Tohoku Math. J.
64 (2012), 317–331

APPROXIMATION BY CESÀRO MEANS OF NEGATIVE ORDER OF DOUBLE WALSH-KACZMARZ-FOURIER SERIES

KÁROLY NAGY

(Received July 28, 2010)

Abstract. In this article we investigate the rate of the approximation by Cesàro means of the quadratical partial sums of double Walsh-Kaczmarz-Fourier series of a function in the Lebesgue space over the Walsh group. The approximation properties of Cesàro means of negative order of one- and two-dimensional Walsh-Fourier series was discussed earlier by Goginava.

1. Introduction. Now, we give a brief introduction to the Walsh-Fourier analysis [1, 9].

Let denote by $\boldsymbol{N}$ the set of natural numbers, $\boldsymbol{P} := \boldsymbol{N}\backslash\{0\}$. Let denote by $\boldsymbol{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition and every subset is open. The normalized Haar measure on $\boldsymbol{Z}_2$ is given in the way that the measure of a singleton is $1/2$. Let $G := \prod_{k=0}^{\infty} \boldsymbol{Z}_2$, which is called the Walsh group. The elements of G are sequences $x = (x_0, x_1, \ldots, x_k, \ldots)$ with coordinates $x_k \in \{0, 1\}$ $(k \in \boldsymbol{N})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure is the product measure (denoted by μ) and the topology is the pruduct topology. Dyadic intervals are defined by

$$I_0(x) := G, \quad I_n(x) := \{y \in G;\ y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}$$

for $x \in G, n \in \boldsymbol{P}$. They form a base for the neighborhoods of G. Let $0 = (0; i \in \boldsymbol{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \boldsymbol{N}$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$ for $n \in \boldsymbol{N}$, where the n-th coordinate of which is 1 and the rest are zeros.

Let $L^p(G^2)$ denote the usual Lebesgue spaces on G^2 (with the corresponding norm $\|.\|_p$). For the sake of brevity in notation, we agree to write $L^\infty(G^2)$ instead of C, where C is the set of continuous functions on G^2 (for more details see [9, pp. 9–11]) and set $\|f\|_\infty := \sup\{|f(x)|;\ x \in G^2\}$.

For $x \in G$ we define $|x|$ by $|x| := \sum_{j=0}^{\infty} x_j 2^{-j-1}$.

Next, we define the dyadic partial modulus of continuity in $L^p(G^2)$, $1 \le p \le \infty$, of a function $f \in L^p(G^2)$ by

$$\omega_p^1\left(\frac{1}{2^A}, f\right) := \sup_{t \in I_A} \|f(x^1 + t, x^2) - f(x^1, x^2)\|_p\,,$$

2000 *Mathematics Subject Classification.* Primary 42C10.

Key words and phrases. Walsh group, Walsh-Kaczmarz system, double Walsh-Kaczmarz-Fourier series, Cesàro means, rate of convergence, approximation.

$$\omega_p^2\left(\frac{1}{2^A}, f\right) := \sup_{t\in I_A} \|f(x^1, x^2+t) - f(x^1, x^2)\|_p .$$

If the modulus of continuity $\omega(\delta)$ ($\delta = 1/2^A$ for some A) is given, then denote

$$H_p^\omega := \{f \in L^p(G^2); \omega_p^i(f, \delta) = O(\omega(\delta)),\ i = 1, 2\} .$$

The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \boldsymbol{N}) .$$

Let the Walsh-Paley functions be the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i,\quad n_i \in \{0, 1\}$ $(i \in \boldsymbol{N})$, where only a finite number of n_i's are different from zero. Define the order $|n|$ of $n > 0$ by $|n| := \max\{j \in \boldsymbol{N}; n_j \neq 0\}$. Walsh-Paley functions are $w_0 = 1$ and for $n \geq 1$

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} .$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 = 1$ and for $n \geq 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}} .$$

The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions are equal in each dyadic block. Namely,

$$\{\kappa_n; 2^k \leq n < 2^{k+1}\} = \{w_n; 2^k \leq n < 2^{k+1}\}$$

for all $k \in \boldsymbol{P}$ and $\kappa_0 = w_0$.

A relation (given by Skvortsov [12]) between the Walsh-Kaczmarz functions and the Walsh-Paley functions can be given by the transformation $\tau_A : G \to G$ defined by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots)$$

for $A \in \boldsymbol{P}$. By the definition of τ_A, we have

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \boldsymbol{N}, x \in G) .$$

The transformation τ_A is measure-preserving and $\tau_A(\tau_A(x)) = x$ for all $x \in G$ (for more details, see [12]).

In 1948, Šneider [13] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n\to\infty} \frac{D_n^\kappa(x)}{\log n} \geq C > 0$$

holds a.e. A number of pathological properties are due to this "spreadness" property of the kernel. For example, for Walsh-Kaczmarz-Fourier series it is impossible to establish any local test for convergence at any point or on any interval, since the principle of localization does not hold for this system.

On the other hand, the global behavior of the Walsh-Kaczmarz-Fourier series is similar in many aspects to the Walsh-Paley-Fourier series. In 1974, Schipp [10] and Young [16] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [12] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f.

In 1998, Gát [2] proved, for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function. In 2004, Gát's result was generalized by Simon [11].

The Dirichlet kernels are defined by

$$D_n^{\psi} := \sum_{k=0}^{n-1} \psi_k ,$$

where $\psi_n = w_n$ $(n \in \boldsymbol{P})$ or $\psi_n = \kappa_n$ $(n \in \boldsymbol{P})$, $D_0^{\psi} := 0$. The 2^n-th Dirichlet kernels have a closed form (see e.g. [9])

$$D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = D_{2^n}(x) = \begin{cases} 2^n & x \in I_n , \\ 0 & \text{otherwise } (n \in \boldsymbol{N}) . \end{cases} \tag{1}$$

We consider the double systems $\{\psi_n(x^1) \times \psi_m(x^2); n, m \in \boldsymbol{N}\}$ on G^2. Suppose that f is an integrable function on G^2. Its Walsh-(Kaczmarz)-Fourier series is defined by

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hat{f}^{\psi}(k, l)\psi_k(x^1)\psi_l(x^2) ,$$

$(x = (x^1, x^2) \in G^2)$, where $\hat{f}^{\psi}(k, l) := \int_{G^2} f \psi_k \psi_l$ is the (k, l)-th Walsh-(Kaczmarz)-Fourier coefficient of the function f.

