THE IRREDUCIBLE DECOMPOSITIONS OF THE MAXIMAL HILBERT ALGERAS OF THE FINITE CLASS

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(Received June 12, 1952)

The notion of the Hilbert algebras was introduced by H. Nakano [8] and W. Ambrose [1], independently, and their relations were studied by O. Takenouchi [13]. Recently R. Godement [4] has introduced the double unitary representations of a unimodular locally compact group by the central Radon measure of the positive type, but their research is clearly reduced to the one of the maximal Hilbert algebras.

The object of the present paper is to obtain the decompositions of the maximal Hilbert algebras of the finite class by the method due to Godement in the theory of the double unitary representations of a group [4]. But it is remarkable to obtain the irreducibility of the decompositions without any separability condition, and these results may be suggestive to the central decompositions of the arbitrary W^*-algebras.

As we have extended the notion of the $\natural$-oparation, introduced by J. Dixmier [2], to the arbitrary W^*-algebras in the previous papers [10,11], we can decompose the arbitrary maximal Hilbert algebra in this way, but we cannot yet prove the irreducibility in this case. Therefore, we shall discuss these problems in the form of the extension of the Plancherel formula to a unimodular group in the next paper [12].

Finally, some related problems are announced by Takenouchi [13], but it seems to be necessary to introduce the separability condition there.

1. Fundamental properties of Hilbert algebras.

Following Nakano [8],

DEFINITION 1.1. A linear manifold $\mathfrak{A}$ of a Hilbert space $\mathfrak{H}$ is called a *Hilbert algebra*, if (1) $\mathfrak{A}$ is dense in $\mathfrak{H}$; (2) $\mathfrak{A}$ is an algebra with complex coefficients; (3) for any $a \in \mathfrak{A}$ there is an adjoint element $a^* \in \mathfrak{A}$ such that

(1.1) $$< ab,c > = < b,a^*c > , \quad < ba,c > = < b,ca^* >;$$

(4) for any $a \in \mathfrak{A}$ there exists a positive number α_a such that $\| ax \| \leqq \alpha_a \| x \|$ for all $x \in \mathfrak{A}$; (5) by (1) and (4), for every $a \in \mathfrak{A}$ we obtain uniquely a bounded linear operator L_a on $\mathfrak{H}$ such that $L_a x = ax$ for all $x \in \mathfrak{A}$. (6) For an element $x \in \mathfrak{H}$, if $L_a x = 0$ for all $a \in \mathfrak{A}$, then we have $x = 0$.

DEFINITION 1.2. A Hilbert algebra $\mathfrak{A}$ is said to be *maximal*, if there is no *extension* except itself, that is, a Hilbert algebra containing $\mathfrak{A}$ as a subalgebra.

As can be easily verified, we have ([8; § 1])

LEMMA 1.1. (1) $L_{a^*} = L_a^*$; (2) $L_{\alpha a} = \alpha L_a$, $L_{a+b} = L_a + L_b$, $L_{ab} = L_a L_b$; (3) $< a,b > = < b^*, a^* >$, $\|a^*\| = \|a\|$.

By (3) of the above lemma, we know that, if we put $Sa = a^*$ for all $a \in \mathfrak{A}$, then S is an *involution* in $\mathfrak{H}$.[1]

We see easily (by the above definition and lemma): for any $a \in \mathfrak{A}$, there exists a positive number β_a such that $\|xa\| \leqq \beta_a \|x\|$ for all $x \in \mathfrak{A}$. Therefore we obtain uniquely a bounded linear operator R_a on $\mathfrak{H}$, and

$$L_a b = R_b a = ab \quad \text{for } a,\ b \in \mathfrak{A}. \tag{1.2}$$

Thus we can define the operators L_x, R_x for any $x \in \mathfrak{H}$ by

$$L_x a = R_a x, \quad R_x a = L_a x \quad \text{for } a \in \mathfrak{A}; \tag{1.3}$$

if these operators L_x and R_x are bounded, then we say $x \in \mathfrak{H}$ is a *bounded* element. For the bounded elements of $\mathfrak{H}$, the following facts are known ([13, § 1] and 4; Chap. I]):

