phi_{\Lambda_{m+1}}(x-t)$$

and therefore $D_{-\lambda_{m+1}}\Phi_{\Lambda_{m+1}}(x-t) = -\Phi_{\Lambda_m}(x-t)$. Thus

$$A_m = -\int_{x_0}^{x} D^{(m+1)}f(t) \cdot D_{-\lambda_{m+1}}\left(\Phi_{\Lambda_{m+1}}(x-t)\right)dt.$$

Proposition 13 below applied to $g = \Phi_{\Lambda_{m+1}}(x-t)$, $f = D^{(m+1)}f(t)$ and $\lambda = \lambda_{m+1}$ gives

$$A_m = -D^{(m+1)}f(t)\,\Phi_{\Lambda_{m+1}}(x-t)\,|_{x_0}^{x} + \int_{x_0}^{x} D_{\lambda_{m+1}}D^{(m+1)}f(t) \cdot \Phi_{\Lambda_{m+1}}(x-t)\,dt,$$

and the result is proven since $D_{\lambda_{m+1}}D^{(m+1)}f(t) = D^{(m+2)}f(t)$. $\qquad\square$

PROPOSITION 13. *Let λ be a complex number and let $f, g : [a,b] \to \mathbf{C}$ be continuously differentiable. Then for any $x_0, x \in [a,b]$ with $x > x_0$, holds*

$$\int_{x_0}^{x} f(t) \cdot D_{-\lambda}g(t)\,dt = f(t) \cdot g(t)\,|_{x_0}^{x} - \int_{x_0}^{x} D_\lambda f(t) \cdot g(t)\,dt.$$

PROOF. Partial integration yields $\int_{x_0}^{x} fg'dt = f \cdot g\,|_{x_0}^{x} - \int_{x_0}^{x} f'g\,dt$. Then

$$\int_{x_0}^{x} f(t) \cdot D_{-\lambda}g(t)\,dt = \int_{x_0}^{x} f(t) \cdot \left(g'(t) + \lambda g\right)dt = f \cdot g\,|_{x_0}^{x} - \int_{x_0}^{x} f'g\,dt + \lambda\int_{x_0}^{x} f(t) \cdot g\,dt,$$

which gives the statement. $\qquad\square$

The next result gives a simple sufficient condition such that the "Taylor polynomial", defined by (31) below, converges to f. This criterion is based on estimates of derivatives $D^{(2n)}f(t)$ of even order motivated by the results in Section 2 for the Fourier-Laplace coefficients of a polyharmonic function of infinite order. It is also instructive to compare the result with the classical polynomial case (see e.g. [19] for a different approach).

THEOREM 14. *Let $\lambda_n, n \in \mathbf{N}_0$, be complex numbers such that $\overline{\lim}_{n\to\infty}|\lambda_n|/n \leq \beta$ for some $\beta \geq 0$. Assume that $f \in C^\infty[x_0, x_0+\gamma]$ with $\gamma > 0$ satisfies the following property: there exist constants $\sigma \geq 0$ and $C > 0$ such that*

$$(30) \qquad \left|D^{(2n)}f(t)\right| \leq C \cdot (2n)! \cdot \sigma^{2n}$$

for all $t \in [x_0, x_0+\gamma]$ and $n \in \mathbf{N}_0$. Then

$$(31) \qquad s_{2n-1}(x) := \sum_{k=0}^{2n-1} D^{(k)}f(x_0)\,\Phi_{\Lambda_k}(x-x_0)$$

converges uniformly to $f(x)$ on the interval $[x_0, x_0+\delta]$ for a suitable positive $\delta < \gamma$.

PROOF. Define $s_n = \sum_{k=0}^{n} D^{(k)}f(x_0)\,\Phi_{\Lambda_k}(x-x_0)$. Then $f(x) = s_n(x) + R_n(x)$ by Taylor's formula (29) where

$$(32) \qquad R_n(x) = \int_{x_0}^{x} D^{(n+1)}f(t) \cdot \Phi_{\Lambda_n}(x-t)\,dt.$$

For the convergence of $s_{2n-1}(x)$ to $f(x)$, it suffices to show that $R_{2n-1}(x) \to 0$ for $n \to \infty$. Note that the integration parameter t in (32) satisfies $x_0 \le t \le x$, so we have $x - t \ge 0$ and $0 \le x - t \le x - x_0$. We shall show uniform convergence $R_{2n-1}(x) \to 0$ for all $x \in [x_0, x_0 + \delta]$ where $\delta > 0$ will be specified later. Clearly we have $|x - t| \le x - x_0 \le x_0 + \delta - x_0 \le \delta$. By Theorem 7, for given $\varepsilon > 0$ there exists a natural number $\alpha > 0$ such that

$$\left| \Phi_{\Lambda_{2n-1}}(x - t) \right| \le \frac{e^{\alpha|x-t|}}{(2n-1)!} \left(\frac{e^{(1+\varepsilon)\beta|x-t|} - 1}{(1+\varepsilon)\beta} \right)^{2n-1} \le \frac{e^{\delta\alpha}}{(2n-1)!} \left(\frac{e^{(1+\varepsilon)\beta\delta} - 1}{(1+\varepsilon)\beta} \right)^{2n-1}$$

for all natural numbers n. This in connection with (30) leads to the estimate:

$$|R_{2n-1}(x)| \le C |x - x_0| (2n)! \sigma^{2n} \frac{e^{\delta\alpha}}{(2n-1)!} \left(\frac{e^{(1+\varepsilon)\beta\delta} - 1}{(1+\varepsilon)\beta} \right)^{2n-1}.$$

Now we make $\delta > 0$ so small such that $\sigma \left(e^{(1+\varepsilon)\beta\delta} - 1 \right) / (1+\varepsilon)\beta < 1$. Then $R_{2n-1}(x)$ converges uniformly on $[x_0, x_0 + \delta]$ to zero.   □

PROPOSITION 15. *Let $\lambda_n, n \in N_0$, be real numbers such that $\overline{\lim}_{n\to\infty} |\lambda_n| / n \le \beta$ for some $\beta > 0$. Let $f \in C^\infty [x_0, x_0 + \gamma]$ with $\gamma > 0$ and assume that there exist constants $C > 0$ and $\sigma > 0$ such that*

$$(33) \qquad \left| D^{(2n)} f(t) \right| \le C \cdot (2n)! \sigma^{2n} \text{ for all } t \in [x_0, x_0 + \gamma].$$

Then for every $\varepsilon > 0$ there exist constants $C_2 > 0$ and $\delta > 0$ such that

$$(34) \qquad \left| D^{(2n+1)} f(t) \right| \le C_2 (2n+1)! (\sigma + \varepsilon)^{2n+1}$$

for all $t \in [x_0, x_0 + \delta]$ and for all natural numbers n.

PROOF. Let $\varepsilon_0 > 0$. Then there exists $\alpha > 0$ such that

$$(35) \qquad |\lambda_n| \le \alpha + \beta (1 + \varepsilon_0) n \text{ for all } n \in N_0.$$

Let $\gamma > 0$ and $\varepsilon > 0$ as in the proposition. Clearly we can find $\delta > 0$ small enough so that $2\delta < \gamma$, and

$$(36) \qquad e^{2\beta(1+\varepsilon_0)\delta} \sigma < \sigma + \varepsilon.$$

The assumption (33) implies the estimate

$$(37) \qquad \left| D^{(2n)} f(t) \right| + \left| D^{(2n+2)} f(s) \right| \le C (2n+2)! \sigma^{2n} (1 + \sigma^2)$$

for all $s, t \in [x_0, x_0 + 2\delta]$. Theorem 32 in the appendix provides the estimate

$$\left| D^{(2n+1)} f(x) \right| \le 2 \max \left\{ \frac{2}{\delta}, \delta \right\} e^{(|\lambda_{2n}| + |\lambda_{2n+1}|)\delta}$$

$$\times \left(\max_{t \in [x_0, x_0 + 2\delta]} \left| D^{(2n)} f(t) \right| + \max_{t \in [x_0, x_0 + 2\delta]} \left| D^{(2n+2)} f(t) \right| \right)$$

for all $x \in [x_0, x_0 + \delta]$. Now (35) and (37) imply that

$$\left| D^{(2n+1)} f(x) \right| \le 2 \max \left\{ \frac{2}{\delta}, \delta \right\} C e^{2\alpha\delta + \beta(1+\varepsilon_0)\delta + 4\beta(1+\varepsilon_0)n\delta} (2n+2)! \sigma^{2n} (1 + \sigma^2)$$

for all $x \in [x_0, x_0 + \delta]$ and for all $n \in N_0$. The statement is now obvious since (36) implies that

$$(2n + 2)e^{4\beta(1+\varepsilon_0)n\delta}\sigma^{2n} \leq A (\sigma + \varepsilon)^{2n}$$

for a suitable constant A and for all $n \in N_0$. □

The next theorem is the main result of this subsection:

