COMPACT SET IN UNIFORM SPACE AND FUNCTIONS SPACES*)

BY

HISAHARU UMEGAKI

The purpose of this paper is to discuss some compactness problems in uniform space and in a space of continuous functions whose domain and range are both uniform spaces. It is known that a uniform structure in uniform space may be represented by a family of pseudo-metrics. Using this we shall prove a convex linear topological space can be imbedded into a direct product of normed spaces (§ 1). We shall next prove a compactness theorem of the Kolmogoroff-Tulajkov type in uniform space (§ 2). We introduce furthermore some topologies into the space of continuous functions and prove a compactness theorem of the Ascoli-Arzelà type, and as an application we shall prove similar theorem for character group of topological group.

§ 1. **On the uniform structure.** Let E be a uniform space defined by the uniform structure $\{V_\alpha\}_{\alpha\in\mathfrak{A}}$. After A. Weil, for each V_α we shall define pseudo-metric d_α such that:

$$d_\alpha(p, q) \geqq 0, \ d_\alpha(p, p) = 0, \ d_\alpha(p, q) \leqq d_\alpha(p, r) + d_\alpha(r, q)$$

and $p = q \longleftrightarrow d_\alpha(p, q) = 0$ for all $\alpha\in\mathfrak{A}$.

We define a structure by $W_{\alpha\varepsilon} = \{(p, q); d_\alpha(p, q) < \varepsilon\}$; then $\{W_{\alpha\varepsilon}\}_{\alpha, \varepsilon > 0}$ is equivalent to $\{V_\alpha\}$. We can replace the triangle condition of $\{d_\alpha\}$ by the following: for each $\alpha\in\mathfrak{A}$, there exists $\beta_\alpha = \beta\in\mathfrak{A}$ such that $d_\alpha(p, q) \leqq d_\beta(p, r) + d_\beta(r, q)$.

The same consideration can be applied for linear topological space (l. t. s.).

Let L be a l. t. s. defined by the neighbourhood (nbd.) system $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ of the origin θ. D. H. Hyers [1] has proved that there exists a family of pseudo-norms $\{|\cdot|_\alpha\}$ satisfying the following conditions:

(a) for every $x\in L$ and $\alpha\in\mathfrak{A}$, $|x|_\alpha \geqq 0$.

(b) for every real λ, $x\in L$ and $\alpha\in\mathfrak{A}$, $|\lambda x|_\alpha = |\lambda| \cdot |x|_\alpha$.

(c) for every $\alpha\in\mathfrak{A}$, there exists $\beta_\alpha = \beta\in\mathfrak{A}$ such that $|x+y|_\alpha \leqq |x|_\beta + |y|_\beta$ for all $x, y\in L$.

*) Received April 3, 1950.

(d) $x = \theta \longleftrightarrow |x|_\alpha = 0$ for all $\alpha \in \mathfrak{A}$.

Then the topology of L is equivalent to the topology defined by $\{|\cdot|_\alpha\}$ L is convex if and only if

(c′) for every $\alpha \in \mathfrak{A}$, $|x+y|_\alpha \leqq |x|_\alpha + |y|_\alpha$.

REMARK. We can introduce another equivalent family of the norms satisfying the conditions (a), (c′), (d) and the following: if $\lambda_n \to 0$ or $\| x_n \|_\alpha \to 0$, then $\| \lambda_n x \|_\alpha \to 0$ or $\| \lambda x_n \|_\alpha \to 0$ for all $\alpha \in \mathfrak{A}$. By this norm, L is convex if and only if L satisfies (b).

Moreover, for a topological algebra A, we can introduce a family of pseu do-norms $| \cdot |_\alpha$ satisfying the conditions (a), (b), (c), (d) and the following: for any $\alpha \in \mathfrak{A}$ there exists $\beta \in \mathfrak{A}$ such that $| x \cdot y |_\alpha \leqq | x |_\beta \cdot | y |_\beta$.