The n-th Cesàro (C, α)-mean and the kernel of the double Walsh-(Kaczmarz)-Fourier series of a function f are defined by

$$\sigma_n^{\psi,\alpha}(f; x^1, x^2) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_{k,k}^{\psi}(f, x^1, x^2) ,$$

$$\boldsymbol{K}_n^{\psi,\alpha}(x^1, x^2) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} D_k^{\psi}(x^1) D_k^{\psi}(x^2)$$

for $(x^1, x^2) \in G^2$, where $\psi_n = w_n$ $(n \in \boldsymbol{P})$ or $\psi_n = \kappa_n$ $(n \in \boldsymbol{P})$, and

$$A_0^{\alpha} := 1 , \quad A_n^{\alpha} := \frac{(\alpha + 1) \cdots (\alpha + n)}{n!} , \quad (\alpha \neq -1, -2, \dots) .$$

It is well known that

$$A_n^{\alpha} = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} ,$$

$$A_n^{\alpha} - A_{n-1}^{\alpha} = A_n^{\alpha-1} ,$$

$$A_n^{\alpha} \sim n^{\alpha}$$

[18]. The n-th Cesàro (C, α) kernel of the one-dimensional Walsh-(Kaczmarz)-Fourier series is defined by

$$K_n^{\psi,\alpha}(x) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} D_k^{\psi}(x) \quad (x \in G).$$

In the papers [14, 15], Tevzadze has studied the uniform convergence of Cesàro means of negative order of Walsh-Fourier series. In particular, he proved a criterion in terms of modulus of continuity and variation of a function $f \in C_W([0,1])$ for the uniform summability by the Cesàro method of negative order of Walsh-Fourier series. In [4], Goginava gave sufficient conditions for the convergence of Cesàro means of negative order of Walsh-Fourier series in the space $L^p([0,1]^N)$, $1 \le p \le \infty$. In the papers [4, 14, 15], the results were established without any estimation of the approximation. In 2002, Goginava [5] studied the rate of the approximation by Cesàro $(C, -\alpha)$-means (where $0 < \alpha < 1$) of Walsh-Fourier series of a function in $L^p([0,1])$.

In 2004, the rate of the approximation by Cesàro $(C, -\alpha)$-means (where $0 < \alpha < 1$) of double Walsh-Fourier series of a function in $L^p([0,1]^2)$ was studied by Goginava in [6]. Analogical questions for double trigonometric Fourier series was studied by Zhizhiashvili [17], and recently by Goginava [7].

2. The main results. In this article we investigate the rate of the approximation by $(C, -\alpha)$-means (where $0 < \alpha < 1$) of double Walsh-Kaczmarz-Fourier series of a function in L^p $(1 \le p \le \infty)$. We show that the approximation behavior of the Cesàro $(C, -\alpha)$-means of double Walsh-Kaczmarz-Fourier series is so good as the approximation behavior of the double Walsh-Cesàro $(C, -\alpha)$-means (where $0 < \alpha < 1$).

We note that the rate of the approximation of $(C, 1)$-means of one-dimensional Walsh-Kaczmarz-Fourier series for functions in $L^p(G)$ $(1 \le p \le \infty)$ was studied by Skvortsov [12] earlier.

THEOREM 2.1. *Let $f \in L^p(G^2)$ for $1 \le p \le \infty$ and $0 < \alpha < 1$. Then*

$$\begin{aligned}\|\sigma_n^{\kappa,-\alpha}(f) - f\|_p \le c\Bigg(&|n|2^{|n|\alpha}\omega_p^1\Big(\frac{1}{2^{|n|-1}}, f\Big) + |n|2^{|n|\alpha}\omega_p^2\Big(\frac{1}{2^{|n|-1}}, f\Big)\\ &+ \sum_{l=0}^{|n|-2} 2^{l-|n|}\omega_p^1(1/2^l, f) + \sum_{l=0}^{|n|-2} 2^{l-|n|}\omega_p^2(1/2^l, f)\Bigg)\end{aligned}$$

holds, where c depends only on p and α.

The following corollary follows from the proof of Theorem 2.1.

COROLLARY 2.2. *Let $f \in L^p(G^2)$ for $1 \le p \le \infty$ and $0 < \alpha < 1$. Then for any $k \in \boldsymbol{N}$ the inequality*

$$\|\sigma_{2^k}^{\kappa,-\alpha}(f) - f\|_p \le c\Big(2^{k\alpha}\omega_p^1\Big(\frac{1}{2^{k-1}}, f\Big) + 2^{k\alpha}\omega_p^2\Big(\frac{1}{2^{k-1}}, f\Big)$$

$$+\sum_{l=0}^{k-2} 2^{l-k}\omega_p^1(1/2^l, f) + \sum_{l=0}^{k-2} 2^{l-k}\omega_p^2(1/2^l, f)\Big)$$

holds, where c depends only on p and α.

To prove Theorem 2.1, we need the following lemmas of Glukhov [3] and Goginava [6]:

LEMMA 2.3 (Glukhov [3]). *Let* $\alpha_1, \ldots, \alpha_n$ *be real numbers. Then*

$$\frac{1}{n}\left\|\sum_{k=1}^{n}\alpha_k\prod_{j=1}^{d}D_k^w\right\|_1 \le \frac{c}{\sqrt{n}}\left(\sum_{k=1}^{n}\alpha_k^2\right)^{1/2},$$

where c is an absolute constant.