THEOREM 16. *Let $\lambda_n, n \in N_0$, be real numbers with the property that $\overline{\lim}_{n\to\infty} |\lambda_n| / n \leq \beta$ for some $\beta > 0$. Let $f \in C^\infty[x_0, x_0 + \gamma]$ with $\gamma > 0$ and assume that there exist constants $C > 0$ and $\sigma > 0$ such that*

$$\left| D^{(2n)} f (t) \right| \leq C \cdot (2n)! \sigma^{2n}$$

for all $t \in [x_0, x_0 + \gamma]$ and $n \in N_0$. Then the series

$$\sum_{n=0}^{\infty} D^{(n)} f (x_0) \, \Phi_{\Lambda_n} (z - x_0)$$

defines an analytic extension of f and it converges compactly and absolutely in the disc in C with center x_0 and radius

$$\frac{1}{\beta} \ln \left(1 + \frac{\beta}{\sigma} \right).$$

PROOF. By Proposition 15, for each $\varepsilon > 0$ the estimate $\left| D^{(n)} f (x_0) \right| \leq C_2 n! (\sigma + \varepsilon)^n$ holds for all natural numbers n. Thus

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{\left| D^{(n)} f (x_0) \right|}{n!}} \leq \sigma + \varepsilon .$$

Now let ε go to 0. By Theorem 8, $\sum_{n=0}^{\infty} D^{(n)} f (x_0) \, \Phi_{\Lambda_n} (z - x_0)$ converges for all z as stated in the theorem. By Theorem 14, the series represents the function f. □

4. Analytic extensions of Fourier-Laplace coefficients. Let us recall that $f_{k,l} (r)$ is the Fourier-Laplace coefficient of the function $f \in C (A (r_0, r_1))$ defined for all values $r \in (r_0, r_1)$, cf. formula (4). Using the transformation $r = e^v$ with $v \in (\log r_0, \log r_1)$ we can define a function

$$\widetilde{f}_{k,l} (v) := f_{k,l}(e^v).$$

Let us look at a simple example:

EXAMPLE 17. Let $f (x) = \log |x|$ be defined on the annular region $\mathbf{R}^d \setminus \{0\}$. Recalling that $Y_{0,1} (\theta) = 1/\sqrt{\omega_{d-1}}$, the Fourier-Laplace coefficient $f_{0,1}$ defined in (4) satisfies $f_{0,1} (r) = \sqrt{\omega_{d-1}} \log r$. Thus $f_{0,1}$ has an analytic extension to the slit plane $C \setminus (-\infty, 0]$ and $\widetilde{f}_{0,1} (v) = \sqrt{\omega_{d-1}} \log e^v = \sqrt{\omega_{d-1}} v$ is defined for every complex number $v \in C$.

The next observation is very useful: the differential operator L_k^p defined in (9) in Section 2 can be transformed to a linear differential operator with *constant coefficients* in the variable v for $r = e^v$. We cite the following theorem ([22, Theorem 10.34]):

THEOREM 18. *Let $0 \leq r_0 < r_1 \leq \infty$ and let $g : (r_0, r_1) \to C$ be a C^∞-function.*
Define

$$\widetilde{g} : (\log r_0, \log r_1) \to C, \quad \widetilde{g}(v) := g(e^v).$$

Then

$$\left[L_k^p(g) \right](e^v) = e^{-2pv} \left[M_{k,p}(\widetilde{g}) \right](v)$$

for any $v \in (\log r_0, \log r_1)$, where

$$(38) \qquad M_{k,p} = \prod_{j=0}^{p-1} \left(\frac{d}{dv} - (k+2j) \right) \prod_{j=0}^{p-1} \left(\frac{d}{dv} - (-k-d+2+2j) \right).$$

For given $k \in N_0$ and dimension d, let us define the exponents

$$(39) \qquad \lambda_{2j}(k,d) = k + 2j \text{ and } \lambda_{2j+1}(k,d) = -k - d + 2 + 2j \text{ for } j \in N_0.$$

For notational simplicity we will often suppress the dependence on k and d and we simply
write λ_n with $n \in N_0$. In accordance with the notations in Section 3, we shall define

$$\Phi_n(v) := \Phi_{(\lambda_0,\ldots,\lambda_n)}(v),$$
$$D^{(n)}g(v) := \left(\frac{d}{dv} - \lambda_0 \right) \cdots \left(\frac{d}{dv} - \lambda_{n-1} \right) g(v).$$

Now we will prove the following result:

THEOREM 19. *Let $f : A(r_0, r_1) \to C$ be polyharmonic of infinite order and type
$\tau \geq 0$ and define $\widetilde{f}_{k,l}(v) := f_{k,l}(e^v)$ for $v \in (\log r_0, \log r_1)$. Then, given $v_0 \in (\log r_0, \log r_1)$
and $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$(40) \qquad \left| D^{(n)} \widetilde{f}_{k,l}(v_0) \right| \leq C \cdot n! \cdot \left[e^{v_0}(\tau + \varepsilon) \right]^n$$

for all $n \in N_0$ and for all $k \in N_0, l = 1, \ldots, a_k$.

PROOF. Clearly, $\lambda_n, n \in N_0$, defined as above, are real numbers with the property that
$\overline{\lim}_{n\to\infty} |\lambda_n|/n = 1$. Let $v \in (\log r_0, \log r_1)$ and $\varepsilon > 0$. We want to apply Proposition 15 for
$\beta = 1$ and to the function $\widetilde{f}_{k,l}$ and $\sigma = e^{v_0}(\tau + \varepsilon)$. Thus we want to show that the following
estimate holds: There exists $\gamma > 0$ and $C > 0$ (independent of k, l) such that

$$\left| D^{(2p)} \widetilde{f}_{k,l}(v) \right| \leq C \cdot (2p)! \left[e^{v_0}(\tau + \varepsilon) \right]^{2p}$$

for all $v \in [v_0, v_0 + \gamma]$ and $p \in N_0$. Theorem 18 shows that

$$(41) \qquad D^{(2p)} \widetilde{f}_{k,l}(v) = M_{k,p}(\widetilde{f}_{k,l})(v) = e^{2pv} L_k^p(f_{k,l})(e^v)$$

for all $v \in (\log r_0, \log r_1)$. Let us take $\gamma > 0$ and $\varepsilon_1 > 0$ small enough so that $e^{v_0+\gamma}(\tau + \varepsilon_1) < e^{v_0}(\tau + \varepsilon)$ and $v_0 + \gamma < \log r_1$. Theorem 1 shows that, for $\varepsilon_1 > 0$ and for the
subinterval $[v_0, v_0 + \gamma] \subset (\log r_0, \log r_1)$, there is positive number C such that

$$\left| L_k^p(f_{k,l})(e^v) \right| \leq C(2p)!(\tau + \varepsilon_1)^{2p}$$

for all $v \in [v_0, v_0 + \gamma]$ and all $p \in N_0, k \in N_0, l = 1, \ldots, a_k$. Since $e^v \leq e^{v_0 + \gamma}$, we obtain from (41) the estimate

$$\left| D^{(2p)} \widetilde{f}_{k,l} (v) \right| \leq e^{2pv} \left| L_k^p (f_{k,l}) (e^v) \right| \leq C e^{2p(v_0 + \gamma)} (2p)! (\tau + \varepsilon_1)^{2p}$$
$$\leq C (2p)! \left[e^{v_0} (\tau + \varepsilon) \right]^{2p} .$$

Thus the assumptions of Proposition 15 are satisfied and the theorem is proven. □

Here is the main result about the analytical extension in the present section.