After these preparations we shall prove the following theorem:

THEOREM 1.1. *A convex l. t. s. can be imbedded into the closed linear manifold of the direct product of normed linear spaces.*

PROOF. Let $N_\alpha = \{x ; \| x \|_\alpha = 0\}$, then N_α is a closed linear subspace of L for each $\alpha \in \mathfrak{A}$. Since, if $y \in x + N_\alpha$, then $\| x \|_\alpha = \| y \|_\alpha$, the quotient spaces $L_\alpha = L - N_\alpha$ are linear and normed by the norm $\| x_\alpha \| = \| x \|_\alpha$ for $x_\alpha \equiv x + N_\alpha$. Then the natural mapping from L onto L_α is homomorphic and isometric. Let ΠL_α be the direct product of L_α, $\alpha \in \mathfrak{A}$, and L' be a set of all the elements of L_α such that $x' = \{x_\alpha\}$ with $x_\alpha = x + N_\alpha$, $x \in L$. The topology of ΠL_α is defined by $|x'| = \| x_\alpha \|$, $\alpha \in \mathfrak{A}$, where $x' \in \Pi L_\alpha$, $x' = x$. Then ΠL_α is a l. t. s. and L' is a closed linear manifold of ΠL_α which is topologically isomorphic with L by the natural mapping.

From the proof we can easily see the following corollary, when we remind that N_α is a closed ideal in our case:

COROLLARY. *A topological algebra A, satisfying the conditions* (a), (b), (c′), (d) *and the condition* (e): $| xy |_\alpha \leqq | x |_\alpha \cdot | y |_\alpha$, *can be imbedded into the closed subalgebra of the direct product of normed algebras.*

§ 2. **A compactness theorem in the uniform space.** In this section we prove a theorem of the Kolmogoroff-Tulajkov type in uniform space, (proved by in the case of B-space P. Phillips [2]). We shall now introduce the concept of boundedness and complete continuity into the uniform space. It is called that a subset S of a uniform space E is precompact if the completion $\bar{S}$ is compact (N. Bourbaki [3]).

DEFINITION 1. A subset S of E is said to be bounded if, for any $\alpha \in \mathfrak{A}$ there exists a constant $M_\alpha > 0$ such that $d_\alpha(S) < M_\alpha$, where $d_\alpha(S)$ is the least upper bound of $d_\alpha(p, q)$ for $p, q \in S$.

This definition of boundedness is coincident in l. t. s. with that of Banach and J. von Neumann. It is clear that any uniform space is uniformly homeomorph with bounded uniform space, and any precompact set is bounded.

DEFINITION 2. A transformation from E into E' is said to be completely continuous if it transforms every bounded set of E into a precompact set of E' (c. c. t. = completely continuous transformation).

We suppose that $\Pi = \{\pi\}$ has the Moore-Smith property with respect to the order "$\geq$", and for every $\pi \in \Pi$ there corresponds a c. c. t. T_π from E into E. Then it is easily seen that the uniform limit of T_π is also c. c. t. Moreover the sequence $\{T_\pi\}$ satisfies the following conditions.

(1) $T_\pi p$ convergs to p for every $p \in E$,

(2) for every $\alpha \in \mathfrak{A}$ and every $\epsilon > 0$, there exist $\Pi_{\alpha\varepsilon} \in \Pi$, $\beta_{\alpha\varepsilon} = \beta \in \mathfrak{A}$ and $\delta_{\alpha\varepsilon} > 0$ such that

$$d_\beta(p, q) < \delta_{\alpha\varepsilon} \rightarrow d_\alpha(T_\pi p, T_\pi q) < \epsilon \text{ for all } \pi \geq \pi_{\alpha\varepsilon}.$$