LEMMA 2.4 (Goginava [6]). *Let* $0 < \alpha < 1$ *and* $p = 2^k, 2^{k+1}, \ldots$. *Then*

$$\int_{G^2}\left|\sum_{\nu=1}^{2^k} A_{p-\nu}^{-\alpha-1}D_\nu(x^1)D_\nu(x^2)\right|d\mu(x) \le c(\alpha) < \infty, \quad k = 1, 2, \ldots.$$

By the method of the proof of Lemma 2.4 we get the following lemma:

LEMMA 2.5. *Let* $0 < \alpha < 1$, $p = 2^k, 2^{k+1}, \ldots$. *Then*

$$\int_G\left|\sum_{l=1}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1}D_l^w\right| \le c(\alpha)$$

for $k = 1, 2, \ldots$.

LEMMA 2.6 (Goginava [6]). *Let* $0 < \alpha < 1$. *Then the inequality*

$$\int_{G^2}\left|\sum_{\nu=1}^{n} A_{n-\nu}^{-\alpha-1}D_\nu(x^1)D_\nu(x^2)\right|d\mu(x) \le c(\alpha)\log n$$

holds.

We prove the following lemma:

LEMMA 2.7. *Let* $f \in L^p(G^2)$ *for* $1 \le p \le \infty$ *and* $0 < \alpha < 1$. *Then*

$$\left\|\frac{1}{A_{n-1}^{-\alpha}}\int_{G^2}\sum_{k=0}^{2^{|n|-1}-1} A_{n-k}^{-\alpha-1}D_k^\kappa(x^1)D_k^\kappa(x^2)(f(.+x)-f(.))d\mu(x)\right\|_p$$

$$\le c\left(\sum_{l=0}^{|n|-2} 2^{l-|n|}\omega_p^1(1/2^l, f) + \sum_{l=0}^{|n|-2} 2^{l-|n|}\omega_p^2(1/2^l, f)\right)$$

holds, where c depends only on p and α.

PROOF. Set

$$\|I\|_p := \left\|\frac{1}{A_{n-1}^{-\alpha}}\int_{G^2}\sum_{k=0}^{2^{|n|-1}-1} A_{n-k}^{-\alpha-1}D_k^\kappa(x^1)D_k^\kappa(x^2)(f(.+x)-f(.))d\mu(x)\right\|_p.$$

To discuss $\|I\|_p$, we use the following equality [12]:

$$D_n^{\kappa}(x) = D_{2^{|n|}}(x) + r_{|n|}(x) D^{w}_{n-2^{|n|}}(\tau_{|n|}(x)) . \tag{2}$$

For $\|I\|_p$ we write

$$\|I\|_p = \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{|n|-2} \sum_{k=2^l}^{2^{l+1}-1} A_{n-k}^{-\alpha-1} D_k^{\kappa}(x^1) D_k^{\kappa}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p$$

$$= \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{|n|-2} \sum_{k=0}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} D_{2^l+k}^{\kappa}(x^1) D_{2^l+k}^{\kappa}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p$$

$$\le \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{|n|-2} \sum_{k=0}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} D_{2^l}(x^1) D_{2^l}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p$$

$$+ \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{|n|-2} \sum_{k=0}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} D_{2^l}(x^1) r_l(x^2) D_k^{w}(\tau_l(x^2))(f(.+x) - f(.))d\mu(x) \right\|_p$$

$$+ \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{|n|-2} \sum_{k=0}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} r_l(x^1) D_k^{w}(\tau_l(x^1)) D_{2^l}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p$$

$$+ \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{|n|-2} \sum_{k=0}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} r_l(x^1) r_l(x^2) D_k^{w}(\tau_l(x^1)) D_k^{w}(\tau_l(x^2))(f(.+x) - f(.))d\mu(x) \right\|_p$$

$$=: \|I_1\|_p + \|I_2\|_p + \|I_3\|_p + \|I_4\|_p .$$

From the equation (1) and by generalized Minkowski inequality, we have

$$\begin{aligned} &\left\| \int_{G^2} D_{2^l}(x^1) D_{2^l}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p \\ &\qquad \le \int_{G^2} D_{2^l}(x^1) D_{2^l}(x^2) \|f(.+x) - f(.)\|_p d\mu(x) \\ &\qquad \le \omega_p^1(1/2^l, f) + \omega_p^2(1/2^l, f) . \end{aligned} \tag{3}$$

This gives

$$\begin{aligned} \|I_1\|_p &\le \frac{1}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} \sum_{k=0}^{2^l-1} |A_{n-2^l-k}^{-\alpha-1}| \left\| \int_{G^2} D_{2^l}(x^1) D_{2^l}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p \\ &\le \frac{c}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} (\omega_p^1(1/2^l, f) + \omega_p^2(1/2^l, f)) \sum_{k=0}^{2^l-1} |A_{n-2^l-k}^{-\alpha-1}| \\ &\le \frac{c}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} (\omega_p^1(1/2^l, f) + \omega_p^2(1/2^l, f)) 2^l |A_{n-2^{l+1}+1}^{-\alpha-1}| \end{aligned}$$

$$\leq c \sum_{l=0}^{|n|-2} 2^{l-|n|}(\omega_p^1(1/2^l, f) + \omega_p^2(1/2^l, f)) .$$