THEOREM 20. *Let $f : A (r_0, r_1) \to C$ be polyharmonic of infinite order and type $\tau \geq 0$ and define $\widetilde{f}_{k,l} (v) := f_{k,l} (e^v)$ for $v \in (\log r_0, \log r_1)$. Then, given $v_0 \in (\log r_0, \log r_1)$, the series*

$$\sum_{n=0}^{\infty} D^{(n)} \widetilde{f}_{k,l} (v_0) \cdot \Phi_n (v - v_0)$$

defines an analytic extension of $\widetilde{f}_{k,l}$, and it converges compactly and absolutely in the disk with center v_0 and radius

$$\ln \left(1 + \frac{1}{e^{v_0} \cdot \tau} \right) .$$

If f is polyharmonic of infinite order and type 0, then $\widetilde{f}_{k,l} (v)$ is an entire function and the Fourier-Laplace coefficient $f_{k,l}$ possesses an analytic extension to the slit plane $C_- := C \setminus (-\infty, 0]$.

PROOF. Theorems 19 and 16 show the first statement. If f is polyharmonic of infinite order and type 0, the convergence radius is infinite and $\widetilde{f}_{k,l}$ is entire. Now define $g (z) = \widetilde{f}_{k,l} (\log z)$ for all z in the slit plane C_-. Then for $r \in (r_0, r_1)$ we have

$$g (r) = \widetilde{f}_{k,l} (\log r) = f_{k,l} (e^{\log r}) = f_{k,l} (r) .$$

□

5. Analytic extensions of Fourier-Laplace coefficients for odd dimension. Assume that the dimension d of the underlying euclidean space is odd. Then for any fixed $k \in N_0$ the exponents

(42) $$\lambda_{2j} (k) := k + 2j \quad \text{and} \quad \lambda_{2j+1} (k) := -k - d + 2 + 2j$$

defined in (39) are pairwise different and $|\lambda_m (k) - \lambda_n (k)| \geq 1$ for all $m \neq n$. Since $\lambda_n (k)$ are pairwise different, the defining equality (16) for the fundamental function Φ_n implies that

(43) $$\Phi_n (v) = \sum_{j=0}^{n} \frac{e^{\lambda_j (k) v}}{q_n' (\lambda_j (k))} \quad \text{where} \quad q_n' (\lambda_j (k)) = \prod_{\substack{s=0 \\ s \neq j}}^{n} (\lambda_j (k) - \lambda_s (k)) .$$

Here the polynomial

$$q_n (\lambda) = \prod_{j=0}^{n} (\lambda - \lambda_j)$$

is the symbol of the linear differential operator L defined in (13) for which the notation $L(\lambda)$ would be more traditional.

If f is polyharmonic of infinite order and type τ and $v_0 \in (\log r_0, \log r_1)$ then, according to Theorem 20, the series

$$f_{k,l}\left(e^v\right) = \widetilde{f}_{k,l}(v) = \sum_{n=0}^{\infty} D^{(n)} \widetilde{f}_{k,l}(v_0) \Phi_n(v - v_0)$$

converges for v in a neighborhood of v_0. It follows that

$$f_{k,l}\left(e^v\right) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} D^{(n)} \widetilde{f}_{k,l}(v_0) \frac{e^{\lambda_j(k)\cdot(v-v_0)}}{q_n'\left(\lambda_j(k)\right)}.$$

Substituting $e^v = r$ back we arrive at

$$f_{k,l}(r) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} D^{(n)} \widetilde{f}_{k,l}(v_0) \frac{e^{-\lambda_j(k)v_0}}{q_n'\left(\lambda_j(k)\right)} r^{\lambda_j(k)}.$$

In the following, we want to prove that this double series converges compactly and absolutely, even for complex values r, in the punctured plane C^* provided that f is polyharmonic of infinite order and type 0.

First we need an estimate for $\left|q_n'\left(\lambda_j\right)\right|$:

PROPOSITION 21.   *Let $\lambda_0, \ldots, \lambda_n$ be real numbers such that $|\lambda_s - \lambda_t| \geq \alpha > 0$ for all $s, t \in \{0, \ldots, n\}, s \neq t$. Then for $q_n(z) = (z - \lambda_0) \cdots (z - \lambda_n)$ we have*

$$\left|q_n'\left(\lambda_j\right)\right| = \lim_{z \to \lambda_j} \left|\frac{q_n(z)}{z - \lambda_j}\right| \geq \frac{\alpha^n n!}{2^n} \quad \text{for all}\ \ j = 0, \ldots, n.$$

PROOF.   We may assume that $\lambda_0 < \cdots < \lambda_n$. Then

$$\lim_{z \to \lambda_j} \frac{q_n(z)}{(z - \lambda_j)} = \left(\lambda_j - \lambda_0\right) \cdots \left(\lambda_j - \lambda_{j-1}\right) \left(\lambda_j - \lambda_{j+1}\right) \cdots \left(\lambda_j - \lambda_n\right).$$

Using $\lambda_0 < \cdots < \lambda_n$ we obtain an estimate for $\lambda_{k+l} - \lambda_k$ as

$$\lambda_{k+l} - \lambda_k = \lambda_{k+l} - \lambda_{k+l-1} + \lambda_{k+l-1} - \lambda_{k+l-2} + \lambda_{k+l+2} - \cdots + \lambda_{k+1} - \lambda_k \geq l \cdot \alpha.$$

Finally we obtain

$$\lim_{z \to \lambda_j} \left|\frac{q_n(z)}{(z - \lambda_j)}\right| \geq \alpha^n j!\, (n - j)! = \frac{\alpha^n n!}{\binom{n}{j}} \geq \alpha^n n! \frac{1}{2^n}.$$

$\square$

The next result strengthens Theorem 20 for odd dimension $d > 1$. For example, in the case that f is polyharmonic of infinite order and type 0, it follows that the Fourier-Laplace coefficients possess an analytic extension to the punctured plane C^* instead of the slit plane $C \setminus (-\infty, 0]$. In Theorem 23 below we shall give an explicit representation of the Fourier-Laplace coefficients giving a proof of formula (7) mentioned in the introduction.

THEOREM 22. *Let $d > 1$ be odd and $\lambda_j(k)$ as in (42). Let $f : A(r_0, r_1) \to \mathbf{C}$ be polyharmonic of infinite order and type $\tau < 1/2r_0$. Then for any v_0 with $r_0 < e^{v_0} < \min\{r_1, 1/2\tau\}$, the series*

$$(44) \qquad F_{k,l}(z) := \sum_{n=0}^{\infty} \sum_{j=0}^{n} D^{(n)} \widetilde{f}_{k,l}(v_0) \frac{e^{-\lambda_j(k)v_0}}{q_n'(\lambda_j(k))} z^{\lambda_j(k)}$$

converges compactly and absolutely in the annulus $\{z \in \mathbf{C}; 0 < |z| < 1/2\tau\}$ and $F_{k,l}(r) = f_{k,l}(r)$ for all $r \in (r_0, \min\{r_1, 1/2\tau\})$.