THEOREM 2. *A set S in E is precompact if and only if*

(1°) *S is bounded,*

(2°) *$\{T_\pi p\}$ converges uniformly to p in S.*

PROOF. Suppose that S is precompact. For any $\alpha \in \mathfrak{A}$ and any $\epsilon > 0$, there exist $\beta \in \mathfrak{A}$ and $\delta_\varepsilon > 0$ such that $d_\beta(p, q) < \delta_\varepsilon$ implies $d_\alpha(p, q) < \epsilon/3$ and $d_\alpha(T_\pi p) < \epsilon/3$ for all $\pi \geq \pi_\beta$. Since S is precompact, there exist $a_{\beta_i} \in S$, $i = 1, 2, \ldots, n$, such that $S \subset \bigcup_{i=1}^n V_{\beta\delta\varepsilon}(a_{\beta_i})$ that is, for any $p \in S$ there exists a_{β_i} such that $d_\beta(a_{\beta_i}, p) < \delta_\varepsilon$. By the condition (2), there is $\pi_0 \in \Pi$ such that $d_\beta(a_{\beta_i}, T_\pi a_{\beta_i}) < \delta_\varepsilon$ for all $\pi \geq \pi_0$ and for all β_i. Hence, $d_\alpha(T_\pi a_{\beta i}, T_\pi p) < \epsilon/3$ and $d_\alpha(a_{\beta_i}, T_\pi a_{\beta i}) < \epsilon/3$ for all $\pi \geq \pi_0$, π_β. Thus $d_\alpha(p, T_\pi p) \leqq d_\alpha(p, a_{\beta i}) + d_\alpha(a_{\beta i}, T_\pi a_{\beta_i}) + d_\alpha(T_\pi a_{\beta i}, T_\pi p) < \epsilon$ for all $\pi \geq \pi_0$, π_β and $p \in S$. Thus (2°) holds.

Conversely, suppose that the set S satisfies the conditions (1°) and (2°). Then, for any $\alpha \in \mathfrak{A}$ and any $\epsilon > 0$ there exists $\pi_\alpha \in \Pi$ such that $d_\alpha(p, T_{\pi_\alpha} p) < \epsilon/3$ for all $p \in S$. Since $T_{\pi_\alpha}(S)$ is precompact, there exist $a_{\alpha_i} \in S$, $i = 1, 2, \ldots, n$, such that $T_{\pi_\alpha}(S) \subset \bigcup_{i=1}^n V_{\alpha\varepsilon}(T_{\pi_\alpha} a_{\alpha_i})$, that is, for any $p \in S$ there is an a_{α_i} such that $d_\alpha(T_{\pi_\alpha} p, T_{\pi_\alpha} a_{\alpha_i}) < \epsilon/3$. Consequently,

$$d_\alpha(a_{\alpha_i}, p) \leqq d_\alpha(a_{\alpha_i}, T_{\pi_\alpha} a_{\alpha_i}) + d_\alpha(T_{\pi_\alpha} a_{\alpha_i}, T_{\pi_\alpha} p) + d_\alpha(T_{\pi_\alpha} p, p) < \epsilon.$$

Thus the set S is precompact, and then the theorem is proved.

§ 3. **Topology of the family of continuous transformations.** We shall generalize the Ascoli-Arzelà theorem. Let X be a topological space (particularly, when we consider the uniformity in X, we introduce a family of the pseudometric $\{d_\sigma\}_\Sigma$), and Y be a uniform space with topology defined by $\{d_\alpha\}_{\mathfrak{A}}$, and further be a family of all the continuous transformations from X into Y. We define the uniform topology in C as follows: for any subset M of X, $\rho_{\alpha M}(f, g) = \sup_{x \in M} d_\alpha(f(x), g(x))(1 + d_\alpha(f(x), g(x)))$. Denote the uniform topology in C by τ_W, τ_K or τ_S according as M is a finite set, a compact set or the whole space X respectively, and say weak, compact or strong topologies, respectively. It is clear that $\tau_s \geqq \tau_K \geqq \tau_W$, where $\tau_1 \geqq \tau_2$ denote that τ_1 is stronger than τ_2. The topology τ_A in C is admissible, if $f(x)$ is continuous on (f, x) in the product topology of τ_A and X. If X is compact, $\tau_S = \tau_K = \tau_A$. If X is discrete, $\tau_K = \tau_W$. Arens [4] and Myers [5] have proved that if X is locally compact, $\tau_K \rightarrow \tau_A$. While Arens has proved that always $\tau_A \geqq \tau_K$. We will now define the equi-continuity in C and prove the compactness theorem of Ascoli-Arielà type in C.

