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# RICCI SOLITONS AND REAL HYPERSURFACES IN A COMPLEX SPACE FORM

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**Abstract.** We prove that a real hypersurface in a non-flat complex space form does not admit a Ricci soliton whose potential vector field is the Reeb vector field. Moreover, we classify a real hypersurface admitting so-called "$\eta$-Ricci soliton" in a non-flat complex space form.

**1. Introduction.** A *Ricci soliton* is a natural generalization of an Einstein metric and is defined on a Riemannian manifold $(M, g)$ by

$$\frac{1}{2}\mathscr{L}_V g + \mathrm{Ric} - \lambda g = 0 \tag{1}$$

where $V$ is a vector field (the potential vector field) and $\lambda$ a constant on $M$. Obviously, a trivial Ricci soliton is an Einstein metric with $V$ zero or Killing. Compact Ricci solitons are the fixed points of the Ricci flow: $(\partial/\partial t)g = -2\,\mathrm{Ric}$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. Hamilton [4] and Ivey [5] proved that a Ricci soliton on a compact manifold has constant curvature in dimension 2 and 3, respectively. If the vector field $V$ is the gradient of a potential function $f$, then $g$ is called a gradient Ricci soliton. Due to Perelman's result [16, Remark 3.2], we find that in a compact Ricci soltion, the potential vector field is written as the sum of a gradient and a Killing vector field. We refer to [3] for details about Ricci solitons or gradient Ricci solitons.

In [6], it was proved that there are no real hypersurfaces with parallel Ricci tensor in a non-flat complex space form $\tilde{M}_n(c)$ with $c \neq 0$ when $n \geq 3$. Furthermore, Kim [8] proved that this is also true when $n = 2$. These results imply, in particular, that there do not exist Einstein real hypersurfaces in a non-flat complex space form.

In this situation, we study on Ricci solitons of real hypersurfaces in a non-flat complex space form. Then we prove that a real hypersurface $M$ in a non-flat complex space form $\tilde{M}_n(c)$ with $c \neq 0$ does not admit a Ricci soliton whose soliton vector field is the Reeb vector field $\xi$ (Corollary 7). In this context, we define so called "$\eta$-*Ricci soliton*" $(\eta, g)$, which satisfies

$$\frac{1}{2}\mathscr{L}_\xi g + \mathrm{Ric} - \lambda g - \mu\eta \otimes \eta = 0$$

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for constants $\lambda$, $\mu$. Then we first prove that a real hypersurface $M$ which admits an $\eta$-Ricci soliton in a non-flat complex space form $\tilde{M}_n(c)$ is a Hopf-hypersurface. Moreover, we classify those $\eta$-Ricci soliton real hypersurfaces in a non-flat complex space form (Theorem 6).

**2. Real hypersurfaces in Kähler manifolds.** In this paper, all manifolds are assumed to be connected and of class $C^\infty$ and the real hypersurfaces are supposed to be oriented.

First, we give a brief review of several fundamental notions and formulas which we will need later on.

Let $\tilde{M}_n$ be a complex $n$-dimensional Kähler manifold and $M$ a real hypersurface of $\tilde{M}_n$. We denote by $\tilde{g}$ and $J$ a Kähler metric tensor and its Hermitian structure tensor, respectively. For any vector field $X$ tangent to $M$, we put

$$JX = \phi X + \eta(X)N\,, \quad JN = -\xi\,, \tag{2}$$

where $\phi$ is a (1,1)-type tensor field, $\eta$ is a 1-form and $\xi$ is a unit vector field on $M$. The induced Riemannian metric on $M$ is denoted by $g$. Then by properties of $(\tilde{g}, J)$, we see that the structure $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $M$, that is, from (2) we can deduce:

$$\phi^2 = -I + \eta \otimes \xi\,, \quad \eta(\xi) = 1\,, \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{4}$$

for all vector fields on $M$.

