# ON NON-CONNECTED MAXIMALLY ALMOST PERIODIC GROUPS[*]

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All groups which we shall consider in this note are locally compact. If $G$ is a group, let $G^0$ be the connected component of the identity in $G$. Under the restriction of connectedness a necessary and sufficient condition for $G$ to be maximaly almost periodic (m.a.p. for simplicity) is, as well known, that $G$ is decomposed into a direct product of a compact group and a vector group. In the present note, we shall consider the non-connected case where however $G/G^0$ is compact. Firstly, an analogous proposition as above is not valid in this case (cf. Remark 1), and only the space of $G$ is decomposed into the topological direct product of the space of a compact subgroup and that of a closed subgroup which is isomorphic with a vector group (cf. Remark to Theorem A). Moreover, the criterion for the maximally almost periodicity of $G$ is given by that of $G^0$. In fact, if $G^0$ is m.a.p., $G$ is also m.a.p. (Theorem B).

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Lemma 1[1]. *Let $N$ be a closed normal subgroup of $G$. If $N$ is a vector group and $G/N$ is a compact group, $G$ contains a compact subgroup $K$ such as $K{\cdot}N=G$, $K \cap N=\{e\}$*[2].

Theorem A. *Let $G/G^0$ be compact and $G^0$ be m.a.p. Then $G$ contains a compact group $K$ and a vector group $N$ such as*

$$G=K{\cdot}N, \qquad K\cap N=\{e\}.$$

Proof. As $G^0$ is connected and m.a.p., $G^0=K_1\times N$, where $K_1$ is a compact normal subgroup of $G$ and $N$ is a vector group. As $K_1$ is compact, $G^0/K_1\,(\simeq N)$ is a closed normal subgroup of $G/K_1$ and $(G/K_1)/(G^0/K_1)$ is ccmpact. By Lemma 1, there exists a compact subgroup $K$ of $G$ containing $K_1$ such as

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1) This lemma is obtained by K. Iwasawa. For the proof see Iwasawa [2], esp. Lem. 3.8.

2) In this note $e$ denotes the identity of groups.

$G/K_1 = K/K_1 \cdot G^0/K_1$, $K/K_1 \cap G^0/K_1 = \{e\}$. Thence $K \cap G^0 \subset K_1$, and in particular $K \cap N = \{e\}$. Therefore $G = K \cdot G^0 = K \cdot K_1 \cdot N = K \cdot N$, $K \cap N = \{e\}$.

Lemma 2. *Let $G^0$ be m.a.p. Lie group, and let $G/G^0$ be compact. As $G^0$ is m.a.p. and connected, we can put $G^0 = K \times N$, where $K$ is a characteristic compact subgroup $G$ and $N$ is a vector group. If here we take suitable $N$, $N$ is also a normal subgroup of $G$.*

Proof. Let $Z$ be the center of $G^0$, and let $Z^0$ be the connected component of the identity in $Z$. It is easy to see that $Z^0 = T \times N$, where $T$ is torous group and is the connected component of the identity in the center of $K$. Transformation by an arbitrary element $g$ of $G$ induces an automorphism $A_g$ of $Z^0$: $Z^0 \ni x \rightarrow g^{-1} x g$. $A_g$ is a linear transformation of local vector group $V$ which is a sufficiently small neighborhood of the identity in $Z^0$, and since $T$ is a normal subgroup of $G$, $V \cap T$ is invariant by arbitrary $A_g$. Let $A$ be the group consisting of the totality of $A_g$. Since for any element $g$ of $G^0$, $A_g$ is an identity transformation, $A$ is isomophic with a factor group of $G/G^0$, which is compact and 0-dimensional. On the other hand, $A$ is a matrix group. This means that $A$ is a finite group. As any representation of a finite group is completely reducible, $V$ contains a local linear subspace $N$ which is locally isomorphic with $N$. On computing $N$ explicitly by making use of the coordinates in $V$ and the matrix form of $A_g$, it is easy to see that $N'$ generates a closed subgroup $M$ which is isomorphic with a vector group and that $G^0 = K \times M$. As $M$ is invariant by all of $A$, $M$ is a normal subgroup of $G$.