Definition 3. The set S in C is said to be equi-continuous if, for any $\alpha \in \mathfrak{A}$, and any $\epsilon > 0$ there exist $\sigma \in \Sigma$ and $\delta > 0$ such that $d_\sigma(x, x') < \delta$ implies $d_\alpha(f(x), f(x')) < \epsilon$ for all $f \in S$. The set S is said to be equi-continuous (K) if, for any $\alpha \in \mathfrak{A}$, any $\epsilon > 0$ and any compact set $K \subset X$, there exsit $\sigma \in \Sigma$ and $\delta > 0$ such that $x, x' \in K$ and $d_\alpha(x, x') < \delta$ imply $d_\alpha(f(x), f(x')) < \epsilon$ for all $f \in S$. The set S is said to be equi-continuous (p) if, for any $\alpha \in \mathfrak{A}$, any $\epsilon > 0$ and any $x_0 \in X$ there exists a nbd. $V(x_0)$ of x_0 such that $x \in V(x_0)$ implies $d_\alpha(f(x_0), f(x)) < \epsilon$ for all $f \in S$.

Theorem 3. (1) *if C is equi-continuous (p), then the topologies τ_W and τ_K are equivalent and both admissible.* (2) *If X is precompact and C is equi-continuous, then the topologies τ_W, τ_K and τ_S are equivalent and all admissible.*

Proof (1). It is sufficient to see that τ_W is admissible, since the weak topology which is admissible is compact open (this is clear by Arens [4], Theorem 2). For any $\epsilon > 0$, any $x_0 \in X$ and any $f_0 \in C$, let $V(f_0)$ denote the nbd. of f in the τ_W-topology as the set of all $f \in C$ such that $d_\alpha(f_0(x_0), f(x_0)) < \epsilon/2$. Since C is equi-continuous (p), there exists a nbd. $V(x_0)$ of x_0 in X such that $d_\alpha(f(x_0), f(x)) < \epsilon/2$ for all $x \in V(x_0)$ and $f \in C$. Hence, $d_\alpha(f_0(x_0), f(x)) < \epsilon$ for all $x \in V(x_0)$ and all $f \in V(f_0)$. Thus $f(x) = (f, x)$ is continuous in $\tau_W \times X$.

(2). From (1) it is sufficient to see that τ_W and τ_S are equivalent. Since C is equi-continuous, for any $\alpha \in \mathfrak{A}$ and any $\epsilon > 0$ there exist $\sigma \in \Sigma$ and $\delta > 0$ such that $d_\sigma(x, y) < \delta$ implies $d_\alpha(f(x), f(y)) < \epsilon/3$ for all $f \in C$. Since X is precomp ct, there are finite points $x_1, \ldots, x_n$ such that for any $x \in X$ $d_\sigma(x_i, x) < \delta$ holds for some x_i.

Hence,

$$d_\alpha(f(x), g(x)) \leqq d_\alpha(f(x), f(x_i)) + d_\alpha(f(x_i), g(x_i)) + d_\alpha(g(x_i), g(x)) < \epsilon$$

Thus, $d_{\alpha M}(f, g) < \epsilon/3$ implies $d_\alpha(f, g) < \epsilon$ where $M = \{x_1, \cdots, x_n\}$. This proves the theorem.