In the relation between the ambient space and its real hypersurface, the Gauss and Weingarten formula for $M$ are given as

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N\,,$$
$$\tilde{\nabla}_X N = -AX$$

for any tangent vector fields $X$, $Y$, where $\tilde{\nabla}$ and $\nabla$ denote the Levi-Civita connection of $(M_n(c), \tilde{g})$ and $(M, g)$, respectively and $A$ is the shape operator field. From (2) and $\tilde{\nabla} J = 0$, we obtain

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi\,, \tag{5}$$

$$\nabla_X \xi = \phi AX\,. \tag{6}$$

We define a vector field $U$ on $M$ by $U = \nabla_\xi \xi$. Then, from (6), we easily observe that

$$g(U, \xi) = 0\,, \quad g(U, A\xi) = 0\,, \quad \|U\|^2 = g(U, U) = \alpha_2 - \alpha_1^2\,, \tag{7}$$

where $\alpha_1 = g(A\xi, \xi)$ and $\alpha_2 = g(A^2\xi, \xi)$. From (4), we have the following lemma immediately.

LEMMA 1. *$A\xi = \alpha_1 \xi$ if and only if $\|U\|^2 = 0$.*

Now we suppose that the ambient space $\tilde{M} = \tilde{M}_n(c)$ is a complex space form. Then we have the following Gauss and Codazzi equations:

$$\begin{aligned} R(X, Y)Z =& \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ & + g(AY, Z)AX - g(AX, Z)AY\,, \end{aligned} \tag{8}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{9}$$

for any tangent vector fields $X, Y, Z$ on $M$. From (8), we get for the Ricci tensor $S$ of type (1,1):

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X\,, \tag{10}$$

where $h$ (= trace of $A$) denotes the mean curvature. Then we have the relation $\mathrm{Ric}(X, Y) = g(SX, Y)$.

We prepare some more results which are needed later to prove ours. Let $M$ be a *Hopf hypersurface*, which means that the Reeb vector field $\xi$ is a principal curvature vector field ($A\xi = \alpha_1\xi$), in a non-flat complex space form $\tilde{M}_n(c)$, ($c \neq 0$). Then we already know that $\alpha_1$ is a constant (cf. [7], [10], [11]). Differentiating $A\xi = \alpha_1\xi$ covariantly, we get

$$(\nabla_X A)\xi = \alpha_1\phi AX - A\phi AX$$

by using (6). Use the Codazzi equation (9) to obtain again

$$(\nabla_\xi A)X = \frac{c}{4}\phi X + \alpha_1\phi AX - A\phi AX$$

for any vector field $X$ on $M$. Since $\nabla_\xi A$ is self-adjoint, by taking the anti-symmetric part of the above equation, we have the relation:

$$2A\phi AX - \frac{c}{2}\phi X = \alpha_1(\phi A + A\phi)X\,.$$

Here we assume that $AX = fX$, $X \perp \xi$, $\|X\| = 1$. Then it follows that

$$(2f - \alpha_1)A\phi X = \left(f\alpha_1 + \frac{c}{2}\right)\phi X\,.$$

The case $2f = \alpha_1$ yields $f^2 = -c/4$, which determines the horosphere in $H_n\boldsymbol{C}$ (cf. [1]). In fact the shape operator of the horosphere is written as $A = I + \eta \otimes \xi$. Hence, we have the following lemma.

LEMMA 2. *For a Hopf hypersurface $M$ in a non-flat complex space form $\tilde{M}_n(c)$, $\phi X$ is a principal direction if $X(\perp \xi)$ is a principal direction.*

Takagi [17], [18] classified the homogeneous real hypersurfaces of $P_n\boldsymbol{C}$ into six types $\mathrm{A}_1$, $\mathrm{A}_2$, B, C, D, E. Cecil and Ryan [2] extensively studied a Hopf hypersurface which is realized as tubes over certain submanifolds in $P_n\boldsymbol{C}$ by using its focal map. By making use of those results and the mentioned work of R. Takagi, M. Kimura [9] proved the local classification theorem for Hopf hypersurfaces of $P_n\boldsymbol{C}$ whose all principal curvatures are constant.