LEMMA 1.2. (1) *If x is bounded, then Sx is bounded and*

$$R_{Sx} = R_x^*, \quad L_{Sx} = L_x^*. \tag{1.4}$$

(2) *L_x and R_x are related by*

$$R_{Sx} = SL_xS, \quad L_{Sx} = SR_xS. \tag{1.5}$$

(3) *For the bounded x,y, the product*:

$$xy = L_x y = R_y x \tag{1.6}$$

is well defined and xy is also a bounded element in $\mathfrak{H}$ satisfying

$$L_{xy} = L_x L_y, \quad R_{xy} = R_y R_x. \tag{1.7}$$

Thus we know that the set $\widetilde{\mathfrak{A}}$ of all bounded elements forms an algebra containing $\mathfrak{A}$ as a subalgebra. Moreover, Takenouchi [13; Theorems 1 and 2] has proved that for each Hilbert algebra $\mathfrak{A}$, there exists a unique maximal Hilbert algebra $\widetilde{\mathfrak{A}}$, and this $\widetilde{\mathfrak{A}}$ is characterised by the set of all bounded elements of $\mathfrak{H}$.

In the sequel we treat only a maximal Hilbert algebra $\mathfrak{H}_0$ in $\mathfrak{H}$, and denote $\mathbf{L}_0 = \{L_a;\ a \in \mathfrak{H}_0\}$ and $\mathbf{R}_0 = \{R_a;\ a \in \mathfrak{H}_0\}$, and the W^*-algebras[2] generated by $\mathbf{L}_0$ and $\mathbf{R}_0$ will be denoted by $\mathbf{L}$ and $\mathbf{R}$, respectively. Then, the following facts are known (cf. [13; § 1 and § 3] and [4; Chap. I]):

LEMMA 1.3. (1) *$L_x \in \mathbf{L}$, $R_x \in \mathbf{R}$; if $A \in \mathbf{L}$ (or $\in \mathbf{R}$), then Ax is also bounded and*

$$AL_x = L_{Ax}, \qquad (AR_x = R_{Ax}), \tag{1.8}$$

$$L_x A = L_{SA^*Sx}. \quad (R_x A = R_{SA^*Sx}). \tag{1.9}$$

(2) *$\mathbf{L}_0$ (or $\mathbf{R}_0$) is a two-sided ideal in $\mathbf{L}$ (or $\mathbf{R}$).*

Then the following important theorem is proved by [13; Theorem 5] (which is proved analogously by [4; Theorem 1]).

THEOREM 1.1. $\mathbf{R}' = \mathbf{L}$ and $\mathbf{L}' = \mathbf{R}$.

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1) The involution S is such an operator on $\mathfrak{H}$ that $S^2 = I$, $(Sx, Sy) = (y, x)$ and $S(\alpha x + \beta y) = \bar{\alpha} Sx + \bar{\beta} Sy$ for $x, y \in \mathfrak{H}$.

2) By a *W^*-algebra*, we shall mean a weakly closed operator algebra in a Hilbert space, and by a *C^*-algebra* a uniformly closed one, in the terminology of Segal [[illegible]].

Therefore $\mathbf{Z} = \mathbf{R} \cap \mathbf{L}$ is the *center* of the $\mathbf{R}$ and $\mathbf{L}$.

Next we shall show

THEOREM 1.2. $\mathbf{R} = S\mathbf{L}S$.

PROOF. It is sufficient to prove that if P be a projection in $\mathbf{L}$, then SPS is a projection in $\mathbf{R}$, because S is an isometry and the set of projections in a W^*-algebra $\mathbf{R}$ is uniformly dense in $\mathbf{R}$[3]. Let $P \in \mathbf{L}$ be a projection, then clearly $(SPS)^* = SPS$, $(SPS)^2 = SPS$, so that SPS is also a projection.

$$< SPS\, L_a x, y > = < SP(SL_a x), y > = < Sy, PR_{a^*} x^* >$$
$$= < Sy, R_a{}^* PSx > = < R_a y^*,\ PSx >,$$
$$< L_a\, SPSx, y > = < SPSx, L_a{}^* y > = < R_a y^*, PSx >,$$

therefore we have $SPS \in \mathbf{R}$, that is, the proof is completed.