REMARK. From Theorem 3 and Arens [4] it follows that if Y is the (0, 1)-interval and C is equi-continuous (p), then X is locally compact.

THEOREM 4. *If the set S in C is precompact on τ_S, then S is equi-continuous* (p).

PROOF. From the assumption, for any $\alpha \in \mathfrak{A}$ and $\epsilon > 0$ there exist $f_1, \cdots, f_n \in S$ such that for any $f \in S$ $d_\alpha(f, f_i) < \epsilon/3$ holds for some f_i. Since f_i ($i = 1, 2, \cdots, n$) are continuous, for any $x_0 \in X$ there exists a nbd. $V(x_0)$ of x_0 satisfying $d_\alpha(f_i(x_0), (x)) < \epsilon/3$, $i = 1, 2, \cdots, n$. Hence we have $d_\alpha(f(x_0), f(x)) \leqq d_\alpha(f(x_0), f_i(x_0)) + d_\alpha(f_i(x_0), f_i(x)) + d_\alpha(f_i(x), f(x)) < \epsilon$ for all $x \in V(x_0)$, that is, C is equi-continuous (p), which is the required.

From this proof it follows

COROLLARY 4.1. (1) *When X is compact or its uniform structure is unique, if a set S in C is precompact on τ_S, then S is equi-continuous.* (2) (Arens) *If a set S in C is precompact in τ_K, then the set S is equi-continuous* (k).

Let us prove the converse of the preceding theorem. For a set S in C, we write $S(x) = \{f(x); f \in S\}$.

THEOREM 5. (1) *If a set S in C is equi-continuous* (p) *and $S(x)$ is precompact for all $x \in X$ then S is precompact on τ_K.* (2) *A set S in C is precompact on τ_K if and only if the set S is equi-continuous* (k) *and $S(x)$ is precompact for all $x \in X$.*

PROOF. It is sufficient to prove (1) only, since (2) follows from (1) and Corollary 4. . If the set is not precompact on τ_K, there exists a compact set K in X, an index $\alpha \in \mathfrak{A}$, a constant $N_0 > 0$ and a sequence $\{f_i\} \subset S$ such that $\rho_{\alpha K}(f_i, f_j) > N_0$ ($i \neq j$). Consequentely, there exists a sequence $\{x_{ij}\} \subset K$ such that $d_\alpha(f_i(x_{ij}), f_j(x_{ij})) > N_0$ ($i \neq j$). Since K is compact, there exists a cluster point $x_0 \in K$ of $\{x_{ij}\}$. For any $\epsilon > 0$ there exists a nbd. $V(x_0)$ of x_0 such that $d_\alpha(f(x), f(x_0)) < \epsilon/3$ for all $x \in V(x_0)$ and for all $f \in S$. Since for all $x \in X$ $\{f_i(x)\}$ is precompact, there exist $f_{n_1}, f_{n_2}, \cdots, f_{n_m}$ in S such that for any f_i, $d_\alpha(f_i(x_0), f_{n_k}(x_0)) < \epsilon/6$ holds for some f_{n_k}. There exist i, j such that $d_\alpha(f_i(x_0), f_j(x_i)) \leqq d_\alpha(f_i(x_0), f_{n_k}(x_0)) + d_\alpha(f_{n_k}(x_0), f_j(x_0)) < \epsilon/3$. Hence,
$d_\alpha(f_i(x_{ij}), f_j(x_{ij})) \leqq d_\alpha(f_{ij}), f_i(x_0)) + d_\alpha(f_i(x_0), f_j(x_0)) + d_\alpha(f_i(x_0), f_j(x_{ij})) < \epsilon$.

This is a contradiction.