Remark to Theorem A. As $K$ is compact, it is easy to deduce from Theorem A that the space of $G$ is topological direct product of those of $K$ and $N$.

Lemma 3. *Let $H$ be a finite matrix group of degree $r$ and order $s$, $H = \{A_1, A_2, \cdots, A_s\}$. Let $G$ be the linear group consisting of all matrices of the form*

$$\begin{pmatrix} & & x_1 \\ & A_i & x_2 \\ & & \vdots \\ & & x_r \\ 0\,0 & \cdots\,0 & 1 \end{pmatrix}$$

*where $x_j$ are real numbers $(j = 1, \cdots, r)$ and $i = 1, \cdots, s$. Then $G$ is m.a.p.*

Proof. We put

$$G_1 = \left\{ \begin{pmatrix} & & x_1 \\ & A_i & x_2 \\ & & \vdots \\ & & x_r \\ 0\,0 & \cdots\,0 & 1 \end{pmatrix};\ x_j \text{ are complex numbers, } j = 1, \cdots, r;\ i = 1, \cdots, s. \right\}$$

$$G_B = \left\{ \begin{pmatrix} BA_i B^{-1} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{matrix} \\ 0\,0 \cdots 0 & 1 \end{pmatrix} ;\ x_j \text{ are complex numbers, } j = 1, \cdots, r;\ i = 1, \cdots, s. \right\}.$$

Taking a suitable $B$ we can get a group $G_2$ whose element has a following form

$$\begin{pmatrix} A_1^i & & & & x_1 \\ & A_2^i & & 0 & x_2 \\ & & \ddots & & \vdots \\ 0 & & & A_t^i & x_r \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

where for each $k = 1, 2, \cdots, t$, $A_i \to A_i^k$ is an irreducible representation of the finite group $H$. Let $u_k$ be the degree of this irreducible representation.

Put

$$G_3 = \left\{ \begin{pmatrix} A_{i_1}^1 & & & & x_1 \\ & A_{i_2}^2 & & 0 & x_2 \\ & & \ddots & & \vdots \\ 0 & & & A_i^i & x_r \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} ;\ x_j \text{ are complex numbers, } j = 1, \cdots, r,\ i_l = 1, \cdots, s;\ l = 1, \cdots, t, \right\}$$

and

$$G_4^k = \left\{ \begin{pmatrix} A_i^k & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_{u_k} \end{matrix} \\ 0\,0 \cdots\cdots 0 & 1 \end{pmatrix} ;\ x_j \text{ are complex numbers, } j = 1, \cdots, u_k,\ i = 1, \cdots, s. \right\}.$$

Put

$$G_5 = \left\{ \begin{pmatrix} R_i & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{matrix} \\ 0\,0 \cdots\cdots 0 & 1 \end{pmatrix} ;\ x_j \text{ are comlex numbers, } j = 1, \cdots, s;\ i = 1, \cdots, s. \right\}$$

where $A_i \to R_i$ is the regular representation of the finite group $H$. Put further

$$G_{5c} = \left\{ \begin{pmatrix} CR_i C^{-1} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{matrix} \\ 0\,0 \cdots\cdots 0 & 1 \end{pmatrix} ;\ x_j \text{ are complex numbers, } j = 1, \cdots, s;\ i = 1, \cdots, s \right\}.$$

