SOME TRIGONOMETRICAL SERIES, IV[1]

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1. This paper concerns the problem proposed by O. Szász[2]: is the series $\sum a_n \cos nt$ continuous at $t = 0$ or uniformly convergent at $t = 0$ if $\sum a_n$ converges and $na_n \to 0$? Answering this problem we prove the following theorems.

THEOREM 1. *There is a sequence (a_n) such that $na_n \to 0$, $\sum a_n$ converges and the series*

(1) $$\sum a_n \cos nt$$

does not converge in the neighborhood of $t = 0$.

THEOREM 2. *There is a sequence (a_n) such that $na_n \to 0$, (1) converges for all t, but (1) is not continuous at $t = 0$.*

THEOREM 3. *There is a sequence (a_n) such that $na_n \to 0$, $\sum a_n$ converges but (1) is not uniformly convergent at $t = 0$.*

Theorem 2 is proved by Hardy and Littlewood[3] for sine series. For cosine series, proof is similar.

Another problem of O. Szász is negatively answered as follows:

THEOREM 4. *There is a seqnence (a_n) such that*

$$(n+1)s_{n+1} - ns_n \geqq -p \qquad (n = 1, 2, \dots)$$

where $s_n = a_1 + a_2 + \cdots + a_n$ and p is a positive constant and that (1) is not uniformly convergent at $t = 0$.

2. Proof of Theorem 1. The series

$$\sum_n (-1)^n \frac{\cos 2nt}{2n \log(2n)}$$

does not converge at $t = \pi/2$, and then there is an integer n_1 such that

$$\left| \sum_{2n < n_1} (-1)^n \frac{\cos 2nt}{2n \log(2n)} \right| > 1$$

at $t = \pi/2$. Similarly, the series

$$\sum_n (-1)^n \frac{\cos 4nt}{4n \log(4n)}$$

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1) Some trigonometrical series I, II, III will appear in the Journal of Mathematics, vol. 1, No. 2-3, 1953.

2) O. Szász, Bull. Amer. Math. Soc., 50 (1944).

3) Hardy-Littlewood, Proc. London Math. Soc., 18 (1918).

does not converge at $t = \pi/4$, and then there is an integer n_2 such that

$$\left| \sum_{n_1 < 4n < n_2} (-1)^n \frac{\cos 4nt}{4n \log (4n)} \right| > 1.$$

Let n_3 and n_4 be integers such that

$$\left| \sum_{n_2 < 2n < n_3} (-1)^n \frac{\cos 2nt}{2n \log (2n)} \right| > 1,$$

$$\left| \sum_{n_3 < 4n < n_4} (-1)^n \frac{\cos 4nt}{4n \log (4n)} \right| > 1.$$

Further the series

$$\sum (-1)^n \frac{\cos 8nt}{8n \log (8n)}$$

does not converge at $t = \pi/8$, and then there is an integer n_k such that

$$\left| \sum_{n_4 < 8n < n_5} (-1)^n \frac{\cos 8nt}{8n \log (8n)} \right| > 1.$$

Let n_6, n_7, n_8 be integers such that

$$\left| \sum_{n_5 < 2n < n_6} (-1)^n \frac{\cos 2nt}{2n \log (2n)} \right| > 1,$$

$$\left| \sum_{n_6 < 4n < n_7} (-1)^n \frac{\cos 4nt}{4n \log (4n)} \right| > 1,$$

$$\left| \sum_{n_7 < 8n < n_3} (-1)^n \frac{\cos 8nt}{8n \log (8n)} \right| > 1.$$

Thus proceeding we can determine (n_k). Putting

$$s(k, i; t) = \sum_{n_i < 2kn < n_i + 1} (-1)^n \frac{\cos 2^k nt}{2^k n \log (2^k n)},$$

consider the series $(n_0 = 0)$

$$\begin{aligned} & s(1, 0; t) + s(2, 1; t) \\ & + s(1, 2; t) + s(2, 3; t) + s(3, 4; t) \\ & + s(1, 5; t) + s(2, 6; t) + s(3, 7; t) + s(4, 8; t) \\ & + \cdots\cdots \end{aligned}$$

Writing out each term as a sum of cosines, we get a cosine series where there are no overlapping terms. If we denote this by $\sum a_n \cos nt$, then $na_n \to 0$ and $\sum a_n$ converges, since we can take $n_k > 2^k$. Thus the theorem is proved.