As mentioned in Introduction, a real hypersurface $M$ in a non-flat complex space form $\tilde{M}_n(c)$ does not admit Einstein metric. In this context, M. Kon [10] studied and classified *pseudo-Einstein* (or $\eta$-Einstein) real hypersurfaces in a complex space form. The term means that there are constants $\lambda$ and $\mu$ such that

$$S = \lambda I + \mu\eta \otimes \xi\ ,$$

where $I$ denotes the identity transformation. Later, Cecil and Ryan [2], Montiel [12] developed the results for $P_n\boldsymbol{C}$, $H_n\boldsymbol{C}$, respectively. In reality, they classified those real hypersurfaces in $P_n\boldsymbol{C}$ or $H_n\boldsymbol{C}$ for $n \geq 3$ and for smooth functions $\lambda$ and $\mu$.

THEOREM 3 ([2], [10] ). *Let $M^{2n-1}$ $(n \geq 3)$ be a real hypersurface of $P_n\boldsymbol{C}$ with Fubini-study metric of constant holomorphic sectional curvature* 4. *Then $M$ is pseudo-Einstein if and only if $M$ is locally congruent to one of the following*:

(A$_1$) *a geodesic hypersphere of radius $r$, where* $0 < r < \pi/2$,

(A$_2$) *a tube of radius $r$ over a totally geodesic $P_l\boldsymbol{C}$ $(1 \leq l \leq n-2)$, where* $0 < r < \pi/2$ *and* $\cot^2 r = k/(n-k-1)$,

(B) *a tube of radius $r$ over a complex quadric $Q^{n-1}$ and $P_n\boldsymbol{R}$, where* $0 < r < \pi/4$ *and* $\cot^2 2r = n-2$.

For the case $H_n\boldsymbol{C}$, Berndt [1] proved the classification theorem for Hopf hypersurfaces whose all principal curvatures are constant.

THEOREM 4 (Montiel [12]). *Let $M^{2n-1}$ $(n \geq 3)$ be a real hypersurface of $H_n\boldsymbol{C}$ with Bergman metric of constant holomorphic sectional curvature* $-4$. *Then $M$ is pseudo-Einstein if and only if $M$ is locally congruent to one of the following*:

(A$_0$) *a horosphere*,

(A$_1$) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\boldsymbol{C}$*,

(A$_2$) *a tube over a totally geodesic $H_l\boldsymbol{C}$* $(1 \leq l \leq n-2)$.

A real hypersurface of type A$_1$, A$_2$ (without extra restriction $\cot^2 r = k/(n-k-1)$) in Theorem 3 and of type A$_0$, A$_1$, A$_2$ in Theorem 4 are simply called a real hypersurface of type A. There are many characterizations of real hypersurfaces of type A (cf. [14]). In particular, Okumura ([15]) (resp. Montiel and Romero ([13])) proved that $\phi A = A\phi$ if and only if $M$ is locally congruent to one of type A in $P_nC$ (resp. $H_nC$).

**3. Real hypersurfaces with Ricci solitons in a complex space form.** In view of those results of Einstein or pseudo-Einstein (or $\eta$-Einstein) real hypersurfaces in a complex space form, we introduce *$\eta$-Ricci soliton* on real hypersurfaces in a Kähler manifold:

DEFINITION 1. Let $M$ be a real hypersurface in a Kähler manifold $\tilde{M}_n$. If $M$ satisfies

$$\frac{1}{2}\mathscr{L}_\xi g + \mathrm{Ric} - \lambda g - \mu\eta \otimes \eta = 0 \tag{11}$$

for constants $\lambda$, $\mu$, then we say that $M$ admits an *$\eta$-Ricci soliton* (*with the soliton vector field* $\xi$). When $\mu = 0$, it includes a Ricci soliton with the soliton vector field $\xi$.

