THE PARACOMPACTNESS OF CW-COMPLEXES

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The category of paracompact normal spaces is a sufficiently broad class in which some methods of algebraic topology may be applied. It is well-known that this category contains all metric spaces, compact normal spaces and fully normal Hausdorff spaces. Simplicial complexes with the weak topology and CW-complexes in the sense of J. H. C. Whitehead [5, § 5][1] especially play important roles in the algebraic topology. Author [3, Lemma 4], J. Dugundji [2, Theorem 4] and D. G. Bourgin [1, Theorem 3] have been proved that any simplicial complex with the weak topology is paracompact. The purpose of the present note is to prove that any CW-complex is paracompact.

1. Notations. Let X be a space and W be a subset of X and $\mathfrak{V} = \{V_j\}$ be a family of subsets $V_j \subset X$ and $f: Y \to X$ be a continuous map of a space Y into X. We shall use the following notations·

$$\overline{\mathfrak{V}} = \{\overline{V}_j\} \quad (\overline{V}_j \text{ denotes the closure of } V_j \text{ in } X),$$

$$\mathfrak{V} \cap W = \{V_j \cap W\}, \ \mathrm{St}(W; \mathfrak{V}) = \cup V_j (V_j \cap W \neq \phi, \ V_j \in \mathfrak{V}),$$

$$\mathrm{St}(\mathfrak{V}) = \{\mathrm{St}(V_j; \mathfrak{V}) \mid V_j \in \mathfrak{V}\}, \ f^{-1}(\mathfrak{V}) = \{f^{-1}(V_j) \mid V_j \in \mathfrak{V}\}.$$

And $\mathfrak{U} > \mathfrak{V}$ means that each element of $\mathfrak{U}$ is contained in some element of $\mathfrak{V}$.

Let E^n be the subset of n-dimensional Euclidean space defined by

$$-1 \leqq x_i \leqq 1 \qquad (i = 1, \ldots, n).$$

The boundary of E^n is denoted by S^{n-1}. For a point $x \in S^{n-1}$ and a real number $t\,(0 \leqq t \leqq 1)$ let (x, t) denotes a point which divides the segment joining x to the center $0 = (0, \cdots, 0)$ of E^n in the ratio $t : 1 - t$.

For a given point $(x_0\ t_0)$ $(x_0 \in S^{n-1},\ 0 < t_0 < 1)$ let V be an open set of S^{n-1} which contains x_0 and let ε be a number such that $0 < \varepsilon < \min(t_0, 1 - t_0)$. Then the open set $W(x_0, t_0) = \{(x, t) \mid x \in V, |t - t_0| < \varepsilon\}$ is called a regular open set of E^n with the center (x_0, t_0) and the bottom $\mathrm{B}[W(x_0, t_0)] = V$ and the breadth $\delta[W(x_0, t_0)] = \varepsilon$.

2. Two lemmas. Here we shall prove two lemmas which are used in the proof of our main theorem.

LEMMA 1. *Let P be a union[2] of at most $(n+1)$-dimensional element E^r_λ which are mutually disjoint. Let $\mathfrak{U} = \{U_\alpha\}$ be any open covering of P. Let*

1) Numbers in brackets refer to the references cited at the end of this note.

2) P is topologized so that each E^r_λ with its own topology is both open and closed in P.

$\mathfrak{V}=\{V_j\}$ *be a given open covering of n-skeleton* P^n *such that for each element* V_j *of* $\mathfrak{V}$ *there corresponds an element* $U_{\alpha(j)}$ *of* $\mathfrak{U}$ *which contains the star set* $\mathrm{St}(\bar{V}_j;\bar{\mathfrak{V}})$ *and for each* $(n+1)$*-element* E_λ^{n+1}, $\mathfrak{V}\cap\dot{E}_\lambda^{n+1}$ *is a finite covering. Then there exists an open covering* $\mathfrak{W}=\{W_j,\ W'_\alpha\}$ *which satisfies the conditions*:

(1) $\qquad W_j\cap P^n=V_j$ *and* $W'_\alpha\subset P^{n+1}-P^n$,

(2) $\qquad \mathfrak{W}\cap E_\lambda^{n+1}$ *is a finite covering,*

(3) $\qquad \mathrm{St}(\bar{\mathfrak{W}})>\mathfrak{U}$ *and in particular* $\mathrm{St}(\bar{W}_j;\bar{\mathfrak{W}})\subset U_{\alpha(j)}$.

This lemma will be easily obtained from the following lemma.