Some other properties of the maximal Hilbert algebra can be proved here, but these will be discussed in the following paper [12], related to the extension of the Plancherel formula to the unimodular locally compact group.

2. The $\natural$-operation.

The $\natural$-operation in a W^*-algebra is introduced by Dixmier [2] and this notion is successfully applied by Godement [4] to the double unitary representation of a group. We shall follow the method of Godement with some modifications.

LEMMA 2.1. *Let U be a unitary operator in $\mathbf{L}$, then $V = SUS$ is a unitary operator in $\mathbf{R}$.*

PROOF. $< (SUS)(SUS)x, y > = < SUSx, SUSy > = < x, y >$. $SUS \in \mathbf{R}$ will be easily proved by the method analogous to Theorem 1.2.

For these pairs of unitary operators U and V, we define

DEFINITION 2.1. $x \in \mathfrak{H}$ is called *central* if

(2.1) $$UVx = x \quad \text{for all } U \in \mathbf{L}.$$

Denote the set of all such elements by $\mathfrak{H}^\natural$. But as remarked by Godement [4], there exists a case of $\mathfrak{H}^\natural = (0)$.

Then by the ergodic theorem, (analogously to [4; Theorem 3]), we have

THEOREM 2.1. *Let $x \in \mathfrak{H}$, and let K_x be the smallest closed convex hull of UVx $(U \in \mathbf{L})$ then Kx has a unique common point $x^\natural$ with $\mathfrak{H}^\natural$; $x^\natural$ is a projection of x to $\mathfrak{H}^\natural$, and $x^\natural$ is characterised by the one of the minimal norm in K_x.*

Clearly the operation $x \rightarrow x^\natural$ is a linear continuous mapping of $\mathfrak{H}$ on $\mathfrak{H}^\natural$, which is reduced to the identity on $\mathfrak{H}^\natural$.

LEMMA 2.1. *The $\natural$-operation conserves the boundedness of $\mathfrak{H}$. In order that*

3) J. von Neumann, Zur Algebra..., Math. Ann, 202(1932), p. 391.

abounded $a \in \mathfrak{H}$ *should be in* $\mathfrak{H}^\natural$, *it is necessary and sufficient that*
(2.2) $$L_a = R_a,$$
that is, $L_a \in \mathbf{Z}$.

PROOF. Let $a \in \mathfrak{H}$ be bounded, then by Lemma 1.3, UVa is also bounded; let β_a is such a number that $\| xa \| \leqq \beta_a \| x \|$, then for a bounded x,

$$\| L_x UVa \| = \| VL_x Ua \| = \| VL_{SU^*Sx}\, a \| = \| L_{SU^*Sx}\, a \| \leqq \beta_a \| SU^*Sx \| = \beta_a \| x \|,$$

this inequality can be extended to the convex hull K_x; especially we have
(2.3) $$\| L_x a^\natural \| \leqq \beta_a \| x \|,$$
this shows $a^\natural$ is bounded, and $\| R_{a^\natural} \| \leqq \| R_a \|$. The relation (2.2) can be proved as follows: let $a \in \mathfrak{H}^\natural$, then $UVa = a$, that is, $Ua = V^*a$. As be well-known, the set of all the unitary operators $U \in \mathbf{L}$ generates L[4], so that

$$< L_x a, y > = \lim < Ua, y > = \lim < V^*a, y > = \lim < a, SUS\ y >$$
$$= \lim < USy, Sa > = < L_x Sy, Sa > = < a, SL_x Sy > = < R_x a, y >,$$

that is, $L_a = R_a$. $L_a \in Z$ will be easily verified, so that we obtained the proof.