Taking a suitable $C$ we can get a group $G_6$ whose element has a follwing form

$$\begin{pmatrix} R_i^1 & & & x_1 \\ & & 0 & x_2 \\ R_i^2 & & & \vdots \\ 0 & \ddots & & \vdots \\ & & R_i^p & x_s \\ 0 & 0 \cdots & 0 & 1 \end{pmatrix}$$

where for each $h = 1, 2, \cdots, p$, $A_i \to R_i^h$ is an irreducible representation of $H$. As is well known, any irreducible representation of $H$ is equivalent to one of the reprsentation $A_i \to R_i^h$. Let $A_i \to A_i^k$ be equivalent to $A_i \to R_i^{\varphi(k)}$. Put

$$G_6^k = \left\{ \begin{pmatrix} & & 0 \\ & & \vdots \\ R_i^1 \cdots\cdots\cdots & & 0 \\ \ddots & & x_{m+1} \\ & R_i^{\varphi(k)} \cdots & x_{m+ku} \\ & \ddots & 0 \\ & & \vdots \\ & R_i^p & 0 \\ 0\,0 \cdots\cdots & 0 & 1 \end{pmatrix} ;\ \begin{array}{l} x_j \text{ are complex numbers,} \\ j = m+1, \cdots, m+u_k,\ i=1, \cdots, s, \\ m = u_1 + u_2 + \cdots + u_{k-1}. \end{array} \right\}$$

for each $k = 1, 2, \cdots, t$. Clearly, $G \subset G_1$, $G_1 \cong G_2$, $G_2 \subset G_3$, $G_3 = G_4^1 \times G_4^2 \times \cdots \times G_4^t$,

$$G_4^k \cong G_6^k,\ G_6^k \subset G_6,\ k = 1, 2, \cdots, t, \text{ and } G_6 = G_5.$$

Therefore our proof is completed when we can prove that $G_5$ is m.a.p. Put, for this purpose,

$$D_1 = \left\{ \begin{pmatrix} & & n_1 \\ & & n_2 \\ & E & \vdots \\ & & n_s \\ 0 \cdots\cdots & 0 & 1 \end{pmatrix} ;\ n_j \text{ are integrs, } j = 1, 2, \cdots, s \right\},$$

$$D_2 = \left\{ \begin{pmatrix} & & n_1\sqrt{2} \\ & & n_2\sqrt{2} \\ & E & \vdots \\ & & n_s\sqrt{2} \\ 0 \cdots\cdots & 0 & 1 \end{pmatrix} ;\ n_j \text{ are integers, } j = 1, 2, \cdots, s \right\}$$

where $E$ is an identity matrix of degree $s$. Since each component of the matrix $R_i$ is an integer, we can see easily that $D_1$ and $D_2$ are both normal subgroup of $G_5$. Moreover, $D_1 \cap D_2 = \{e\}$, and $G_5/D_i$ are compact groups for $i = 1, 2$. Thus $G_5$ is an m.a.p. group, which completes our proof.

Corollary to Lemma 3. *If $G/G^0$ is compact, and if $G^0$ is isomorphic with a vector group, then $G$ is m.a.p.*

Proof. Let $G^0$ be an $r$-dimensional vector group. The transformation by any element $g$ of $G$ induces an automorphism $A_g$ of $G^0$.

$$A_g: \qquad G^0 \ni a \to g^{-1}ag.$$

As $G^0$ is a vector group, $A_g$ is a linear transformation and its matrix may be also denoted by $A_g$. Let $A$ be the group consisting of $A_g$. $A$ is a finite group (cf. the proof of Lemma 2). By Lemma 1, $G = K\cdot G^0$, $K \cap G^0 = \{e\}$. It is easy to see that $G \ni g = kg'$, $(k \in K,\ g' = (x_1, x_2, \cdots, x_r) \in G^0)$,

$$\longrightarrow \begin{pmatrix} & & x_1 \\ & A_k & x_2 \\ & & \vdots \\ & & x_r \\ 0\,0 & \cdots\,0 & 1 \end{pmatrix}$$

is a representation of $G$. Let $Z$ be the kernel of this representation. Then $Z \cap G^0 = \{e\}$, and by Lemma 3 $G/Z$ is m.a.p. On the other hand, $G/G^0$ is m.a.p. Thus we have proved that $G$ is m.a.p.