LEMMA 1$'$. *Let E be an n-element and S its boundary. Let* $\mathfrak{U}=\{U_\alpha\}$ *be any open covering of E and let* $\mathfrak{V}=\{V_j\}$ *be a given finite open covering of S such that for each element V of* $\mathfrak{V}$ *there corresponds an element* $U_{\alpha(j)}$ *of* $\mathfrak{U}$ *which contains the star set* $\mathrm{St}(\bar{V}_j;\bar{\mathfrak{V}})$. *Then there exists a finite open covering* $\mathfrak{W}=\{W_j,\ W'_\alpha\}$ *such that* $W_j\cap S=V_j$, $W'_\alpha\subset E-S$, $\mathrm{St}\,\bar{\mathfrak{W}}>\mathfrak{U}$ *and in particular* $\mathrm{St}(\bar{W}_j;\bar{\mathfrak{W}})\subset U_{\alpha(j)}$.

PROOF. We may assume that $E=E^n$, $S=S^{n-1}$. For a fixed element V_j of $\mathfrak{V}$, let $\bar{V}_{j_0},\cdots,\bar{V}_{j_p}$ be element of $\bar{\mathfrak{V}}$ such that $\bar{V}_j\cap\bar{V}_{j_i}\neq\phi$.
We set

$$U_j^*=\bigcap_{i=0}^{p}U_{\alpha(j_i)}\cap U_{\alpha(j)}.$$

By the assumption $\mathrm{St}(\bar{V}_j;\bar{\mathfrak{V}})\subset U_{\alpha(j)}$, hence we have $\bar{V}_j\subset U_{\alpha(j)}$. Therefore there exists a real number $t_0(0<t_0<1)$ such that for any $x\in S$, if $x\in V_j$ and $0\leqq t\leqq 2t_0$ then $(x,\ t)\in U_j^*$.

Now we set

$$W_j=\{(x,\ t)|x\in V,\ 0\leqq t<t_0\},$$
$${}^1W_j=\{(x,\ t)|x\in V,\ 0\leqq t<2t_0\}.$$

Since $\mathfrak{U}$ is an open covering of compact metric space E, there exists a positive number ε such that any subset W of E with the diameter $d(W)<\varepsilon$ is contained in some element of $\mathfrak{U}$.

Next, for each point $(x,\ t_0)$ $(x\in S)$ let $W^*(x,\ t_0)$ be the intersection of all elements of $\{{}^1W_j\}$ which contain $(x,\ t_0)$. Then $W^*(x,\ t_0)$ is non-empty open set. Therefore there exists a regular open set $W(x,\ t_0)$ such that $W(x,\ t_0)\subset W^*(x,\ t_0)$ and $d[W(x,\ t_0)]\leqq\rho=\varepsilon/6$. Since $S_1=\{(x,\ t_0)|x\in S\}$ is compact, S_1 is covered by finite elements of $\{W(x,\ t_0)|x\in S\}$, say $\{W(x_\alpha,\ t_0)\}$ $(\alpha=1,\cdots,q)$. Let $2\delta_0=\mathrm{Min}_\alpha\,\delta[W(x_\alpha,\ t_0)]$ and we define W'_α and ${}^1W'_\alpha$ by

$$W'_\alpha=\{(x,\ t)|x\in B_\alpha,\ |t-t_0|<\delta_0\},$$
$${}^1W'_\alpha=\{(x,\ t)|x\in B_\alpha,\ |t-t_0|<2\delta_0\},$$

where B_α is the bottom of $W(x_\alpha,\ t_0)$.

For each point $(x,\ t_1)$ $(t_1=t_0+\delta_0)$, let $W^*(x,\ t_1)$ be the intersection of all elements of $\{{}^1W'_\alpha\}$ which contain $(x,\ t_1)$. Then $W^*(x,t_1)$ is a non-empty open set. Hence there exists a regular open set $W(x, t_1)$ such that $\overline{W(x, t_1)} \subset W^*(x, t_1)$ and $\delta[W(x, t_1)] \leqq \delta_0/2$. Since $S_2 = \{(x, t_1) | x \in S\}$ is compact, S_2 is covered by finite elements of $\{W(x, t_1) | x \in S\}$, say $\{W(x_\beta, t_1)\}(\beta = 1, \cdots, r)$. Let $\delta_1 = \underset{\beta}{\mathrm{Min}} \{\delta[W(x_\beta, t_1)]\}$ and we define W''_β by

$$W''_\beta = \{(x, t) | x \in B_\beta, |t - t_1| < \delta_1\},$$

where B_β is the bottom of $W(x_\beta, t_1)$.

For each point(x, t) $(x \in S,\ t_1 + \delta_1 \leqq t \leqq 1)$ we associate a spherical neighborhood $W'''(x, t)$ with the diameter $\leqq \delta_1/2$. Since $E' = \{(x, t) | x \in S, t_1 + \delta_1 \leqq t \leqq 1\}$ is compact, E' is covered by finite elements of $\{W'''(x, t) | x \in S,\ t_1 + \delta_1 \leqq t \leqq 1\}$, say $\{W'''_\gamma\}$ $(\gamma = 1, \cdots, s)$.