By using (6), we find that

$$(\mathscr{L}_\xi g)(X,Y) = g((\phi A - A\phi)X, Y)\,.$$

Suppose that $M$ admits an $\eta$-Ricci soliton. Then from (11), by using the relation above and (10), we have

$$\left(A^2 - hA - \frac{1}{2}(\phi A - A\phi) - \frac{c}{4}(2n+1) + \lambda\right)X = -\left(\mu + \frac{3}{4}c\right)\eta(X)\xi\,. \tag{12}$$

First, we prove that $\xi$ is a principal curvature vector. If we put $X = \xi$ in (12), then we get

$$A^2\xi - hA\xi - \frac{1}{2}U + \left(\lambda + \mu + \frac{c}{2}(1-n)\right)\xi = 0 \tag{13}$$

from (6). Take the $\xi$-component of (13) to get

$$\alpha_2 - \alpha_1 h = -\left(\lambda + \mu + \frac{c}{2}(1-n)\right), \tag{14}$$

and then (13) gives

$$A^2\xi = hA\xi + \frac{1}{2}U + (\alpha_2 - \alpha_1 h)\xi\,. \tag{15}$$

If we take an inner product (15) with $U$, then we get

$$g(A\xi, AU) = \frac{1}{2}\|U\|^2\,, \tag{16}$$

where we have used the equalities $g(\xi, U) = g(A\xi, U) = 0$.

We put

$$Q := A^2 - hA - \frac{1}{2}(\phi A - A\phi) - \left(\frac{c}{4}(2n+1) - \lambda\right)I\,, \tag{17}$$

where $I$ denotes the identity transformation. Then we see that $Q$ is a symmetric operator, and (12) is rewritten as

$$QX = -\left(\mu + \frac{3}{4}c\right)\eta(X)\xi\,.$$

Now, we compute $AQ - QA$. Then from (17), we have

$$\frac{1}{2}(\phi A^2 + A^2\phi)X - A\phi AX = -\left(\mu + \frac{3}{4}c\right)(\eta(X)A\xi - \eta(AX)\xi)\,. \tag{18}$$

Putting $X = \xi$, then it follows that

$$\frac{1}{2}\phi A^2\xi - AU = -\left(\mu + \frac{3}{4}c\right)(A\xi - \alpha_1\xi)\,.$$

Applying $\phi$ and using (3), then we get

$$A^2\xi = \alpha_2\xi - 2\phi AU + \left(2\mu + \frac{3}{2}c\right)U\,. \tag{19}$$

From (15) and (19), we obtain

$$hA\xi - \alpha_1 h\xi - \left(2\mu + \frac{3}{2}c - \frac{1}{2}\right)U + 2\phi AU = 0\,,$$

or

$$2AU + \left(2\mu + \frac{3}{2}c - \frac{1}{2}\right)\phi U - hU = 0 \tag{20}$$

by applying $\phi$. Taking an inner product $U$ with (20) and using (4) we get

$$2g(AU, U) = h\|U\|^2\,. \tag{21}$$

Take an inner product $A\xi$ with (20) to get

$$g(AU, A\xi) = \frac{1}{2}\left(2\mu + \frac{3}{2}c - \frac{1}{2}\right)\|U\|^2\,. \tag{22}$$

Hence, together with (16), we obtain

$$\left(2\mu + \frac{3}{2}c - \frac{3}{2}\right)\|U\|^2 = 0\,,$$

which together with Lemma 1 yields that $A\xi = \alpha_1\xi$ if $(2\mu + 3c/2 - 3/2) \neq 0$.