PROOF. Let K be a compact subset of $\{z \in \mathbf{C}; 0 < |z| < 1/2\tau\}$. Then there exists $\rho \in (0, 1/2\tau)$ with $K \subset \{z \in \mathbf{C}; 0 < |z| \le \rho\}$. Let v_0 satisfy $r_0 < e^{v_0} < \min\{r_1, 1/2\tau\}$. If necessary, we can make ρ larger such that

$$(45) \qquad e^{v_0} < \rho < 1/2\tau.$$

Then $2\rho\tau < 1$ and therefore there exists $\varepsilon > 0$ such that $2\rho(\tau + \varepsilon) < 1$. Theorem 19 provides the estimate

$$\left| D^{(n)} \widetilde{f}_{k,l}(v_0) \right| \le Cn! e^{nv_0} (\tau + \varepsilon)^n.$$

Since $1/\left| q_n'(\lambda_j) \right| \le 2^n/n!$, we obtain

$$A(z) := \sum_{n=0}^{\infty} \sum_{j=0}^{n} \left| D^{(n)} \widetilde{f}_{k,l}(v_0) \frac{e^{-\lambda_j(k)v_0}}{q_n'(\lambda_j(k))} z^{\lambda_j(k)} \right| \le \sum_{n=0}^{\infty} C e^{nv_0} 2^n (\tau + \varepsilon)^n \sum_{j=0}^{n} \left| z e^{-v_0} \right|^{\lambda_j(k)}.$$

Moreover

$$\sum_{j=0}^{n} \left| z e^{-v_0} \right|^{\lambda_j(k)} \le \left(\left| z e^{-v_0} \right|^k + \left| z e^{-v_0} \right|^{-k-d+2} \right) \sum_{j=0}^{[n/2]} \left| z e^{-v_0} \right|^{2j}.$$

Clearly $\left| z e^{-v_0} \right|^{2j} \le \left(\rho e^{-v_0} \right)^{2j}$ for $|z| \le \rho$ and for $j = 0, \ldots, [n/2]$. Since $\rho e^{-v_0} > 1$ by (45), we estimate $\left(\rho e^{-v_0} \right)^{2j} \le \left(\rho e^{-v_0} \right)^n$ and we obtain

$$A(z) \le \left(\left| z e^{-v_0} \right|^k + \left| z e^{-v_0} \right|^{-k-d+2} \right) \sum_{n=0}^{\infty} C(n+1)(2(\tau + \varepsilon)\rho)^n$$

for all $z \in K$. This series converges since $2\rho(\tau + \varepsilon) < 1$. $\qquad \square$

THEOREM 23. *Let $d > 1$ be odd and $\lambda_j(k)$ as in (42). Let $f : A(r_0, r_1) \to \mathbf{C}$ be polyharmonic of infinite order and type $\tau < 1/2r_0$. Then for each $k \in \mathbf{N}_0, l = 1, \ldots, a_k$, there exist complex numbers $a_{k,l,j}$ with $j \in \mathbf{N}_0$ such that*

$$(46) \qquad f_{k,l}(z) = z^k \sum_{j=0}^{\infty} a_{k,l,2j} z^{2j} + z^{-k-d+2} \sum_{j=0}^{\infty} a_{k,l,2j+1} z^{2j}$$

converges compactly and absolutely in the annulus $\{z \in \mathbf{C}; 0 < |z| < 1/2\tau\}$. *The power series*

$$f_{k,l}^{(1)}(z) := \sum_{j=0}^{\infty} a_{k,l,2j} z^{2j} \quad and \quad f_{k,l}^{(2)}(z) := \sum_{j=0}^{\infty} a_{k,l,2j+1} z^{2j}$$

have convergence radius at least $1/2\tau$.

PROOF. 1. First we define the coefficients $a_{k,l,j}$. Since $\tau < 1/2r_0$, there exists $v_0 \in (\log r_0, \log(1/2\tau))$ and we can assume that $e^{v_0} < r_1$. Then $e^{v_0} < 1/2\tau$ and we can find $\varepsilon > 0$ such that $e^{v_0}(\tau + \varepsilon) < 1/2$. We put

$$(47) \qquad a_{k,l,j} := e^{-\lambda_j(k)v_0} \sum_{n=j}^{\infty} \frac{D^{(n)} \widetilde{f}_{k,l}(v_0)}{q_n'(\lambda_j(k))}.$$

Using the estimate (40) in Theorem 19, we see that

$$\sum_{n=j}^{\infty} \left| \frac{D^{(n)} \widetilde{f}_{k,l}(v_0)}{q_n'(\lambda_j(k))} \right| \leq C \sum_{n=j}^{\infty} \left[e^{v_0}(\tau + \varepsilon) \right]^n \frac{n!}{\left| q_n'(\lambda_j(k)) \right|},$$

and the last series is converging using the ratio test for $b_n := n! / \left| q_n'(\lambda_j(k)) \right|$

$$\frac{b_{n+1}}{b_n} = \frac{n+1}{\left| \lambda_{n+1}(k) - \lambda_j(k) \right|} \to 1$$

and the fact that $e^{v_0}(\tau + \varepsilon) < 1/2$. So far we have proven that the coefficients $a_{k,l,j}$ are well defined.

2. Using (47) and the fact that $1/\left| q_n'(\lambda_j(k)) \right| \leq 2^n/n!$, we obtain

$$(48) \qquad \left| a_{k,l,j} \right| \leq C e^{-\lambda_j(k)v_0} \sum_{n=j}^{\infty} \left[2e^{v_0}(\tau + \varepsilon) \right]^n = C e^{-\lambda_j(k)v_0} \frac{[2e^{v_0}(\tau + \varepsilon)]^j}{1 - 2e^{v_0}(\tau + \varepsilon)}.$$

Using the definition of $\lambda_{2j}(k)$ and $\lambda_{2j+1}(k)$, we obtain the estimate

$$(49) \qquad \left| a_{k,l,2j} \right| \leq C e^{-kv_0} \frac{[2(\tau + \varepsilon)]^{2j}}{1 - 2e^{v_0}(\tau + \varepsilon)},$$

$$(50) \qquad \left| a_{k,l,2j+1} \right| \leq C e^{(k+d)v_0} \frac{[2(\tau + \varepsilon)]^{2j+1}}{1 - 2e^{v_0}(\tau + \varepsilon)}.$$

It follows that $\overline{\lim}_{j \to \infty} \sqrt[2j]{|a_{k,l,2j}|} \leq 2(\tau + \varepsilon)$ and $\overline{\lim}_{j \to \infty} \sqrt[2j]{|a_{k,l,2j+1}|} \leq 2(\tau + \varepsilon)$ for any $\varepsilon > 0$, from which we conclude that the power series $f_{k,l}^{(1)}$ and $f_{k,l}^{(2)}$ have convergence radius at least $1/2\tau$.

3. By Theorem 22 the series

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} D^{(n)} \widetilde{f}_{k,l}(v_0) \frac{e^{-\lambda_j(k)v_0}}{q_n'(\lambda_j(k))} z^{\lambda_j(k)}$$

converges compactly on each compact subset K of $\{z \in \boldsymbol{C}; 0 < |z| < 1/2\tau\}$. So we may rearrange the series and the series

$$\sum_{j=0}^{\infty} z^{\lambda_j(k)} e^{-\lambda_j(k)v_0} \sum_{n=j}^{\infty} \frac{D^{(n)} \widetilde{f}_{k,l}(v_0)}{q_n'(\lambda_j(k))} = \sum_{j=0}^{\infty} a_{k,l,j} z^{\lambda_j(k)}$$

converges compactly in $\{z \in \boldsymbol{C}; 0 < |z| < 1/2\tau\}$. The decomposition (46) follows by splitting the sum over odd and even indices. The proof is complete.   □

REMARK 24.   The coefficients $a_{k,l,j}$ do not depend on the special value v_0 since the coefficients in (46) are unique. The coefficients $a_{k,l,j}$ in (47) are well defined provided that f is polyharmonic of infinite order and type $< 1/r_0$. However, for the estimate (48) we needed that the type τ is smaller than $1/2r_0$.