By this lemma and the properties stated before, we obtain by the reasons similar to Godement [4; Chap. I. § II],

THEOREM 2.2. *Let* $a \in \mathfrak{H}$ *be bounded, and consider the smallest weakly closed convex set* $\mathbf{K}_a$, *generated by* UL_aU^{-1}, *in* $\mathbf{L}$, *then* $\mathbf{K}_a$ *intersects the center* $\mathbf{L}_0^\natural = \mathbf{Z} \cap \mathbf{L}_0$ *at a unique point* $L_x^\natural$.

PROOF. As we can easily see, $L_{UVa} = UL_aU^{-1}$, so that the proof is sufficient by the one of [4; Theorem 4]. Therefore we have

LEMMA 2.2. *In* $\mathbf{L}_0$ *we can define the* $\natural$*-operation possessing the following properties*: (1) $L_a^\natural = L_{a^\natural}$, (2) $L_a \in \mathbf{Z}$, then $L_a^\natural = L_a$, (3) $(L_aL_b)^\natural = (L_bL_a)^\natural$, (4) *if* L_a *is hermitean* (*or hermitean positive*) *then* $L_a^\natural$ *is also*. (5) *if* $L_a \in \mathbf{Z}$, *then* $(L_aL_b)^\natural = L_aL_b^\natural$ *for* $L_b \in \mathbf{L}_0$.

But it may occur to be $L_a^\natural = 0$ for any $L_a \in \mathbf{L}_0$.

Now, we shall assume that $\mathbf{L}$ be *of the finite class*, in the sense of Dixmier [2], that is, for any partially isometric operator $W \in \mathbf{L}$, $W^*W = I$ implies $WW^* = I$, where I is the identity operator in $\mathbf{L}$. In the sequel we shall say such a Hilbert algebra is *of the finite class*. Then by the Dixmier theorem [2; Theorem 10], $\mathbf{L}$ has a $\natural$-operation to $\mathbf{Z}$, moreover we see by [2; Theorem 18], the above $\natural$-operation in $\mathbf{L}_0$ coincides the one of $\mathbf{L}$. That is, we have

THEOREM 2.3. *Let* $\mathbf{L}$ *be of the finite class, then* $\mathbf{L}$ *has the* $\natural$*-operation to* $\mathbf{Z}$, *and for* $L_a \in \mathbf{L}_0, L_a^\natural = L_a^\natural$.

3. Decomposition of a maximal Hilbert algebra of the finite class. As be proved by Godement [4; Lemma 15], in a W^*-algebra $\mathbf{M}$ of the

4) J. von Neumann, loc cit. Theorem 2.

finite class, there exists the one-to-one correspondence between the maximal two-sided ideals $\mathbf{m}$ in $\mathbf{M}$ and the maximal ideals $\mathbf{m}^{\natural}$ in its center $\mathbf{Z}$; if we introduce a trace[5)] χ of $\mathbf{M}$ by

$$\chi(A) = \chi(A^{\natural}), \tag{3.1}$$

where χ is a character of $\mathbf{Z}$, then the above one-to-one correspondence is characterized by the correspondence of the traces of $\mathbf{M}$ and the characters of $\mathbf{Z}$, given by (3.1). In this sence we shall call such traces the *characters* of $\mathbf{M}$.

It is well-known that the set of all characters of $\mathbf{Z}$ is a compact (totally disconnected) Hausdorff space $\overline{\Omega}$ in the weak topology; if we introduce the weak topology in the set of all characters X of $\mathbf{L}$, then by the above remarked, we obtain a homeomorphism between $\overline{\Omega}$ and $\overline{\mathrm{X}}$, so that $\overline{\mathrm{X}}$ becomes compact. Let us now contract the character χ in X to the $*$-algebra $\mathbf{L}_0$, then we can consider this as a trace σ_χ on $\mathbf{L}_0$; if we introduce also the weak topology in the set of all traces σ_χ on $\mathbf{L}_0$, obtained by such process, then the mapping $\chi \rightarrow \sigma_\chi$ is continuous, and X is compact, so that the image X_0 of $\overline{\mathrm{X}}$ is also compact. Omitting the trace $\sigma \equiv 0$ on $\mathbf{L}_0$ we obtain a locally compact space X, which plays a role of the dual object of the required decomposition.