REMARK. (1) The Theorem 3 and 5 can be applied to the central group in R. Godement's sense [6]. That is, it is a necessery and sufficient condition for a locally compact group G being a central group that the left and right uniform structures in G are coincide (or equivalently, the group of all inner autcmorphisms of G is equi-continuos (p)) and for each element of G the conjugate class is always precompact. (2) If X is countable compact, then a set S in C is precompact in τ_s if and only if S is equi-continuous (p) and $S(x)$ is preccmpact for all $x \epsilon X$.

When Y is compact, we write $C = X^*$ for τ_K. If X is compact, then X X^* is locally compact if and only if locally equi-continuous. We can prove similar theorem for the dual group of discrete group.

THEOREM 6. *If X is discrete, then X^* is compact.*

PROOF. Since X^* is equi-continuous (p) and Y is compact, X^* is precompact by Theorem 5. Consequently, if we show that X^* is complete, then X^* is compact. For any Cauchy directed set $\{f_\nu\}$ in X^*, $\{f_\nu(x)\}$ is also in Y for each $x \epsilon X$. Since Y is complete, $\{f_\nu(x)\}$ converges to $f_0(x)$ on each $x \epsilon X^*$. Thus X^* is complete.

REMARK. On the completeness, more generally, when X is a topological space and Y is a complete uniform space, then τ_S is complete, consequently τ_K and τ_W are also.

THOREM 7. *If X is compact, then X can be imbedded in a compact subset of X^{**}. In this case the range Y is real space or* (0, 1)*-interval, and* $X^{**} = (X^{**})$.

PROOF. For any $x \epsilon X^*$ we put $f(x) = u_x(f)$. Since X is compact and X^* is admissible, $u_x(f)$ is continuous on X^*, that is $u_x \epsilon X^{**}$. We consider the correspondence $x \longleftrightarrow u_x$ between X and X^{**} and denote it by T. For any distinct points x_1 and x_2 in X there exists $f \epsilon C$ such that $f(x_1) \neq f(x_2)$. Hence T is one-to-one correspondence. We shall now show the continuity of T. Let K^* be any compact set of X^*. From the Theorem 5, K^* is equi-continuous. Hence, for any $x_0 \epsilon X$ and any $\epsilon > 0$, there exists a nbd. $V(x_0)$ of x_0 such that $|f(x_0) - f(x)| < \epsilon$ for $x \epsilon V(x_0)$ and $f \epsilon K^*$, that is, $|u_{x_0}(f) - u_x(f)| < \epsilon$; whence T is continuous. Since X is compact, it is bicontinuous. Thus X is homeomorphic with a compact set $T(x)$ of X^{**}

We can state above theorem in the following generalized form. Let f be a transformation from X to Y. For this f we define a relation $[x_1 \overset{R}{\sim} x_2]$

$\equiv [f(x_1) = f(x_2)]$ in X and denote quotient space $X_f (= X_R)$ of X with respect to the relation R (c. f. Bourbaki [3]). Then canonical mapping T from X onto X_f is continuous. Hence, if X is compact, X^{**} is also. If $f \epsilon X^*$, then for any distinct points $x_1, x_2 \epsilon X_f$, $f(x_1) \neq f(x_2)$. Thus we have the following Corollary:

COROLLARY 6. 1. *If X is compact and Y is a topological space, then for any $f \epsilon X^*$, X_f can be imbedded on a compact subset in X.*

REFERENCES

1. D. H. Hyers, Pseudo-normed linear spaces and abelian groups, Duke Math. J. 5 (1939).
2. P. S. Phillips, On linear transformations, Trans. Amer. Math. Soc. 48 (1940).
3. N. Bourbaki, Topologie générale, Actualités Sci. Indus, 858 (1940).
4. R. F. Arens, A topology for space of transformations, Ann. of Math. 47. (1946).
5. S. Myers, Equi-continuous set of mappings, Ann. of Math. 47 (1946).
6. R. Godement, Analyse harmonique dans les groupes centraux. C. R. 225 (1947).

MATHEMATICAL INSTITUTE,
KYÛSHÛ UNIVERSITY, HUKUOKA.