6.   Analytic extensions of polyharmonic functions of infinite order for odd dimension.   We recall some notations and basic facts. We have defined $q(z) := z_1^2 + \cdots + z_d^2$ for $z = (z_1, \ldots, z_d) \in \boldsymbol{C}^d$ and clearly the following inequality holds for all $z \in \boldsymbol{C}^d$:

$$|q(z)| \leq |z_1|^2 + \cdots + |z_d|^2 =: |z|_{\boldsymbol{C}^d}^2 .$$

Note that $q(z)$ is the analytic extension of $|x|^2 = x_1^2 + \cdots + x_d^2$. The *Lie norm* $L_+(z) \in [0, \infty)$ is defined by the equation

$$L_+(z)^2 = |z|_{\boldsymbol{C}^d}^2 + \sqrt{|z|_{\boldsymbol{C}^d}^4 - |q(z)|^2} \quad \text{for } z \in \boldsymbol{C}^d$$

(see e.g. [2], [4], [31], [36]). Note that $|z|_{\boldsymbol{C}^d} \leq L_+(z)$ for all $z \in \boldsymbol{C}^d$. In [25] the following estimate is established (see also [14]):

$$(51) \qquad \sum_{l=1}^{a_k} |Y_{k,l}(z)|^2 \leq \frac{a_k}{\omega_{d-1}} \left(|z|^2 + \sqrt{|z|^4 - |q(z)|^2} \right)^k = \frac{a_k}{\omega_{d-1}} (L_+(z))^{2k}$$

for all $z \in \boldsymbol{C}^d$. Using the Cauchy Schwarz inequality, one obtains

$$(52) \qquad \sum_{l=1}^{a_k} |Y_{k,l}(z)| \leq \sqrt{a_k} \sqrt{\sum_{l=1}^{a_k} |Y_{k,l}(z)|^2} \leq \frac{a_k}{\sqrt{\omega_{d-1}}} (L_+(z))^k .$$

Now we define

$$L_-(z) := \sqrt{|z|_{\boldsymbol{C}^d}^2 - \sqrt{|z|_{\boldsymbol{C}^d}^4 - |q(z)|^2}}$$

for $z \in \boldsymbol{C}^d$. Then $0 \leq L_-(z) \leq L_+(z)$ and it is easy to see that

$$L_+(z) L_-(z) = |q(z)| \text{ for all } z \in \boldsymbol{C}^d.$$

In analogy to the Lie ball we define the *Lie annulus* as the set

$$\widetilde{A}(r_0, r_1) := \left\{ z \in \boldsymbol{C}^d; r_0 < L_-(z) \text{ and } L_+(z) < r_1 \right\}.$$

In [4, p. 95] it is shown that $\widetilde{A}(r_0, r_1)$ is the harmonicity hull of the annular domain $A(r_0, r_1)$. It can be shown that $\widetilde{A}(r_0, r_1)$ is connected. On the other hand, the complement of $\widetilde{A}(r_0, r_1)$

in C^d is connected as well, in contrast to the fact that the complement of the annular region $A(r_0, r_1)$ in R^d consists of two connected components.

It is known that a polyharmonic function of infinite order and type 0 can be extended to a *multi-valued analytic* function on the harmonicity hull (see [2]) and to a *single-valued analytic* function on $\ker(\widetilde{A}(r_0, r_1))$, the *kernel* (in [4, p. 131] *noyau*) of the harmonicity hull (see [4, p. 135] for details) which is clearly contained in the set $\widetilde{A}(r_0, r_1) \setminus q^{-1}((-\infty, 0])$.

We present now our main result about analytical extendibility of polyharmonic functions of infinite order and type $\tau < 1/2r_1$ on the annular region $A(r_0, r_1)$.

THEOREM 25.   *Let $d > 1$ be odd and let $f : A(r_0, r_1) \to C$ be polyharmonic of infinite order and type $\tau < 1/2r_1$. Then there exist analytic functions*

$$f_1 : \{z \in C^d; L_+(z) < r_1\} \to C,$$
$$f_2 : \{z \in C^d; r_0 < L_-(z) \le L_+(z) < 1/2\tau\} \to C$$

such that

$$F(z) = f_1(z) + (z_1^2 + \cdots + z_d^2)^{(2-d)/2} f_2(z)$$

is an analytic extension of f. Here F is defined for all $z \in \widetilde{A}(r_0, r_1) \setminus q^{-1}((-\infty, 0])$.

PROOF.   1.   Let us recall that $f(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l}(r) Y_{k,l}(\theta)$ for $x = r\theta$, and let $\lambda_j(k)$ be as in (42). By Theorem 23, each Fourier-Laplace coefficient $f_{k,l}(r)$ can be expanded in a series of type $\sum_{j=0}^{\infty} a_{k,l,j} r^{\lambda_j(k)}$, and hence

$$(53) \qquad f(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \sum_{j=0}^{\infty} a_{k,l,j} r^{\lambda_j(k)} Y_{k,l}(\theta) .$$

Moreover $Y_{k,l}(x) = r^k Y_{k,l}(\theta)$ for $x = r\theta$. We consider even and odd indices j in (53) and define two functions f_1 and f_2 such that $f(x) = f_1(x) + r^{2-d} f_2(x)$, where

$$f_1(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \sum_{j=0}^{\infty} a_{k,l,2j} r^{2j} Y_{k,l}(x) ,$$
$$f_2(x) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \sum_{j=0}^{\infty} a_{k,l,2j+1} r^{-2k+2j} Y_{k,l}(x) .$$

We shall show that $f_1(x)$ can be analytically extended for all z with $L_+(z) < r_1$ and that $f_2(x)$ can be analytically extended for all z with $r_0 < L_-(z)$ and $L_+(z) < 1/2\tau$.

2.   The function $r^2 = x_1^2 + \cdots + x_d^2$ has the analytic extension $q(z) = z_1^2 + \cdots + z_d^2$ for $z = (z_1, \ldots, z_d) \in C^d$. The polynomial $Y_{k,l}(x)$ has the analytic extension $Y_{k,l}(z)$.

Next we show that

$$F_1(z) := \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \sum_{j=0}^{\infty} a_{k,l,2j} q(z)^j Y_{k,l}(z)$$

converges absolutely for all z with $L_+(z) \le \rho$ for any $0 < \rho < r_1$. Since $\rho < r_1$ and $\tau < 1/2r_1$, we can find $v_0 \in (\log r_0, \log r_1)$ such that $\rho < e^{v_0} < 1/2\tau$. Choose $\varepsilon > 0$ such

that $2e^{v_0}(\tau + \varepsilon) < 1$. We use now (49) and the estimate $|q(z)| \leq |z|^2_{\mathbb{C}^d} \leq L^2_+(z) \leq \rho^2$ and we obtain

$$|F_1(z)| \leq C \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \sum_{j=0}^{\infty} |Y_{k,l}(z)| e^{-kv_0} \frac{[2(\tau + \varepsilon)]^{2j}}{1 - 2e^{v_0}(\tau + \varepsilon)} \rho^{2j}.$$

Since $\rho e^{-v_0} < 1$ and $2e^{v_0}(\tau + \varepsilon) < 1$, the series $\sum_{j=0}^{\infty} [2(\tau + \varepsilon)\rho]^{2j}$ converges and there exists a constant C_1 such that

$$|F_1(z)| \leq C_1 \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} |Y_{k,l}(z)| e^{-kv_0} \leq C_1 \sum_{k=0}^{\infty} e^{-kv_0} \frac{a_k}{\sqrt{\omega_{d-1}}} (L_+(z))^k$$

where we have used (52). Since $L_+(z) \leq \rho$ and $\rho e^{-v_0} < 1$, we see that the last sum converges.