Now following Godement [4] we shall introduce

DEFINITION 3.1. A *double unitary representation* of a $*$-algebra $\mathbf{A}$ is a structure $\{\mathfrak{H}, L_A, R_A, S\}$ satisfying the following conditions:

a) $\mathfrak{H}$ is a Hilbert space,

b) $A \rightarrow L_A$, $A \rightarrow R_A$ are two continuous representations of $\mathbf{A}_0$ on $\mathfrak{H}$, such that $L_{A^*} = L_A{}^*$, and $L_A R_B = R_B L_A$ for $A, B \in \mathbf{A}$,

c) S is an involution of $\mathfrak{H}$ such that $R_A = S L_A S$.

As $\sigma \in \mathrm{X}$ is a trace of the $*$-algebra $\mathbf{L}_0$ we obtain a *double unitary representation* of $\mathbf{L}_0$: these notions of the double unitary representations was already studied by Nakamura [7] for the C*-algebra with a unit element, but the most part of his results holds true in the arbitrary $*$-algebra with a trace. That is, $\mathbf{u}_0(\sigma) = \{L_x; \sigma(L_x{}^* L_x) = 0\}$ is a two-sided ideal in $\mathbf{L}_0$, therefore we obtain a canonical mapping $U_x \rightarrow x(\sigma)$ on the quotient algebra $\mathbf{L}_0/\mathbf{u}_0(\sigma)$. Put

$$< x(\sigma),\ y(\sigma) > = \sigma(L_y{}^* L_x), \tag{3.2}$$

then this is an inner product on the quotient algebra; completing with this inner product, we obtain a Hilbert space $\mathfrak{H}(\sigma)$. Moreover if we put

$$L_x(\sigma) y(\sigma) = xy(\sigma), \quad R_x(\sigma) y(\sigma) = yx(\sigma), \tag{3.3}$$

$$S(x(\sigma)) = (Sx)(\sigma), \tag{3.4}$$

then the system $\{\mathfrak{H}(\sigma), L_x(\sigma), R_x(\sigma), S\}$ becomes the double unitary representation of $\mathbf{L}_0$, by the quite analogous reasons to the one of [7]. Thus we can cor-

5) By a *trace* σ of a $*$-algebra $\mathbf{A}$ is a linear functional of $\mathbf{A}$ satisfying $\sigma(A^*A) \geqq 0$ and $\sigma(AB) = \sigma(BA)$, for $A, B \in \mathbf{A}$.

responded to each $\sigma \in X$ a Hilbert space $\mathfrak{H}(\sigma)$. Since we have introduced in X the weak topology, for each $L_x \in \mathbf{L}_0$, $\sigma(L_x)$ is a continuous function with respect to σ; the *vector-function* $x(\sigma)$, defined on X to $\mathfrak{H}(\sigma)$, is continuous: by the construction mentioned above, the set of $x(\sigma)$ is dense in $\mathfrak{H}(\sigma)$, so that the vector functions $x(\sigma)$ form a *fundamental family of the continuous vector-functions* Λ in the sense of Godement [3; Chap. III]. In the sequel we shall freely use the notions of the *continuous sums of the Hilbert spaces* proposed by Godement. Now our present object is

THEOREM 3.1. *Let $\mathfrak{H}_0$ be the maximal Hilbert algebra in a Hilbert space $\mathfrak{H}$, then there exist a locally compact space* X *and a Radon measure $\hat{\mu}$ on* X, *possessing the following properties:*

a) *for any $x, y \in \mathfrak{H}_0$,*

$$< x, y > = \int_X < x(\sigma), y(\sigma) > d\hat{\mu}(\sigma), \tag{3.5}$$

b) *$\mathfrak{H}$ is isomorphic to L^2_Λ.*

The proof of the Theorem 3.1 is mostly dependent on the Godement's, but we shall use the *-algebra $\mathbf{L}_0$ instead of his $\mathbf{L}_0$, which is the *-algebra of the all continuous fonctions with the compact supports on a group: therefore we shall sketch the proof. For details, see [4· Chap. I, § V].