Now we set

$$\mathfrak{W} = \{W_i,\ W'_\alpha,\ W''_\beta,\ W'''_\gamma\},$$

then $\mathfrak{W}$ is a finite open covering of E. By the construction it is obvious that

$$\overline{W}_j \cap \overline{W}''_\beta = \phi,\ \overline{W}'_\alpha \cap \overline{W}'''_\gamma = \phi,\ W_j \cap S = V_j,\ W'_\alpha,\ W''_\beta,\ W'''_\gamma \subset E - S$$

and

$$d[W'_\alpha] \leqq \rho,\ d[W''_\beta] \leqq \rho,\ D[W'''_\gamma] \leqq \rho.$$

Hence

$$d[\mathrm{St}(\overline{W}''_\beta;\ \mathfrak{W})] \leqq 3\rho < \delta,$$

$$d[\mathrm{St}(\overline{W}'''_\gamma;\ \mathfrak{W})] \leqq 3\rho < \delta,$$

therefore $\mathrm{St}(\overline{W}''_\beta;\ \mathfrak{W})$ and $\mathrm{St}(\overline{W}'''_\gamma;\ \mathfrak{W})$ are respectively containd in some element of $\mathfrak{U}$.

Also it is clear that $\overline{V}_i \cap \overline{V}_j \neq \phi \Leftrightarrow \overline{W}_i \cap \overline{W}_j \neq \phi$.

If $\overline{W}_i \cap \overline{W}_j \neq \phi$, then $\overline{V}_i \cap \overline{V}_j \neq \phi$, hence $\overline{W}_i \subset U^*_i \subset U_{\alpha(j)}$.

If $\overline{W}_j \cap W'_\alpha \neq \phi$, then $\overline{V}_j \cap \overline{V}_k \neq \phi$, where $x_\alpha \in V_k$. Hence $W'_\alpha \subset W^*(x_\alpha, t_0) \subset U^*_j \subset U_{\alpha(j)}$. Therefore $\mathrm{St}(\overline{W}_j; \mathfrak{W}) \subset U_{\alpha(j)}$. Similarly we know that $\mathrm{St}(\overline{W}'_\alpha : \mathfrak{W})$ is containd in some element of $\mathfrak{U}$.

Hence the covering $\mathfrak{W}$ is required. Q.E.D.

Now let K be a CW-complex and let the cells in K be indexed and with each m-cell $e^m_\lambda \in K (m = 0, 1, \cdots)$ let us associate an m-element E^m_λ as follows. The points in E^m_λ shall be the pair (x, e^m_λ) for every point x in E^m, and E^m_λ shall have the topology which makes the map $x \to (x, e^m_\lambda)$ a homeomorphism. No two of these elements have a point in common and we unite them into a topological space[2])

$$P = \bigcup_{m, \lambda} E^m_\lambda.$$

Let $f^m_\lambda : E^m \to e^m_\lambda$ be a characteristic map for e^m_λ and let $f : P \to K$ be the map which is given by $f(x, e^m_\lambda) = f^m_\lambda x$ for each point $(x, e^m_\lambda) \in P$. Since $\overline{e}^m_\lambda$ has the identification topology determined by f_λ^m it follows that the weak topology in K is the identification topology determined by f. Let P^n and K^n denote the n-skeleton of P and K. Then $f^{-1}(K^n) = P^n$ and $f \mid P^{n+1} - P^n$ is topological.

LEMMA 2. *Using the above notation if V, W are any subsets of K^n then* $\bar{V} \cap \bar{W} \neq \phi \Leftrightarrow \overline{f^{-1}(V)} \cap \overline{f^{-1}(W)} \neq \phi$.

PROOF. Since $\bar{V} \subset V$, $f^{-1}(V) \subset f^{-1}(\bar{V})$, hence $\overline{f^{-1}(V)} \subset \overline{f^{-1}(\bar{V})} = f^{-1}(\bar{V})$. Hence if $\overline{f^{-1}(V)} \cap \overline{f^{-1}(W)} \neq \phi$ then $f^{-1}(\bar{V}) \cap f^{-1}(\bar{W}) \neq \phi$. Therefore $\bar{V} \cap \bar{W} \neq \phi$.