To discuss $\|I_2\|_p$ we docompose G as a disjoint union. Define the set X_l by $X_l := \{z \in G; z = (z_0, \dots, z_{l-1}, 0, 0, \dots)\}$, that is, X_l has 2^l elements. Then we have

$$G = \bigcup_{z \in X_l} I_l(z) \quad (l \geq 0) . \tag{4}$$

On the sets $I_l(z)$ the function $D_k^w \circ \tau_l$ is constant for $0 \leq k \leq 2^l$. By generalized Minkowski inequality we write that

$$\begin{aligned}
\|I_2\|_p &\leq \frac{1}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} \left\| \int_{G^2} \sum_{k=0}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} D_{2^l}(x^1) r_l(x^2) D_k^w(\tau_l(x^2))(f(.+x) - f(.)) d\mu(x) \right\|_p \\
&= \frac{1}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} 2^l \left\| \sum_{z \in X_l} \int_{I_l \times I_l(z)} \sum_{k=0}^{2^l-1} A_{n-2^l-k-1}^{-\alpha-1} r_l(x^2) D_k^w(\tau_l(x^2))(f(.+x) - f(.)) d\mu(x) \right\|_p \\
&\leq \frac{1}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} 2^l \sum_{z \in X_l} \left| \sum_{k=0}^{2^l-1} A_{n-2^l-k-1}^{-\alpha-1} D_k^w(\tau_l(z)) \right| \left\| \int_{I_l \times I_l(z)} r_l(x^2)(f(.+x) - f(.)) d\mu(x) \right\|_p .
\end{aligned}$$

Set $e_l^1 := (e_l, 0) \in G^2$ and $e_l^2 := (0, e_l) \in G^2$. Since

$$I_l(z) = I_{l+1}(z) \cup I_{l+1}(z + e_l)$$

and

$$r_l(u) = \begin{cases} 1 & u \in I_{l+1}(z) , \\ -1 & u \in I_{l+1}(z) , \end{cases} \qquad \text{or} \qquad r_l(u) = \begin{cases} -1 & u \in I_{l+1}(z) , \\ 1 & u \in I_{l+1}(z) , \end{cases}$$

we obtain

$$\begin{aligned}
&\left\| \int_{I_l \times I_l(z)} r_l(x^2)(f(.+x) - f(.)) d\mu(x) \right\|_p \\
&\quad = \left\| \int_{I_l \times I_{l+1}(z)} r_l(x^2)(f(.+x) - f(.)) d\mu(x) \right. \\
&\qquad \left. + \int_{I_l \times I_{l+1}(z+e_l)} r_l(x^2)(f(.+x) - f(.)) d\mu(x) \right\|_p \\
&\quad \leq 2 \left\| \int_{I_l \times I_{l+1}(z)} (f(.+x) - f(.+x+e_l^2)) d\mu(x) \right\|_p \\
&\quad \leq 2 \int_{I_l \times I_{l+1}(z)} \|f(.+x) - f(.+x+e_l^2)\|_p d\mu(x) \\
&\quad \leq 2\omega_p^2(1/2^l, f) 2^{-l} \int_{I_{l+1}(z)} d\mu(x^2) .
\end{aligned} \tag{5}$$

This yields that

$$\|I_2\|_p \le \frac{c}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} \omega_p^2(1/2^l, f) \sum_{z\in X_l} \int_{I_{l+1}(z)} \left| \sum_{k=0}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} D_k^w(\tau_l(x^2))\right| d\mu(x^2)$$

$$\le \frac{c}{A_{n-2}^{-\alpha}} \sum_{l=0}^{|n|-2} \omega_p^2(1/2^l, f) \left\| \sum_{k=1}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} D_k^w \circ \tau_l \right\|_1$$

$$\le \frac{c}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} \omega_p(1/2^l, f) \left\| \sum_{k=1}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} D_k^w \right\|_1 .$$

By Lemma 2.3 of Glukhov, we have that

$$\left\| \sum_{k=1}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} D_k^w \right\|_1 \le c \frac{2^l-1}{\sqrt{2^l-1}} \left(\sum_{k=1}^{2^l-1} A_{n-2^l-k}^{-2\alpha-2} \right)^{1/2} \le c 2^l A_{n-2^{l+1}}^{-\alpha-1}$$

and

$$\|I_2\|_p \le c \sum_{l=0}^{|n|-2} 2^{l-|n|} \omega_p^2(1/2^l, f) .$$

Analogously,

$$\|I_3\|_p \le c \sum_{l=0}^{|n|-2} 2^{l-|n|} \omega_p^1(1/2^l, f) .$$

At last, we discuss $\|I_4\|_p$. We use the disjoint decomposition (4) of G. We write that

$$\|I_4\|_p \le \frac{1}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} \Bigg\| \int_{G^2} \sum_{k=0}^{2^l-1} A_{n-2^l-k}^{-\alpha-1}$$

$$\times r_l(x^1+x^2) D_k^w(\tau_l(x^1)) D_k^w(\tau_l(x^2)) (f(.+x) - f(.)) d\mu(x) \Bigg\|_p$$

$$= \frac{1}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} \Bigg\| \sum_{z,v\in X_l} \int_{I_l(z)\times I_l(v)} \sum_{k=0}^{2^l-1} A_{n-2^l-k}^{-\alpha-1}$$

$$\times r_l(x^1+x^2) D_k^w(\tau_l(x^1)) D_k^w(\tau_l(x^2)) (f(.+x) - f(.)) d\mu(x) \Bigg\|_p$$

$$\le \frac{1}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|-2} \sum_{z,v\in X_l} \left| \sum_{k=0}^{2^l-1} A_{n-2^l-k}^{-\alpha-1} D_k^w(\tau_l(z)) D_k^w(\tau_l(v)) \right|$$

$$\times \left\| \int_{I_l(z)\times I_l(v)} r_l(x^1+x^2)(f(.+x) - f(.)) d\mu(x) \right\|_p$$

It is easy to show that

$$\begin{aligned}(6)\qquad &\left\| \int_{I_l(z)\times I_l(v)} r_l(x^1)r_l(x^2)(f(.+x)-f(.))d\mu(x)\right\|_p \\ &\quad \le c(\omega_p^1(1/2^l,f)+\omega_p^2(1/2^l,f))\int_{I_{l+1}(z)\times I_{l+1}(v)} d\mu(x)\end{aligned}$$