3. It remains to show that

$$F_2(z) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \sum_{j=0}^{\infty} a_{k,l,2j+1} q(z)^{-k+j} Y_{k,l}(z)$$

converges compactly in $\{r_0 < L_-(z)$ and $L_+(z) < 1/2\tau\}$. Let ρ_0 and ρ_1 be positive numbers such that $r_0 < \rho_0$ and $\rho_1 < 1/2\tau$, and assume that $L_-(z) \geq \rho_0$ and $L_+(z) \leq \rho_1$. Choose v_0 such that $r_0 < e^{v_0} < \rho_0$, so $e^{v_0}/\rho_0 < 1$. Moreover we can assume that $e^{v_0} < 1/2\tau$ in view of our general assumption $r_0 < 1/2\tau$. Then there exists $\varepsilon > 0$ such that

(54) $$2e^{v_0}(\tau + \varepsilon) < 1 \quad \text{and} \quad 2(\tau + \varepsilon)\rho_1 < 1.$$

The estimate (50) gives

$$|F_2(z)| \leq C \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} \sum_{j=0}^{\infty} e^{(k+d)v_0} \frac{[2(\tau + \varepsilon)]^{2j+1}}{1 - 2e^{v_0}(\tau + \varepsilon)} |q(z)|^{-k+j} |Y_{k,l}(z)|.$$

Since $|q(z)| \leq L^2_+(z) \leq \rho_1^2$, we estimate

$$|F_2(z)| \leq C \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} e^{(k+d)v_0} |q(z)|^{-k} |Y_{k,l}(z)| \sum_{j=0}^{\infty} \frac{1}{\rho_1} \frac{[2(\tau + \varepsilon)\rho_1]^{2j+1}}{1 - 2e^{v_0}(\tau + \varepsilon)}.$$

The last series converges since $2(\tau + \varepsilon)\rho_1 < 1$, and is bounded by a constant, say C_1. Further (52) implies

$$|F_2(z)| \leq CC_1 \sum_{k=0}^{\infty} e^{(k+d)v_0} \frac{a_k}{\sqrt{\omega_{d-1}}} \frac{(L_+(z))^k}{|q(z)|^k}.$$

Recall that $L_+(z) L_-(z) = |q(z)|$, so we can estimate $L_+(z) / |q(z)| = 1/L_-(z) \leq 1/\rho_0$. Hence

$$|F_2(z)| \leq CC_1 \sum_{k=0}^{\infty} e^{dv_0} \frac{a_k}{\sqrt{\omega_{d-1}}} \left(\frac{e^{v_0}}{\rho_0} \right)^k$$

and this series converges since $e^{v_0}/\rho_0 < 1$. The proof is complete. $\qquad \square$

Let us illustrate the theorem for the case of a harmonic function f on the annular region $A(r_0, r_1)$. Then f is of type 0 and the conclusion is that we can decompose f as a sum $f_1(x) + |x|^{2-d} f_2(x)$, where $f_1(z)$ is analytic on the Lie ball and $f_2(z)$ is analytic for all z with $L_-(z) > r_0$.

7. Analytic extensions of polyharmonic functions of infinite order for even dimension.

In this section we assume that the dimension d is an even number. Let $k \in N_0$ be fixed. Then the exponents $\lambda_{2j}(k) = k + 2j$ and $\lambda_{2j+1}(k) = -k - d + 2 + 2j$ may be equal and the description of the exponential space $E_{(\lambda_0, \ldots, \lambda_n)}$ defined in (14) is different from the odd case. Clearly $e_{k,2j}(v) := e^{\lambda_{2j}(k)v}$ are solutions and for odd indices $2j + 1$ we obtain the solutions

$$e_{k,2j+1}(v) := e^{\lambda_{2j+1}(k)v} \text{ if } \lambda_{2j+1}(k) \notin \{k + 2l; l \in N_0\},$$

$$e_{k,2j+1}(v) := v \cdot e^{\lambda_{2j+1}(k)v} \text{ if } \lambda_{2j+1}(k) \in \{k + 2l; l \in N_0\}.$$

Then, for suitable coefficients d_j, the fundamental function $\Phi_n(v)$ is an exponential polynomial of the form

$$(55) \qquad \Phi_n(v) = \sum_{j=0}^{n} d_j e_{k,j}(v) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{e^{vz}}{(z - \lambda_0) \cdots (z - \lambda_n)} dz.$$

If f is polyharmonic of infinite order and type τ and $v_0 \in (\log r_0, \log r_1)$ then, using Theorem 20, the series

$$(56) \qquad f_{k,l}(e^v) = \widetilde{f}_{k,l}(v) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} D^{(n)} \widetilde{f}_{k,l}(v_0) \cdot d_j \cdot e_{k,j}(v - v_0)$$

converges for v in a neighborhood of v_0. Now one may try to formulate results analogous to those given in Sections 5 and 6 where the system $e^{\lambda_j(k)v}$ for $j \in N_0$ is now replaced by $e_{k,j}(v)$ for $j \in N_0$. A quick inspection of the proofs shows that one only needs an appropriate estimate for the coefficients d_j which in the odd case have been equal to $1/q_n'(\lambda_j)$ for $j = 0, \ldots, n$. Below we shall show that the coefficients d_j in (56) satisfy the estimate

$$(57) \qquad |d_j| \leq \frac{2^n}{(n-2)!},$$

which is a little bit weaker than in the odd case but still good enough to prove convergence of the involved sums. We shall leave the details to the reader and formulate only one result for the even case:

THEOREM 26. *Let d be even and let $f : A(r_0, r_1) \to C$ be polyharmonic of infinite order and type $\tau < 1/2r_1$. Then there exist analytic functions*

$$f_1 : \{z \in C^d; L_+(z) < r_1\} \to C,$$

$$f_2 : \{z \in C^d; r_0 < L_-(z) \text{ and } L_+(z) < 1/2\tau \text{ and } q(z) \notin (-\infty, 0]\} \to C$$

such that

$$F(z) = f_1(z) + \left(z_1^2 + \cdots + z_d^2\right)^{(2-d)/2} f_2(z)$$

is an analytic extension of f.

Now we proceed to the estimate of the coefficients d_j. They can be computed by the partial fraction decomposition of the integrand in formula (55). The next result addresses this problem:

PROPOSITION 27. *Let $\mu_0, \ldots, \mu_s$ and $v_1, \ldots, v_r$ be distinct real numbers, $n := 2r + s + 1$, and define*

$$Q_n(z) = (z - \mu_0) \cdots (z - \mu_s)(z - v_1)^2 \cdots (z - v_r)^2 .$$

Then the coefficients a_j and c_j in the partial fraction decomposition

$$\frac{1}{Q_n(z)} = \sum_{j=0}^{s} \frac{a_j}{z - \mu_j} + \sum_{j=1}^{r} \frac{b_j}{z - v_j} + \sum_{j=1}^{r} \frac{c_j}{(z - v_j)^2}$$

are non-zero and are given by

$$a_{j_0} = \lim_{z \to \mu_{j_0}} \frac{z - \mu_{j_0}}{Q_n(z)} \text{ and } c_{j_0} = \lim_{z \to v_{j_0}} \frac{(z - v_{j_0})^2}{Q_n(z)} .$$

If $\left| v_{j_0} - \mu_j \right| \geq 2$ for all $j = 1, \ldots, s$, and $\left| v_{j_0} - v_j \right| \geq 2$ for all $j = 1, \ldots, r$ with $j \neq j_0$, then

$$\left| b_{j_0} \right| \leq (n - 1) \left| c_{j_0} \right| .$$

PROOF. It is easy to see that $1 = \lim_{z \to \mu_{j_0}} a_{j_0} Q_n(z) / \left(z - \mu_{j_0} \right)$ and $1 = \lim_{z \to v_{j_0}} c_{j_0} Q_n(z) / \left(z - v_{j_0} \right)^2$. Further b_{j_0} can be computed by residue theory:

$$(58) \qquad b_{j_0} = \text{res}_{v_{j_0}} \frac{1}{Q_n(z)} = \frac{d}{dz} \frac{(z - v_{j_0})^2}{Q_n(z)} (v_{j_0}) .$$

Let us define $P_{j_0}(z) = Q_n(z) / \left(z - v_{j_0} \right)^2$. Clearly

$$P_{j_0}(v_{j_0}) = \lim_{z \to v_{j_0}} \frac{Q_n(z)}{(z - v_{j_0})^2} = \frac{1}{c_{j_0}} .$$

Then (58) is equivalent to

$$b_{j_0} = \frac{d}{dz} \frac{1}{P_{j_0}}(v_{j_0}) = -\left[P_{j_0}(v_{j_0}) \right]^{-2} P'_{j_0}(v_{j_0}) = -\frac{1}{P_{j_0}(v_{j_0})} \frac{P'_{j_0}(v_{j_0})}{P_{j_0}(v_{j_0})} .$$