That is, by the same reasons to [4· Chap. I, § V, 2], we see: let $F(\sigma)$ be a continuous functions on X vanished at the infinity, there corresponds an operator $L_F \in \mathbf{Z}$; the mapping $F \rightarrow L_F$ is the unitary representation of the *-algebra generated by such functions; moreover, if $F(\sigma) = \sigma(L_x)$, then $L_F = L^\natural_x$. Then we have

LEMMA 3.1. *Let $F(\sigma)$ be a continuous function on* X *with a compact support, then we can take a $L_{x_F} \in \mathbf{L}_0$ such that $F(\sigma) = \sigma(L_{x_F})$.*

PROOF. Suppose F be zero outside of a compact set K, then for each point $\sigma_0 \in K$ there exists a $x \in \mathfrak{H}_0$ such that $x(\sigma_0) \neq 0$, therefore the continuous function $\sigma(L_x^* L_x)$ is not zero in a neighborhood of σ_0 and non-negative. By the well-known theorem of the partition of the unit[6], there exists a $L_x \in \mathbf{L}_0$ such that $\sigma(L_x) = 1$ on K; so that

$$F(\sigma) = G(\sigma)\sigma(L_x), \tag{3.6}$$

where $G(\sigma)$ is a continuous function on X, zero outside of K. Therefore, by the above remarked, we have

$$L_F = L_G L^\natural_x. \tag{3.7}$$

But $L^\natural_x = L^\natural_x \in \mathbf{L}_0$ (Theorem 2.3) and $\mathbf{L}_0$ is a two-sided ideal in $\mathbf{L}$ (Lemma 1.3), so that $L_F \in \mathbf{L}_0$, that is, L_F is defined by a bounded element x_F, and we heve $F(\sigma) = \sigma(L_{x_F})$.

The construction of the Radon measure on X is quite analogous to

6) N. Bourbaki, Topologie Générale, Chap. IX, p. 65.

Godement's; that is, for the conitnuous functions with a compact support F, take a such real G satisfying $FG = F$, and put

(3.8) $$I(F) = < x_F, x_G > ,$$

then we can see that $I(F)$ does not depend on such G, and $I(F)$ is a linear functional of the positive type on the space of the continuous functions with compact supports; therefore we obtain a unique Radon measure on $\mathbf{X}$ such that

(3.9) $$I(F) = \int_{\mathbf{X}} F(\sigma) d\hat{\mu}(\sigma).$$

Moreover, we have

(3.10) $$I(F\bar{G}) = \int F(\sigma)\overline{G(\sigma)} d\hat{\mu}(\sigma) = < x_F, x_G > .$$

As we have shown that $x \rightarrow x^\natural$ is a projection of x to $\mathfrak{H}^\natural$ in Theorem 2.1, and as $\mathfrak{H}_0$ is dense in $\mathfrak{H}$, the rest of the proof runs along the same line to the proof of the Theorem 10 of [4]. Therefore we omit the proof.

4. The irreducibility of the decomposition.

As the trace σ on the *-algebra $\mathbf{L}_0$ is introduced by

(4.1) $$\sigma_{\chi}(L_x) = \chi(L_x), \quad \text{for } L_x \in \mathbf{L}_0,$$