(for more details see the inequality (5)). By this we immediately get

$$\begin{aligned}\|I_4\|_p &\le \frac{c}{A_{n-1}^{-\alpha}}\sum_{l=0}^{|n|-2}(\omega_p^1(1/2^l,f)+\omega_p^2(1/2^l,f)) \\ &\quad\times \sum_{z,v\in X_l}\int_{I_{l+1}(z)\times I_{l+1}(v)}\left|\sum_{k=0}^{2^l-1}A_{n-2^l-k}^{-\alpha-1}D_k^w(\tau_l(x^1))D_k^w(\tau_l(x^2))\right|d\mu(x) \\ &\le \frac{c}{A_{n-2}^{-\alpha}}\sum_{l=0}^{|n|-2}(\omega_p^1(1/2^l,f)+\omega_p^2(1/2^l,f))\left\|\sum_{k=1}^{2^l-1}A_{n-2^l-k}^{-\alpha-1}(D_k^w\times D_k^w)\circ(\tau_l\times\tau_l)\right\|_1 \\ &\le \frac{c}{A_{n-1}^{-\alpha}}\sum_{l=0}^{|n|-2}(\omega_p^1(1/2^l,f)+\omega_p^2(1/2^l,f))\left\|\sum_{k=1}^{2^l-1}A_{n-2^l-k}^{-\alpha-1}(D_k^w\times D_k^w)\right\|_1 .\end{aligned}$$

By Lemma 2.3 of Glukhov, we have that

$$\left\|\sum_{k=1}^{2^l-1}A_{n-2^l-k}^{-\alpha-1}(D_k^w\times D_k^w)\right\|_1 \le c2^lA_{n-2^{l+1}}^{-\alpha-1}$$

and

$$\|I_4\|_p \le c\sum_{l=0}^{|n|-2}2^{l-|n|}(\omega_p^1(1/2^l,f)+\omega_p^2(1/2^l,f)) .$$

This completes the proof of Lemma 2.7. □

LEMMA 2.8. *Let $f\in L^p(G^2)$ for $1\le p\le\infty$ and $0<\alpha<1$. Then*

$$\begin{aligned}&\left\|\frac{1}{A_{n-1}^{-\alpha}}\int_{G^2}\sum_{k=2^{|n|-1}}^{2^{|n|}-1}A_{n-k}^{-\alpha-1}D_k^\kappa(x^1)D_k^\kappa(x^2)(f(.+x)-f(.))d\mu(x)\right\|_p \\ &\qquad \le c2^{|n|\alpha}\left(\omega_p^1\Big(\frac{1}{2^{|n|-1}},f\Big)+\omega_p^2\Big(\frac{1}{2^{|n|-1}},f\Big)\right) ,\end{aligned}$$

where c depends only on p and α.

PROOF. Set

$$\|II\|_p := \left\|\frac{1}{A_{n-1}^{-\alpha}}\int_{G^2}\sum_{k=2^{|n|-1}}^{2^{|n|}-1}A_{n-k}^{-\alpha-1}D_k^\kappa(x^1)D_k^\kappa(x^2)(f(.+x)-f(.))d\mu(x)\right\|_p$$

$$= \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} D_{2^{|n|-1}+l}^{\kappa}(x^1) D_{2^{|n|-1}+l}^{\kappa}(x^2)(f(.+x)-f(.))d\mu(x) \right\|_p .$$

By the equation (2) we have that

$$\begin{aligned}
\|II\|_p \le & \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} D_{2^{|n|-1}}(x^1) D_{2^{|n|-1}}(x^2) \right. \\
& \qquad \left. \times (f(.+x)-f(.))d\mu(x) \right\|_p \\
& + \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} D_{2^{|n|-1}}(x^1) r_{|n|-1}(x^2) D_l^w(\tau_{|n|-1}(x^2)) \right. \\
& \qquad \left. \times (f(.+x)-f(.))d\mu(x) \right\|_p \\
& + \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} r_{|n|-1}(x^1) D_l^w(\tau_{|n|-1}(x^1)) D_{2^{|n|-1}}(x^2) \right. \\
& \qquad \left. \times (f(.+x)-f(.))d\mu(x) \right\|_p \\
& + \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} r_{|n|-1}(x^1) r_{|n|-1}(x^2) D_l^w(\tau_{|n|-1}(x^1)) D_l^w(\tau_{|n|-1}(x^2)) \right. \\
& \qquad \left. \times (f(.+x)-f(.))d\mu(x) \right\|_p \\
=: & \|II_1\|_p + \|II_2\|_p + \|II_3\|_p + \|II_4\|_p .
\end{aligned}$$

From the inequality (3) we immediately get that

$$\begin{aligned}
\|II_1\|_p &\le \frac{cA_{n-1}^{\alpha}}{A_{n-2^{|n|-1}}^{\alpha}} (\omega_p^1(1/2^{|n|-1}, f) + \omega_p^2(1/2^{|n|-1}, f)) \\
&\le c(\omega_p^1(1/2^{|n|-1}, f) + \omega_p^2(1/2^{|n|-1}, f)) .
\end{aligned}$$

To discuss $\|II_2\|_p$ we use the disjoint decomposition (4) of G for $l = |n| - 1$. Then

$$\begin{aligned}
\|II_2\|_p \le \frac{2^{|n|-1}}{A_{n-1}^{-\alpha}} \sum_{z \in X_{|n|-1}} & \left\| \int_{I_{|n|-1} \times I_{|n|-1}(z)} \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} r_{|n|-1}(x^2) \right. \\
& \left. \times D_l^w(\tau_{|n|-1}(x^2))(f(.+x)-f(.))d\mu(x) \right\|_p .
\end{aligned}$$