Now we consider the case $2\mu + 3c/2 - 3/2 = 0$: then (20) gives

$$2AU = -\phi U + hU\,. \tag{23}$$

This time we put $X = U$ in (12), then we get

$$A^2U - hAU - \frac{1}{2}(\phi AU - A\phi U) + \left(\alpha_1 h - \alpha_2 - \frac{3}{4}\right)U = 0\,, \tag{24}$$

where we have used (14) and $\mu + 3c/4 = 3/4$. The inner product of (24) and $U$ is the sum of the left hand sides of

$$\begin{aligned} g(A^2U, U) = g(AU, AU) &= g\left(-\frac{1}{2}\phi U + \frac{h}{2}U, -\frac{1}{2}\phi U + \frac{h}{2}U\right) \\ &= \frac{1}{4}\|U\|^2 + \frac{h^2}{4}\|U\|^2\,, \text{ (use (23))} \end{aligned}$$

$$\begin{aligned} g(-hAU, U) &= -\frac{h^2}{2}\|U\|^2\,, \text{ (use (21))} \\ g(AU, \phi U) &= g\left(-\frac{1}{2}\phi U + \frac{h}{2}U, \phi U\right) = -\frac{1}{2}\|U\|^2 \end{aligned}$$

and $(\alpha_1 h - \alpha_2 - 3/4)\|U\|^2$, which is equal to

$$\frac{1}{4}\|U\|^2 + \frac{h^2}{4}\|U\|^2 - \frac{h^2}{2}\|U\|^2 - \frac{1}{2}\|U\|^2 + (\alpha_1 h - \alpha_2 - 3/4)\|U\|^2\,,$$

where $\alpha_2 = \|U\|^2 + \alpha_1^2$ by (7). Since $-4$ times the coefficient is

$$h^2 - 4\alpha_1 h + 4(\alpha_1^2 + \|U\|^2 + 1) = (h - 2\alpha_1)^2 + 4(\|U\|^2 + 1) > 0$$

is not zero, we have $\|U\|^2 = 0$ by (24). Thus, we have the following proposition.

PROPOSITION 5. *If $M$ admits an $\eta$-Ricci soliton, then $\xi$ is a principal curvature vector.*

We assume that $X$ is principal direction orthogonal to $\xi$ in (12). Then since $\xi$ is a principal curvature vector field, we can see that $\phi A = A\phi$ and $M$ is pseudo-Einstein by Lemma 2. Due to the classification theorems of real hypersurfaces in $P_n\boldsymbol{C}$ or $H_n\boldsymbol{C}$ which satisfy $\phi A = A\phi$ ([13], [15]) or which admit pseudo-Einstein structure (Theorems 3 and 4) we have the following theorem.

THEOREM 6. *Let $M$ be a real hypersurface in a non-flat complex space forms $\tilde{M}_n(c)$ with $c \neq 0$. If $M$ admits an $\eta$-Ricci soliton, then $M$ is a Hopf hypersurface and is locally congruent to one of the following real hypersurfaces*: (i) *a geodesic hypersphere in $P_n\boldsymbol{C}$ or $H_n\boldsymbol{C}$, a horosphere in $H_n\boldsymbol{C}$*, (ii) *a homogeneous tube over totally geodesic complex hyperbolic hyperplane $H_{n-1}\boldsymbol{C}$ in $H_n\boldsymbol{C}$*, (iii) *a homogeneous tube of radius $r$ over a totally geodesic $P_l\boldsymbol{C}$ $(1 \leq l \leq n-2)$, where* $0 < r < \pi/2$ *and* $\cot^2 r = k/(n-k-1)$, (iv) *a homogeneous tube over totally geodesic $H_l\boldsymbol{C}$ $(1 \leq l \leq n-2)$.*

Since the equation (11) with $\mu = 0$ is reduced to a Ricci soliton equation, we have the following corollary.

COROLLARY 7. *A real hypersurface in a non-flat complex space form does not admit a Ricci soliton with the soliton vector field $\xi$.*

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