Moreover

$$\frac{P'_{j_0}(v_{j_0})}{P_{j_0}(v_{j_0})} = \sum_{j=0}^{s} \frac{1}{v_{j_0} - \mu_j} + 2 \sum_{j \neq j_0}^{r} \frac{1}{v_{j_0} - v_j} .$$

Since $\left| v_{j_0} - \mu_j \right| \geq 2$ and $\left| v_{j_0} - v_j \right| \geq 2$, we see that

$$\left| \frac{P'_{j_0}(v_{j_0})}{P_{j_0}(v_{j_0})} \right| \leq \frac{1}{2}(s + 1) + r = \frac{1}{2}(s + 1 + 2r) = \frac{1}{2}n \leq n - 1 .$$

It follows that

$$|b_{j_0}| \leq \frac{n-1}{|P_{j_0}(v_{j_0})|} = (n-1)|c_{j_0}|.$$

The proof is complete.   □

We can order the real and distinct numbers $\mu_0, \ldots, \mu_s, v_1, \ldots, v_r$ according to their values, say $\lambda_0 < \cdots < \lambda_{s+r}$. Clearly $|\lambda_j - \lambda_k| \geq \alpha := 2$ for all $k \neq j$. The exponents λ_j have either multiplicity $m_j = 1$ or $m_j = 2$, and $m_0 + \cdots + m_{s+r} = n$. Then

$$\lim_{z \to \lambda_j} \frac{Q_n(z)}{(z-\lambda_j)^{m_j}} = (\lambda_j - \lambda_0)^{m_0} \cdots (\lambda_j - \lambda_{j-1})^{m_{j-1}} (\lambda_j - \lambda_{j+1})^{m_{j+1}} \cdots (\lambda_j - \lambda_{s+r})^{m_{s+r}}.$$

The proof of Proposition 21 shows that $\lambda_{k+l} - \lambda_k \geq l \cdot \alpha$. Thus

$$\lim_{z \to \lambda_j} \left| \frac{Q_n(z)}{(z-\lambda_j)^{m_j}} \right| \geq \alpha^{m_0} j^{m_0} \cdots \alpha^{m_{j-1}} 1^{m_{j-1}} \cdot \alpha^{m_{j+1}} 1^{m_{j+1}} \cdots \alpha^{m_{s+r}} (s+r-j)^{m_{s+r}} =: A_j.$$

The factor $\alpha = 2$ occurs $n - m_j$ times in the integer A_j, which is greater than or equal to $n - 2$. Since $m_l \geq 1$ for $l = 0, 1, \ldots, s+r$, we have clearly a factor

$$j!(s+r-j)! = j!(n-r-1-j)! \leq j!(n-r-j)!$$

in the expression A_j. Further $m_l = 2$ for at least $r - 1$ different factors in A_j which are non-zero integers; so the product of these number is bigger or equal to $l!(r-1-l)!$ for some natural number $l \in \{1, \ldots, r-1\}$. Thus we conclude that

$$\left| A_j \right| \geq 2^{n-2} j!(n-r-j)! l!(r-1-l)! = 2^{n-2} \frac{(n-r)!}{\binom{n-r}{j}} \frac{(r-1)!}{\binom{r-1}{l}}.$$

Since $\binom{n-r}{j} \leq 2^{n-r}$ and $\binom{r-1}{l} \leq 2^{r-1}$, we obtain

$$\left| A_j \right| \geq 2^{-1}(n-r)!(r-1)! = 2^{-1} \frac{(n-1)!}{\binom{n-1}{r-1}} \leq \frac{(n-1)!}{2^n}.$$

It follows that $|a_j| \leq 2^n / (n-1)!$ and $|c_j| \leq 2^n / (n-1)!$. Further $|b_j| \leq (n-1)|c_j|$ and we conclude that $|d_j| \leq 2^n / (n-2)!$.

8. Appendix: Estimate of derivatives of odd order. In this section we collect and prove some results about linear differential operators which are needed in the paper. The following version of Rolle's theorem is well known and the short proof is included for convenience of the reader.

THEOREM 28 (Rolle's Theorem). *Let $f : [a,b] \to \mathbf{R}$ be continuous and differentiable on (a,b). For $\lambda \in \mathbf{R}$ define the differential operator $D_\lambda f := df/dx - \lambda f$. If $e^{-\lambda a} f(a) = e^{-\lambda b} f(b)$, then there exists $\xi \in (a,b)$ with $D_\lambda f(\xi) = 0$.*

PROOF. Define $g(x) := e^{-\lambda x} f(x)$. Then $g(a) = g(b)$. By the theorem of Rolle there exists $\xi \in (a,b)$ with $g'(\xi) = 0$. Now $g'(x) = f'(x) e^{-\lambda x} - \lambda f(x) e^{-\lambda x} = e^{-\lambda x} D_\lambda f(x)$. So $g'(\xi) = 0$ implies that $D_\lambda f(\xi) = 0$.   □

The following result is an analog of the mean value theorem which is indeed a consequence when we let λ go to 0.

THEOREM 29.   *Let* $f : [a, b] \to \mathbf{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b), *and let* $\lambda \neq 0$ *be a real number. Then there exists* $\xi \in (a, b)$ *with*

$$\frac{e^{\lambda a} f(b) - e^{\lambda b} f(a)}{e^{\lambda b} - e^{\lambda a}} \lambda = D_\lambda f(\xi) \, .$$

PROOF.   Define for $\lambda \neq 0$ a function ψ by

$$\psi(x) = f(x) - \frac{f(b) - e^{\lambda(b-a)} f(a)}{e^{\lambda b} - e^{\lambda a}} \left(e^{\lambda x} - e^{\lambda a} \right) \, .$$

Then $\psi(a) = f(a)$ and $\psi(b) = e^{\lambda(b-a)} f(a)$, so $e^{-\lambda b} \psi(b) = e^{-\lambda a} f(a) = e^{-\lambda a} \psi(a)$. By Theorem 28 there exists $\xi \in (a, b)$ with $D_\lambda \psi(\xi) = 0$. Note that $D_\lambda \left(e^{\lambda x} - e^{\lambda a} \right) = \lambda e^{\lambda x} - \lambda \left(e^{\lambda x} - e^{\lambda a} \right) = \lambda e^{\lambda a}$, so we have

$$0 = D_\lambda \psi(\xi) = D_\lambda f(\xi) - \frac{f(b) - e^{\lambda(b-a)} f(a)}{e^{\lambda b} - e^{\lambda a}} \lambda e^{\lambda a} \, .$$

This gives the statement in the theorem.   □

We need the following simple lemma.

LEMMA 30.   *Let* $\lambda \neq 0$ *be a real number and* $a < b$. *Then*

$$\left| \lambda \frac{e^{\lambda a} + e^{\lambda b}}{e^{\lambda a} - e^{\lambda b}} \right| \leq 2 \frac{e^{|\lambda|(b-a)}}{b - a} \, .$$

PROOF.   Recall that $e^x - 1 \geq x$ for all $x \geq 0$. In the first case suppose that $\lambda > 0$. Then $\lambda(b - a) \geq 0$ and

$$e^{\lambda b} - e^{\lambda a} = e^{\lambda a} \left(e^{\lambda(b-a)} - 1 \right) \geq e^{\lambda a} \lambda (b - a) \, .$$

Further $e^{\lambda a} \leq e^{\lambda b}$ since $\lambda > 0$. Hence

$$\lambda \frac{e^{\lambda a} + e^{\lambda b}}{e^{\lambda b} - e^{\lambda a}} \leq \lambda \frac{2 e^{\lambda b}}{e^{\lambda a} \lambda (b - a)} = 2 \frac{e^{\lambda(b-a)}}{b - a} \, .$$