THEOREM B. *Let $G/G^0$ be compact. A necessary and sufficient condition for $G$ to be m.a.p. is that $G^0$ is m.a.p.*

PROOF. Only the sufficiency has to be proved. Taking the same notation as in Theorem A, $G = KN$, $K \cap N = \{e\}$, and $G^0 = K_1 \times N$ where $K_1 \subset K$.

(i) When $G^0$ is m.a.p. Lie group. Then by Lemma 2 $N$ can be taken as a closed normal subgroup of $G$ which is isomorphic with a vector group. $G/N$ is m.a.p., and by Corollary to Lemm 3, $G/K_1$ is also m.a.p. This means that $G$ is m.a.p.

(ii) When $G^0$ is m.a.p. As $K$ is compact, there exists a collection $\{A_\alpha\}$ of normal subgroups of $K$ such that

$$\bigcap A_\alpha = \{e\}, \qquad K/A_\alpha \text{ is a Lie group.}$$

Put $B_\alpha = A_\alpha \cap K_1$. As $B_\alpha$ is contained in $K_1$, any element of $B_\alpha$ is commutative with any element of $N$, and moreover $B_\alpha$ is a normal subgroup of $K$. This means that $B_\alpha$ is a normal subgroup of $G$. As

$$K_1/B_\alpha = K_1/(A_\alpha \cap K_1) \cong K_1A_\alpha/A_\alpha \subset K/A_\alpha\,,$$

$K_1/B_\alpha$ is a Lie group. It is easy to see that $(G/B_\alpha)^0 = G^0/B_\alpha$.

Summalizing the above results, there exists a collection $\{B_\alpha\}$ of normal subgroups of $G$ contained in $G^0$ such that

$$\bigcap_\alpha B_\alpha = \{e\}, \qquad (G/B_\alpha)^0 \text{ is m.a.p. Lie group,}$$
$$G/B_\alpha/(G/B_\alpha)^0 \text{ is compact.}$$

Thus we have proved that $G$ is m.a.p.

PEMARK 1. We can not extend the structure theorem of a connected locally compact m.a.p. group to our non-connected case, and van Kampen's conjecture is not valid[3]. For example[4], consider the following linear group $G$,

$$G=\left\{\begin{pmatrix}\varepsilon & x\\ 0 & 1\end{pmatrix};\ \varepsilon=\pm 1,\ x=\text{real number}\right\}.$$

$G$ is a locally compact group, $G^0$ consists of the matrices of the form $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ and is isomorphic with the additive group of real numbers. Put

$$D_1=\left\{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix};\ n=\text{integer}\right\},$$

$$D_2=\left\{\begin{pmatrix}1 & n\sqrt{2}\\ 0 & 1\end{pmatrix};\ n=\text{integer}\right\}.$$

Then $D_1$ and $D_2$ are normal subgroups of $G$, $G/D_1$ and $G/D_2$ are compact groups, and $D_1\cap D_2=\{e\}$. Therefore $G$ is m.a.p. Assume that $G=K\times N$, where $K$ is a compact group and $N$ is a vector group. Then it is easy to see that $K\sim G/G^0$ and $N=G^0$. Hence $G$ is a commutative group, which is a contradiction.

REMARK 2. That $G^0$ is the component is essential in Theorem A. In fact from the fact that $H$ is an m.a.p. normal subgroup of $G$ and that $G/H$ is a compact group, we cannot deduce in general that $G$ is m.a.p. For example,

$$G=\left\{\begin{pmatrix}\cos\theta & \sin\theta & a\\ -\sin\theta & \cos\theta & b\\ 0 & 0 & 1\end{pmatrix};\ 0\leqq\theta<2\pi;\ a,\ b\ \text{real numbers}\right\},$$

$$H=\left\{\begin{pmatrix}1 & 0 & a\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix};\ a,\ b\ \text{real numbers}\right\}.$$

$H$ is an m.a.p. normal subgroup of $G$, and $G/H$ is compact. But $G$ is not m.a.p.

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3) See van Kampen [3].
4) This example was obtained jointly with H. Tôyama.

## Bibliography

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