where χ is a character of $\mathbf{L}$; hence if we denote

(4.2) $$\mathfrak{u}(\chi) = \{A \in \mathbf{L};\ \chi(A^*A) = 0\},$$

then $\mathfrak{u}(\chi)$ is a maximal two sided ideal in $\mathbf{L}$, as be remarked in §3. By the isomorphism theorem of the algebras, we have the isomorphism between $\mathbf{L}_0$ $\mathfrak{u}_0(\sigma_\chi)$ and $\mathbf{L}/\mathfrak{u}(\chi)$, because $\mathbf{L}_0$ is a two-sided ideal in $\mathbf{L}$ such that $\mathbf{L}_0 \cap \mathfrak{u}(\chi) \neq (0)$. Therefore, if we introduce the double unitary representation $\{\mathfrak{H}(\chi), L_x(\chi), R_x(\chi), S\}$ of the $\mathbf{L}$ by a character χ, then $\{\mathfrak{H}(\sigma), L_x(\sigma), R_x(\sigma), S\}$ is isometrically isomorphic to $\{\mathfrak{H}(\chi), L_x(\chi), R_x(\chi), S\}$; so that we see the irreducibility of the double unitary representation $\{\mathfrak{H}(\sigma), L_x(\sigma), R_x(\sigma), S\}$ of $\mathbf{L}_0$. By the way, we can also see that, if $\chi(L_x) \neq 0$ for $L_x \in \mathbf{L}_0$, then the character χ on $\mathbf{L}$ is uniquely determined by the values on $\mathbf{L}_0$.

The W^*-algebras $\mathbf{L}$ and $\mathbf{R}$ are isomorphic, because $\mathbf{R} = S\mathbf{L}S$ (Theorem 1.2), so that $\mathbf{L}$ and $\mathbf{R}$, or more clearly, $\mathbf{L}_0$ and $\mathbf{R}_0$ play the symmetric parts essentially. Moreover, in the case of the finite class, $L^\natural_x = L^\natural_x$, $R^\natural_x = R^\natural_x$ and $L_{x^\natural} = R_{x^\natural}$ for any bounded element $x \in \mathfrak{H}$ as shown in Lemma 2.1 and Theorem 2.3.; if we introduce a character χ on $\mathbf{R}$ by the character χ of $\mathbf{Z}$, then we see

(4.3) $$\chi(L_x) = \chi(R_x) \quad \text{for any bounded } x \in \mathfrak{H}.$$

Therefore, by the above remarked, the values of the characters on the corresponding elements of $\mathbf{L}$ and $\mathbf{R}$ coincide. As the double unitary representation of a C^*-algebra by a bounded trace is uniquely determined within the isomorphisms [7; Theorem 1]), so that if we identify the double unitary representations of $\mathbf{L}$ and $\mathbf{R}$ by the same character χ, then we can consider that $L_A(\chi)$ are the unitary representation of $\mathbf{L}$ and $R_A(\chi)$ are the one of $\mathbf{R}$.

By this consideration we arrive at the quite similar notions to the central decemposition of a W^*-algebra, which are studied by Godement [3], Kondo [5, 6] and Neumann [9].

Let $\mathfrak{H}$ be isomorphic to L^2_Λ, and denote this isomorphism by $x \sim x(\chi)$. Then we shall introduce

DEFINITION 4.1. A bounded linear operator A on $\mathfrak{H}$ is called *decomposable*, if and only if, $Ax \sim A(\chi)x(\chi)$ for any $x \in \mathfrak{H}$, where $A(\chi)$ is a bounded linear operator on each $\mathfrak{H}(\chi)$; this correspondence will be denoted by $A \sim A(\chi)$.

DEFINITION 4.2. Let $\mathbf{M}$ be a W^*-algebra on $\mathfrak{H}$ and $\mathbf{M}(\chi)$ be a W^*-algebra on $\mathfrak{H}(\chi)$, $\mathbf{M}$ is called *decomposable* if $A \in \mathbf{M}$ is equivalent to $A \sim A(\chi)$, $A(\chi) \in \mathbf{M}(\chi)$; denote this correspondence by $\mathbf{M} \sim \mathbf{M}(\chi)$. This decomposition is called the *factor decomposition* if $\mathbf{M}(\chi)$ are factors except a null set.