For $\lambda < 0$ we argue very similarly: $e^{\lambda a} - e^{\lambda b} = e^{\lambda b} \left(e^{-\lambda(b-a)} - 1 \right) \geq e^{\lambda b} |\lambda| (b - a)$. Further $e^{\lambda b} \leq e^{\lambda a}$ since $\lambda < 0$ and

$$\frac{e^{\lambda a} + e^{\lambda b}}{e^{\lambda a} - e^{\lambda b}} \leq \frac{2 e^{\lambda a}}{e^{\lambda b} |\lambda| (b - a)} = 2 \frac{e^{\lambda(a-b)}}{|\lambda| (b - a)} = 2 \frac{e^{|\lambda|(b-a)}}{|\lambda| (b - a)} \, .$$

□

Next we want to give an estimate of first derivative $D_{\lambda_0} f$ in terms of f and the second derivative $D_{\lambda_0 \lambda_1} f$. For the case $\lambda_0 = \lambda_1 = 0$ this is a well-known result (see [37, Theorem 2.4]) and its extension to exponential polynomials is not difficult:

THEOREM 31. *Let λ_0, λ_1 be real numbers and $f : [a, b] \to \boldsymbol{C}$ be twice continuously differentiable. Then the following estimate holds:*

$$\left| D_{\lambda_0} f(a) \right| \leq 4 \frac{e^{(|\lambda_0|+|\lambda_1|)(b-a)}}{b-a} \max\{|f(a)|, |f(b)|\} + 2 \max_{t \in [a,b]} \left| D_{\lambda_1} D_{\lambda_0} f(t) \right| (b-a) e^{|\lambda_1|(b-a)}.$$

PROOF. Assume at first that λ_0 and λ_1 are not zero and that f is real-valued. We apply Theorem 29 to f and $\lambda = \lambda_0$, so there exists $\xi_1 \in (a, b)$ such that

$$\frac{e^{\lambda_0 a} f(b) - e^{\lambda_0 b} f(a)}{e^{\lambda_0 b} - e^{\lambda_0 a}} \lambda_0 = D_{\lambda_0} f(\xi_1).$$

Then Lemma 30 yields the estimate

$$\left| D_{\lambda_0} f(\xi_1) \right| \leq \frac{e^{\lambda_0 a} + e^{\lambda_0 b}}{\left| e^{\lambda_0 b} - e^{\lambda_0 a} \right|} |\lambda_0| \max\{|f(a)|, |f(b)|\}$$

$$\leq 2 \frac{e^{|\lambda_0|(b-a)}}{b-a} \max\{|f(a)|, |f(b)|\}.$$

Now apply Theorem 29 to the interval $[a, \xi_1]$ for $\lambda = \lambda_1$ and the function $D_{\lambda_0} f$. Then there exists $\xi_2 \in (a, \xi_1) \subset (a, b)$ such that

$$\frac{e^{\lambda_1 a} D_{\lambda_0} f(\xi_1) - e^{\lambda_1 \xi_1} D_{\lambda_0} f(a)}{e^{\lambda_1 \xi_1} - e^{\lambda_1 a}} \lambda_1 = D_{\lambda_1} D_{\lambda_0} f(\xi_2).$$

Thus $e^{\lambda_1 \xi_1} D_{\lambda_0} f(a) = e^{\lambda_1 a} D_{\lambda_0} f(\xi_1) - \left(e^{\lambda_1 \xi_1} - e^{\lambda_1 a} \right)/\lambda_1 \times D_{\lambda_1} D_{\lambda_0} f(\xi_2)$ and

$$\left| D_{\lambda_0} f(a) \right| \leq e^{\lambda_1(a-\xi_1)} \left| D_{\lambda_0} f(\xi_1) \right| + \left| D_{\lambda_1} D_{\lambda_0} f(\xi_2) \right| \frac{\left| 1 - e^{\lambda_1(a-\xi_1)} \right|}{|\lambda_1|}.$$

We estimate

$$\left| \frac{e^{\lambda_1(a-\xi_1)} - 1}{\lambda_1} \right| \leq \sum_{n=1}^{\infty} \frac{|\lambda_1(a-\xi_1)|^n}{|\lambda_1| \, n!} \leq \sum_{n=1}^{\infty} \frac{|\lambda_1|^{n-1}}{n!} (b-a)^n$$

$$\leq (b-a) \sum_{n=1}^{\infty} \frac{|\lambda_1|^{n-1}}{(n-1)!} (b-a)^{n-1} = (b-a) e^{|\lambda_1|(b-a)}.$$

Thus

$$\left| D_{\lambda_0} f(a) \right| \leq 2 e^{|\lambda_1|(b-a)} \frac{e^{|\lambda_0|(b-a)}}{b-a} \max\{|f(a)|, |f(b)|\} + \left| D_{\lambda_1} D_{\lambda_0} f(\xi_2) \right| (b-a) e^{|\lambda_1|(b-a)}.$$

Next consider the case that f is complex-valued. Decompose f into its real and imaginary part, say $f = f_1 + i f_2$ with real-valued functions f_1, f_2. Then

(59) $$\left| D_{\lambda_0} f(a) \right| \leq \left| D_{\lambda_0} f_1(a) \right| + \left| D_{\lambda_0} f_2(a) \right|.$$

Now estimate each summand in (59) as above. Since $\left| f_j(t) \right| \leq |f(t)|$ and $\left| D_{\lambda_1} D_{\lambda_0} f_j(t) \right| \leq \left| D_{\lambda_1} D_{\lambda_0} f(t) \right|$ (note that $D_{\lambda_1} D_{\lambda_0} f_1$ is the real part of $D_{\lambda_1} D_{\lambda_0} f$), we obtain the estimate with the factor 2. The case of arbitrary real numbers λ_0 and λ_1 follows from a continuity argument. $\qquad\square$

THEOREM 32. *Let λ_m, $m \in N_0$ be real numbers and $f \in C^\infty \left[x_0, x_0 + \gamma\right]$ with $\gamma > 0$. Assume that $0 < 2\delta < \gamma$. Then the estimate*

$$\left| D^{(2m+1)} f(x) \right| \leq 2 \max \left\{ \frac{2}{\delta}, \delta \right\} e^{(|\lambda_{2m}| + |\lambda_{2m+1}|)\delta}$$

$$\left(\max_{t \in [x_0, x_0 + 2\delta]} \left| D^{(2m)} f(t) \right| + \max_{t \in [x_0, x_0 + 2\delta]} \left| D^{(2m+2)} f(t) \right| \right)$$

holds for all $x \in [x_0, x_0 + \delta]$.

PROOF. We use the estimate in Theorem 31

$$|D_{\lambda_0} f(a)| \leq 4 e^{(|\lambda_0| + |\lambda_1|)(b-a)} \frac{\max \{|f(a)|, |f(b)|\}}{b - a}$$

$$+ 2 \max_{t \in [a,b]} \left| D_{\lambda_1} D_{\lambda_0} f(t) \right| (b - a) e^{|\lambda_1|(b-a)}$$

for the function

$$D^{(2m)} f(t) = \left(\frac{d}{dt} - \lambda_0 \right) \cdots \left(\frac{d}{dt} - \lambda_{2m-1} \right) f(t)$$

for the exponents $\lambda_0 := \lambda_{2m}$ and $\lambda_1 := \lambda_{2m+1}$, and for $a = x$ and $b := x + \delta$ where it is assumed that $x \in [x_0, x_0 + \delta]$. It follows that $t \in [x, x + \delta]$ is in $[x_0, x_0 + 2\delta]$. Then

$$\left| \left(\frac{d}{dx} - \lambda_{2m} \right) D^{(2m)} f(x) \right| \leq \frac{4}{\delta} e^{(|\lambda_{2m}| + |\lambda_{2m+1}|)\delta} \max_{t \in [x_0, x_0 + 2\delta]} \left| D^{(2m)} f(t) \right|$$

$$+ 2\delta \max_{t \in [x_0, x_0 + 2\delta]} \left| D^{(2m+2)} f(t) \right| e^{|\lambda_{2m+1}|\delta}.$$

Using the trivial inequality $e^{|\lambda_{2m+1}|\delta} \leq e^{(|\lambda_{2m}| + |\lambda_{2m+1}|)\delta}$ the statement follows. □

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