3. Proof of Theorem 2. Let

$$n_j = \text{exp exp exp } j$$

and

$$a_n = \frac{1}{n \log n} \cos \frac{n\pi}{j} \qquad (n_j < n < n_{j+1}.$$

Evidently $na_n \to 0$, and Σa_n converges. For, if we put

$$s_{n_j,k} = \sum_{n_j < n \leqq k} a_n, \qquad (n_j < k < n_{j+1}$$

then

$$s_{n_j,k} = O(j/n_j \log n_j) = o(1)$$

by Abel's lemma and by $\Sigma \sin(n\pi/j) = O(1/\sin(\pi/j))$. Since $\Sigma j/n_j \log n_j$ converges, Σa_n converges.

Similarly, the series

$$(2) \qquad \Sigma a_n \cos nt$$

converges for all $t \neq 0$. For putting

$$s_{n_j,k}(t) = \sum_{n_j < n \leqq k} a_n \cos nt,$$

we have, for $\pi/j < t/2$,

$$s_{n_j,k}(t) = \frac{1}{2} \sum_{r_j < n \leqq k} \frac{1}{n \log n} \left\{ \cos n\left(t - \frac{\pi}{j}\right) + \cos n\left(t + \frac{\pi}{j}\right) \right\}$$
$$= O(1/tn_j \log n_j).$$

Thus we get the convergence of (2), whose sum we denote by $f(t)$.

On the other hand,

$$f(\pi/j) = \sum_{n_j < n < n_{j+1}} \frac{1}{n \log n} \cos^2 \frac{n\pi}{j} + \sum_{\substack{k=1 \\ k \neq j}}^{\infty} \sum_{n_k < n < n_{k+1}} \frac{1}{n \log n} \cos \frac{n\pi}{j} \cos \frac{n\pi}{k}$$
$$= f_1 + f_2,$$

say. Now

$$f_1 > \frac{1}{2} \sum_{n_j < n < n_{j+1}} \frac{1}{n \log n} > \frac{1}{2} (\log \log n_{j+1} - \log \log n_j) > e^j(e-1)/2,$$

for large j, and since

$$\sum_{n_k < n < n_{k+1}} \frac{1}{n \log n} \cos \frac{n\pi}{j} \cos \frac{n\pi}{k} = O\left(\frac{jk}{|j-k|} \frac{1}{n_k \log n_k}\right)$$
$$= O(jk/n_k \log n_k)$$

for $k \neq j$, we have

$$f_2 = O\left(j \sum_{\substack{k=1 \\ k \neq j}}^{\infty} k/n_k \log n_k\right) = O(j).$$

Thus $f(\pi/j) = f_1 + f_2 \to \infty$ as $j \to \infty$, and hence the theorem is proved.

Theorem 3 and 4 may be proved by the above example.

4. Finally we can show that a theorem due to O. Szász is best possible.

Szász' theorem reads as follows:

THEOREM. *If, for a* $\delta\ (1 > \delta > 0)$,

$$\sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) = O(n^\delta)$$

and

$$s_n = \sum_{\nu=1}^{n} a_\nu = O(1/\log n),$$

then $\sum a_n$ *is* (R_1) *summable.*

We can prove that δ cannot be replaced by 1 in the theorem. In fact we have

THEOREM 5. *There is a sequence* (a_n) *such that*

$$\sum_{\nu=n}^{2n} |a_\nu| = o(n),$$

$$s_n = \sum_{\nu=0}^{n} a_\nu = o(1/\log n)$$

and Σa_n *is not* (R_1) *summable.*

PROOF. Let

$$s_n = \frac{1}{\log n \log \log n} \sin \frac{n\pi}{j} \qquad (n_j < n < n_{j+1}),$$

where n_j is the sequence defined in the proof of Theorem 2. Then, as in the proof of Theorem 2, the limit

$$\lim_{t=0} \sum_{n=1}^{\infty} \frac{s_n}{n} \sin nt$$

does not exist. Verification of other conditions is easy.

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