Let us now identify $\mathfrak{H}$ with L^2_Λ, and correspond to $A \in \mathbf{L}$ an *operator-function* $L_A(\chi)$ on X, then clearly $(Ax)(\chi) = L_A(\chi)x(\chi)$ on X, for any $x \in \mathfrak{H}_0$, that is, for any $x \sim x(\chi) \in \Lambda$; as Λ is dense in L^2_Λ, we obtain

$$(Ax)(\chi) = L_A(\chi)x(\chi) \quad \text{a.e. for } x \in \mathfrak{H}, \tag{4.4}$$

this shows that $A \in \mathbf{L}$ is decomposable. Moreover, this operator-function $L_A(\chi)$ is continuous, in the sense that $L_A(\chi)x(\chi)$ is continuous as a vector-function for $x(\chi) \in \Lambda$. Similar facts remain true for the operators in $\mathbf{R}$.

Let $A \in \mathbf{L}$ and $B \in \mathbf{R}$, $A \sim L_A(\chi)$, $B \sim R_B(\chi)$, and let $\mathbf{L}(\chi)$, $\mathbf{R}(\chi)$ be the W^*-algebras generated by $L_A(\chi)$ and $R_B(\chi)$, respectively, then $L_A(\chi)R_B(\chi) = R_B(\chi)L_A(\chi)$ for all $\chi \in$ X, so that we have

$$\mathbf{L}(\chi)' \supseteqq \mathbf{R}(\chi), \quad \mathbf{R}(\chi)' \supseteqq \mathbf{L}(\chi). \tag{4.5}$$

Conversely, if a decomposable operator A has its component $T_A(\chi)$, permutable with $\mathbf{L}(\chi)$ a.e., then we see that A is permutable with $\mathbf{L}$, that is, $A \in \mathbf{R}$. Let $A \sim R_A(\chi)$, then $R_A(\chi) = T_A(\chi)$ a.e.; therefore we can write $\mathbf{L} \sim \mathbf{L}(\chi)$, $\mathbf{R} \sim \mathbf{R}(\chi)$ in the sense of Definition 4.2. Finally, our double unitary representations are obtained by the characters, so it is *irreducible*; clearly this implies

$$\mathbf{L}(\chi)' \cap \mathbf{R}(\chi)' = \alpha I(\chi). \tag{4.6}$$

From (4.5) and (4.6), we see that $\mathbf{L}(\chi)$ and $\mathbf{R}(\chi)$ are factors for all $\chi \in$ X[7].

As remarked above, the double unitary representations of $*$-algebras $\mathbf{L}_0$ and $\mathbf{R}_0$ and the W^*-algebras $\mathbf{L}$ and $\mathbf{R}$ are isometrically isomorphic; therefore summarizing above, we obtain our final

THEOREM 4.1. *Let $\mathfrak{H}_0$ be the maximal Hilbert algebra of the finite class in a Hilbert space $\mathfrak{H}$, and let $\mathbf{L}_0$ and $\mathbf{R}_0$ be the $*$-algebras formed by the L_x and R_x, corresponding to $x \in \mathfrak{H}_0$, respectively. Then there exist a locally compact space* X *and a Radon measure $\hat{\mu}$ on* X, *and we obtain the isomorphism between $\mathfrak{H}$ and L^2_Λ as shown in Theorem* 3.1. *Furthermore, with respect to this*

7) F. J. Murray and J. von Neumann, On rings of operators, Ann. of Math., 37(1936), Lemma 3 1.2, p. 138.

isomorphism, we obtain the irreducible decompositions of the maximal Hilbert algebra $\mathfrak{H}_0$.

By the way, we have obtained

THEOREM 4.2. *Let* $\mathfrak{H}_0$ *be the maximal Hilbert algebra of the finite class in a Hilbert space* $\mathfrak{H}$, *and let* $\mathbf{L}$ *and* $\mathbf{R}$ *be the W*-algebras generated by the* $\mathbf{L}_0$ *and* $\mathbf{R}_0$. *Then we have the factor decompositions of the W*-algebras* $\mathbf{L}$ *and* $\mathbf{R}$.

It is remarkable that the above theorem is deduced without any separability condition, and contains no ambiguity of the null set.

If we assume, furthermore, that $\mathfrak{H}$ is separable, then more precise results will be obtained, but we shall not enter these problems here.

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