The function $\sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} D_l^w \circ \tau_{|n|-1}$ is constant on the set $I_{|n|-1}(z)$ for a fixed $z = (z_0, \dots, z_{|n|-2}, 0, 0, \dots)$. By the inequality (5) we write

$$\begin{aligned}
\|II_2\|_p &\le \frac{2^{|n|-1}}{A_{n-1}^{-\alpha}} \sum_{z \in X_{|n|-1}} \left| \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} D_l^w(\tau_{|n|-1}(z)) \right| \\
&\quad \times \left\| \int_{I_{|n|-1} \times I_{|n|-1}(z)} r_{|n|-1}(x^2)(f(.+x) - f(.)) d\mu(x) \right\|_p \\
&\le \frac{c}{A_n^{-\alpha}} \omega_p^2(1/2^{|n|-1}, f) \sum_{z \in X_{|n|-1}} \left| \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} D_l^w(\tau_{|n|-1}(z)) \right| \int_{I_{|n|}(z)} d\mu(x) \\
&\le c 2^{|n|\alpha} \omega_p^2(1/2^{|n|-1}, f) \left\| \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} D_l^w \circ \tau_{|n|-1} \right\|_1 \\
&\le c 2^{|n|\alpha} \omega_p^2(1/2^{|n|-1}, f) \left\| \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} D_l^w \right\|_1 .
\end{aligned}$$

Lemma 2.5 yields

$$\|II_2\|_p \le c 2^{|n|\alpha} \omega_p^2(1/2^{|n|-1}, f) .$$

Analogously,

$$\|II_3\|_p \le c 2^{|n|\alpha} \omega_p^1(1/2^{|n|-1}, f) .$$

At last, we discuss $\|II_4\|_p$. By the help of the disjoint decomposition (4) of G, the fact that the function $\sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} (D_l^w D_l^w) \circ (\tau_{|n|-1} \times \tau_{|n|-1})$ is constant on the set $I_{|n|-1}(z) \times I_{|n|-1}(v)$ for fixed $z = (z_0, \dots, z_{|n|-2}, 0, 0, \dots)$, $v = (v_0, \dots, v_{|n|-2}, 0, 0, \dots)$ and the inequality (6), we have that

$$\|II_4\|_p \le c 2^{|n|\alpha} (\omega_p^1(1/2^{|n|-1}, f) + \omega_p^2(1/2^{|n|-1}, f)) \left\| \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} (D_l^w D_l^w) \right\|_1 .$$

Since,

$$\begin{aligned}
\left\| \sum_{l=0}^{2^{|n|-1}-1} A_{n-2^{|n|-1}-l}^{-\alpha-1} (D_l^w D_l^w) \right\|_1 &\le \left\| \sum_{l=0}^{2^{|n|-1}} A_{n-2^{|n|-1}-l}^{-\alpha-1} (D_l^w D_l^w) \right\|_1 \\
&\quad + \left\| A_{n-2^{|n|}}^{-\alpha-1} (D_{2^{|n|-1}} D_{2^{|n|-1}}) \right\|_1 .
\end{aligned}$$

Lemma 2.4 and the equation (1) complete the proof of Lemma 2.8. □

Now, we prove our main theorem.

PROOF OF THEOREM 2.1. We carry out the proof for $1 \le p < \infty$. The proof for $p = \infty$ is similar and even simpler. It is simple to calculate

$$\begin{aligned}
\|\sigma_n^{\kappa,-\alpha}(f) - f\|_p &= \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{k=1}^{n} A_{n-k}^{-\alpha-1} D_k^{\kappa}(x^1) D_k^{\kappa}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p \\
&\le \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{k=0}^{2^{|n|-1}-1} A_{n-k}^{-\alpha-1} D_k^{\kappa}(x^1) D_k^{\kappa}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p \\
&+ \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{k=2^{|n|-1}}^{2^{|n|}-1} A_{n-k}^{-\alpha-1} D_k^{\kappa}(x^1) D_k^{\kappa}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p \\
&+ \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{k=2^{|n|}}^{n} A_{n-k}^{-\alpha-1} D_k^{\kappa}(x^1) D_k^{\kappa}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p \\
&=: \|I\|_p + \|II\|_p + \|III\|_p .
\end{aligned}$$

By Lemma 2.7 we obtain that

$$\|I\|_p \le c\left(\sum_{l=0}^{|n|-2} 2^{l-|n|}\omega_p^1(1/2^l, f) + \sum_{l=0}^{|n|-2} 2^{l-|n|}\omega_p^2(1/2^l, f) \right).$$

From Lemma 2.8 we get that

$$\|II\|_p \le c\left(2^{|n|\alpha}\omega_p^1\Big(\frac{1}{2^{|n|-1}}, f\Big) + 2^{|n|\alpha}\omega_p^2\Big(\frac{1}{2^{|n|-1}}, f\Big) \right) .$$

Thus, we have to discuss $\|III\|_p$.

The equality (2) yields that

$$\begin{aligned}
\|III\|_p &= \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{j=0}^{n-2^{|n|}} A_{n-2^{|n|}-j}^{-\alpha-1} D_{2^{|n|}+j}^{\kappa}(x^1) D_{2^{|n|}+j}^{\kappa}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p \\
&\le \left\| \frac{1}{A_{n-1}^{-\alpha}} \int_{G^2} \sum_{j=0}^{n-2^{|n|}} A_{n-2^{|n|}-j}^{-\alpha-1} D_{2^{|n|}}(x^1) D_{2^{|n|}}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p \\
&+ \left\| \frac{A_{n-2^{|n|}-1}^{-\alpha}}{A_{n-1}^{-\alpha}} \int_{G^2} D_{2^{|n|}}(x^1) r_{|n|}(x^2) K_{n-2^{|n|}}^{w,-\alpha}(\tau_{|n|}(x^2))(f(.+x) - f(.))d\mu(x) \right\|_p \\
&+ \left\| \frac{A_{n-2^{|n|}-1}^{-\alpha}}{A_{n-1}^{-\alpha}} \int_{G^2} r_{|n|}(x^1) K_{n-2^{|n|}}^{w,-\alpha}(\tau_{|n|}(x^1)) D_{2^{|n|}}(x^2)(f(.+x) - f(.))d\mu(x) \right\|_p \\
&+ \left\| \frac{A_{n-2^{|n|}-1}^{-\alpha}}{A_{n-1}^{-\alpha}} \int_{G^2} r_{|n|}(x^1+x^2) \boldsymbol{K}_{n-2^{|n|}}^{w,-\alpha}(\tau_{|n|}(x^1), \tau_{|n|}(x^2))(f(.+x) - f(.))d\mu(x) \right\|_p \\
&=: \|III_1\|_p + \|III_2\|_p + \|III_3\|_p + \|III_4\|_p .
\end{aligned}$$