Conversely let us assume that $\bar{V} \cap \bar{W} \ni p$. Then there exists a cell $e_\lambda^m \in K$ containing the point p. Let $g = f \mid E_\lambda^m - \dot{E}_\lambda^m$ then g is a topological map of $E_\lambda^m - \dot{E}_\lambda^m$ onto e_λ^m. Hence there exists the unique point $q \in E_\lambda^m - \dot{E}_\lambda^m$ such that $g(q) = p$. Since $\overline{f^{-1}(V)} \supset \overline{f^{-1}(V) \cap [E_\lambda^m - \dot{E}_\lambda^m]} = \overline{g^{-1}(V)} = g^{-1}(\bar{V})$, $\overline{f^{-1}(V)} \ni q$. And similarly $\overline{f^{-1}(W)} \ni q$. Therefore $\overline{f^{-1}(V)} \cap \overline{f^{-1}(W)} \ni q$.

3. The main result. We shall prove the following our main theorem.

THEOREM. *Any* CW*-complex is paracompact.*

PROOF. Let K be a CW-complex and P, $f: P \to K$ be the same as preceding. Let $\mathfrak{U}_0$ be a given open covering of K.

For a fixed integer n we assume that there is an open covering $\mathfrak{V}^n = \{V_j^n\}$ of K^n which satisfies the conditions:

(1_n) for each element V_j^n of $\mathfrak{V}^n$ there corresponds an element $U_{\alpha(j)}$ of $\mathfrak{U}_0$ such that $\mathrm{St}(\bar{V}_j; \mathfrak{V}') \cap U_{\alpha(j)}$,

(2_n) for each cell $E_\lambda^r (r \leqq n+1)$ $f^{-1}(\mathfrak{V}^n) \cap \dot{E}_\lambda^r$ is a finite covering.

Let us put $\mathfrak{U} = f^{-1}(\mathfrak{U}_0)$, $\mathfrak{V} = f^{-1}(\mathfrak{V}^n)$. Then, by Lemma 2, ($1_n$) implies that $\mathrm{St}(\overline{f^{-1}(V_j^n)}; \mathfrak{V}) \subset f^{-1}(U_{\alpha(j)})$. Therefore, by Lemma 1, there exists a covering $\mathfrak{W} = \{W_j, W_a'\}$ of P^{+1} such that

$$W_j \cap P^n = f^{-1}(V_j^n), \qquad W_a' \subset P^{n+1} - P^n,$$

and

(A) $\qquad \mathrm{St}(\mathfrak{W}) > \mathfrak{U}$ and in particular $\mathrm{St}(\bar{W}_j; \mathfrak{W}) \subset f^{-1}(U_{\alpha(j)})$,

and $\mathfrak{W} \cup \dot{E}_\lambda^r (r \leqq n+2)$ is a finite covering.

We put $V_j^{n+1} = f(W_j)$, $V_a^{n+1} = f(W_a')$, then $f^{-1}(V_j^{n+1}) = W_j$, $f^{-1}(V_a^{n+1}) = W_a'$. Therefore $V^{n+1} = \{V_j^{n+1}, V_a^{n+1}\}$ is an open covering of K^{n+1} and $f^{-1}(\mathfrak{V}^{n+1}) = \mathfrak{W}$. Hence from Lemma 2 and (A) we have

$$\mathrm{St}(\mathfrak{V}^{n+1}) < \mathfrak{U}_0 \text{ and in particular } \mathrm{St}(V_j^{a+1}; \mathfrak{V}^{n+1}) \subset U_{\alpha(j)}.$$

Therefore the covering $\mathfrak{V}^{n+1}$ satisfies the conditions (1_{n+1}) and (2_{n+1}).

For $n = 0$, since K^0 is discrete, the covering $\mathfrak{V}^0 = \mathfrak{U}_0 \cap K^0$ is satisfies the conditions (1_0) and (2_0). Starting with V^0, it follows by induction on n

that there is a sequence of covering $V^n = \{V^n_{\rho_0}, \cdots, V^n_{\sigma_n}\}$ $(\rho_0 \in J_0, \cdots, \rho_n \in J_n)$ of K^n such that $\mathrm{St}(\overline{V}^n_{\rho_i}; \mathfrak{V}^n) \subset U_{\alpha(\rho_i)}$ and $V^{n+1}_{\rho_i} \cup K^n = V^n_{\rho_i}$ $(i = 0, \ldots, n-1;$ $\rho_i \in J_i)$. If we put

$$\mathfrak{V} = \{V_{\rho_0}, V_{\rho_1}, \cdots,\} \qquad (\rho_0 \in J_0, \rho_1 \in J_1, \ldots .)$$

where

$$V_{\rho_k} = \bigcup_{n=k}^{\infty} V^n_{\rho_k},$$

then $\mathfrak{V}$ is an open covering of K and it is obvious that $\mathrm{St}\,\mathfrak{V} > \mathfrak{U}_0$. Thus it has been proved that any open covering of K has a star refinement. Hence K is fully normal. By the definition of CW-complex K, K is a Hausdorff space, hence, by [4, Theorem 1], K is paracompact. Q. E. D.

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