By the inequality (3) we immediately get that

$$\|III_1\|_p \le 2^{|n|\alpha}(\omega_p^1(1/2^{|n|}, f) + \omega_p^2(1/2^{|n|}, f)) .$$

To discuss $\|III_2\|_p$, we use the decomposition (4) of G for $l = |n|$ and the fact that the function $K^{w,-\alpha}_{n-2^{|n|}} \circ \tau_{|n|}$ is constant on the set $I_{|n|}(z)$ (for a fixed $z = (z_1, \dots, z_{|n|-1}, 0, 0, \dots)$). By generalized Minkowski inequality we have that

$$\begin{aligned}\|III_2\|_p &= \left\| \frac{A^{-\alpha}_{n-2^{|n|}-1}}{A^{-\alpha}_{n-1}} \int_{G^2} D_{2^{|n|}}(x^1) r_{|n|}(x^2) K^{\omega,-\alpha}_{n-2^{|n|}}(\tau_{|n|}(x^2))(f(.+x) - f(.)) d\mu(x) \right\|_p \\ &= \left\| \frac{2^{|n|} A^{-\alpha}_{n-2^{|n|}-1}}{A^{-\alpha}_{n-1}} \sum_{z \in X_{|n|}} \int_{I_{|n|} \times I_{|n|}(z)} r_{|n|}(x^2) K^{\omega,-\alpha}_{n-2^{|n|}}(\tau_{|n|}(x^2))(f(.+x) - f(.)) d\mu(x) \right\|_p \\ &\le \frac{2^{|n|} A^{-\alpha}_{n-2^{|n|}-1}}{A^{-\alpha}_{n-1}} \sum_{z \in X_{|n|}} |K^{\omega,-\alpha}_{n-2^{|n|}}(\tau_{|n|}(z))| \left\| \int_{I_{|n|} \times I_{|n|}(z)} r_{|n|}(x^2)(f(.+x) - f(.)) d\mu(x) \right\|_p .\end{aligned}$$

The inequality (5) yields

$$\begin{aligned}\|III_2\|_p &\le c A^{\alpha}_{n-1} \omega_p^2(1/2^{|n|}, f) \sum_{z \in X_{|n|}} \int_{I_{|n|+1}(z)} |A^{-\alpha}_{n-2^{|n|}-1} K^{\omega,-\alpha}_{n-2^{|n|}}(\tau_{|n|}(x))| d\mu(x) \\ &\le c A^{\alpha}_{n-1} \omega_p(1/2^{|n|}, f) \|A^{-\alpha}_{n-2^{|n|}-1} K^{\omega,-\alpha}_{n-2^{|n|}} \circ \tau_{|n|}\|_1 \\ &\le c A^{\alpha}_{n-1} \omega_p(1/2^{|n|}, f) \|A^{-\alpha}_{n-2^{|n|}-1} K^{\omega,-\alpha}_{n-2^{|n|}}\|_1 .\end{aligned}$$

It is well known that

$$\|A^{-\alpha}_{n-1} K^{w,-\alpha}_n\|_1 \le \frac{1}{\alpha} < \infty \tag{7}$$

for $0 < \alpha < 1$ [8] . Thus, the inequality (7) yields

$$\|III_2\|_p \le c 2^{|n|\alpha} \omega_p^2(1/2^{|n|}, f) .$$

Analogously, we have that

$$\|III_3\|_p \le c 2^{|n|\alpha} \omega_p^1(1/2^{|n|}, f) .$$

The discussion of $\|III_4\|_p$ goes analogously. Thus, we write only a few steps of it.

By the disjoint decomposition (4) of G for $l = |n|$ and the fact that the function $\boldsymbol{K}^{w,-\alpha}_{n-2^{|n|}} \circ (\tau_{|n|} \times \tau_{|n|})$ is constant on the set $I_{|n|}(z) \times I_{|n|}(v)$, we write

$$\begin{aligned}\|III_4\|_p = \frac{A^{-\alpha}_{n-2^{|n|}-1}}{A^{-\alpha}_{n-1}} &\sum_{z, v \in X_{|n|}} |\boldsymbol{K}^{\omega,-\alpha}_{n-2^{|n|}}(\tau_{|n|}(z), \tau_{|n|}(v))| \\ &\times \left\| \int_{I_{|n|}(z) \times I_{|n|}(v)} r_{|n|}(x^1) r_{|n|}(x^2)(f(.+x) - f(.)) d\mu(x) \right\|_p\end{aligned}$$

By the inequality (6) we write that

$$\begin{aligned}\|III_4\|_p &\le cA_{n-1}^{\alpha}(\omega_p^1(1/2^{|n|},f)+\omega_p^2(1/2^{|n|},f))\|A_{n-2^{|n|}-1}^{-\alpha}\boldsymbol{K}_{n-2^{|n|}}^{\omega,-\alpha}\circ(\tau_{|n|}\times\tau_{|n|})\|_1\\ &\le cA_{n-1}^{\alpha}(\omega_p^1(1/2^{|n|},f)+\omega_p^2(1/2^{|n|},f))\|A_{n-2^{|n|}-1}^{-\alpha}\boldsymbol{K}_{n-2^{|n|}}^{\omega,-\alpha}\|_1\,.\end{aligned}$$

By Lemma 2.6

$$\|III_4\|_p \le c|n|2^{|n|\alpha}(\omega_p^1(1/2^{|n|},f)+\omega_p^2(1/2^{|n|},f))\,.$$

This completes the proof of Theorem 2.1. □

THEOREM 2.9. *Let f belong to H_1^{ω} and $0<\alpha<1$. Then*

(a)
$$\|\sigma_n^{\kappa,-\alpha}(f)-f\|_1 = O\left(n^{\alpha}\omega\left(\frac{1}{n}\right)\log n\right).$$

(b) *Let $\omega(1/2^A)/(1/2^A)\uparrow\infty$ as $A\to\infty$. Then there exists a function $f\in H_1^{\omega}$ for which*

$$\limsup_{n\to\infty}\frac{\|\sigma_n^{\kappa,-\alpha}(f)-f\|_1}{n^{\alpha}\omega\big(\frac{1}{n}\big)\log n}>0\,.$$

To prove Theorem 2.9 (b) we use the counterexample function given by Goginava in the Walsh case [6].

The construction: Let $\{n_k;k\ge 1\}$ be a monotonicaly increasing sequence of positive integers such that

$$\omega\left(\frac{1}{2^{n_k}}\right)<\frac{1}{2}\omega\left(\frac{1}{2^{n_{k-1}}}\right),\quad k\ge 2$$

and

$$\sum_{l=1}^{k-1}2^{n_l}\omega\left(\frac{1}{2^{n_l}}\right)<\frac{2^{n_k}}{k}\omega\left(\frac{1}{2^{n_k}}\right).$$

Let $\{m_k;k\ge 1\}$ be a sequence of positive integers such that $2^{n_k-1}\le m_k<2^{n_k}$ and $\|D_{m_k}\|_1\ge c\log m_k$. Set

$$f(x,y):=\sum_{k=1}^{\infty}f_k(x,y)\,,$$

where

$$f_k(x,y):=\omega\left(\frac{1}{2^{n_k}}\right)(D_{2^{n_k}}(x)-D_{2^{n_k-1}}(x))(D_{2^{n_k}}(y)-D_{2^{n_k-1}}(y))\,.$$

The function f satiesfies the conditions of Theorem 2.9 (see [6]). The proof of Theorem 2.9 goes analogously as Goginava did in the Walsh case [6, Theorem 5]. Thus, it is left to the readers.

REFERENCES

[1] G. H. AGAEV, N. JA. VILENKIN, G. M. DZHAFARLI AND A. I. RUBINSTEIN, Multiplicative systems of functions and harmonic analysis on zero-dimensional groups, (in Russian), Izd. ("Elm"), Baku, 1981.

[2] G. GÁT, On $(C, 1)$ summability of integrable functions with respect to the Walsh-Kaczmarz system, Studia Math. 130 (2) (1998), 135–148.

[3] V. A. GLUKHOV, Summation of multiple Fourier series in multiplicative systems, (in Russian), Mat. Zametki 39 (1986), 665–673.

[4] U. GOGINAVA, On the convergence and summability of N-dimensional Fourier series with respect to the Walsh-Paley systems in the spaces $L^p([0, 1]^N)$, $p \in [1, \infty]$, Georgian Math. J. 7 (2000) 53–72.

[5] U. GOGINAVA, On the approximation properties of Cesàro means of negative order of Walsh-Fourier series, J. Approx. Theory 115 (2002), 9–20.

[6] U. GOGINAVA, Approximation properties of (C, α) means of double Walsh-Fourier series, Anal. Theory Appl. 20 (1) (2004), 77–98.

[7] U. GOGINAVA, Uniform convergence of Cesàro means of negative order of double trigonometric Fourier series, Anal. Theory Appl. 23 (3) (2007), 255–265.

[8] B. I. GOLUBOV, A. V. EFIMOV AND V. A. SKVORTSOV, Walsh series and transformations, (in Russian), Nauka, Moscow, Math. Appl. (Soviet Ser.) 64, 1987, English transl.: Kluwer Acad. publ., Dordrecht, 1991.

[9] F. SCHIPP, W. R. WADE, P. SIMON AND J. PÁL, Walsh series. An introduction to dyadic harmonic analysis, Adam Hilger, Bristol-New York, 1990.

[10] F. SCHIPP, Pointwise convergence of expansions with respect to certain product systems, Anal. Math. 2 (1976), 65–76.

[11] P. SIMON, (C,α) summability of Walsh-Kaczmarz-Fourier series, J. Approx. Theory 127 (2004), 39–60.

[12] V. A. SKVORTSOV, On Fourier series with respect to the Walsh-Kaczmarz system, Anal. Math. 7 (1981), 141–150.

[13] A. A. ŠNEIDER, On series with respect to the Walsh functions with monotone coefficients, Izvestiya Akad. Nauk SSSR Ser. Mat. 12 (1948), 179–192.

[14] V. I. TEVZADZE, Uniform convergence of Cesàro means of negative order of Walsh-Fourier series, (in Russian), Soobshch. Acad. Nauk. Gruzii SSR 102 (1981), 33–36.

[15] V. I. TEVZADZE, Uniform (C, α) $(-1 < \alpha < 0)$ summability of Fourier series with respect to the Walsh-Paley system, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 22 (2006), 41–61.

[16] W. S. YOUNG, On the a.e. convergence of Walsh-Kaczmarz-Fourier series, Proc. Amer. Math. Soc. 44 (1974), 353–358.

[17] L. V. ZHIZHIASHVILI, Trigonometric Fourier series and their conjugates, (in Russian), Tbilisi, 1993; English transl.: Kluwer Acad. publ., Dordrecht, 1996.

[18] A. ZYGMUND, Trigonometric series, Cambridge Univ. Press, Cambridge, UK, 1959.

INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCES
COLLEGE OF NYÍREGYHÁZA
H-4400 NYÍREGYHÁZA
P.O. BOX 166
HUNGARY

E-mail address: nkaroly@nyf.hu