ON THE GENERATION OF A STRONGLY ERGODIC SEMI-GROUP OF OPERATORS *)

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1. Introduction. *A* one-parameter family of bounded linear operators $T(\xi), 0 < \xi < \infty$, from a complex Banach space X into itself with the property $T(\xi + \eta) = T(\xi)\cdot T(\eta)$ is said to be a semi-group of operators. In the theory of semi-group of operators, a fundamental problem is to characterize the infinitesimal generator which determines the structure of a semi-group of operators.

Such a problem has been discussed by E. Hille ([1], Theorem 12.2.1)[1] and K. Yosida [2] for a semi-group of operators satisfying the following conditions:

(c_1) *$T(\xi)$ is strongly continuous at zero.*

(c_2) $\| T(\xi)\| \leqq 1 + \beta\xi$ *for sufficiently small* ξ,

where β is a constant. Later their results were generalized to a semigroup of operators satisfying only the condition (c_1) by R. S. Phillips ([5], Theorem 2.1) and the present author [3]. This result has later been generalized to a strongly measurable semi-group of operators by W. Feller [7].

In this paper we shall deal with the generation of a semi-group of operators which is strongly ergodic to the identity at zero in the Abel sense (in § 2) and in the (C, 1) sense (in § 3). Our main results in Abel case are contained in Theorems 1 and 2, and those in (C, 1) case in Theorems 3 and 4. The idea of our proof is much due to K. Yosida [2] and W. Feller [7].

2. Semi-group of operators strongly Abel ergodic at zero. Let $\{T(\xi); 0 < \xi < \infty\}$ be a semi-group of operators satisfying the following conditions:

(a) *For each ξ, $0 < \xi < \infty$, $T(\xi)$ is a bounded linear operator from a complex Banach space X into itself and*

$$T(\xi + \eta) = T(\xi)T(\eta) = T(\eta)T(\xi). \tag{2.1}$$

(b) *$T(\xi)$ is strongly measurable in $(0, \infty)$.*

We note that the conditions (a) and (b) imply the boundedness of $\| T(\xi)\|$ in each finite interval $[\varepsilon, 1/\varepsilon]$, $\varepsilon > 0$, and consequently the strong continuity of $T(\xi)$. This result is due to R. S. Phillips [6] and the present author [4]. On the other hand, $\xi^{-1} \log \| T(\xi)\|$ tends to a finite limit or to $-\infty$ as $\xi \to \infty$ and we can always replace $\{T(\xi); 0 < \xi < \infty\}$ by the equivalent semi-group $\{e^{-\alpha\xi}T(\xi); 0 < \xi < \infty\}$, and therefore we may assume the following condition without loss of generality.

(c) $\| T(\xi) \|$ *is bounded at* $\xi = \infty$.

DEFINITION 1. $T(\xi)$ is said to be *strongly Abel-ergodic to the identity at*

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1) Numbers in brackets refer to the references at the end of the paper.

zero if it satisfies the following conditions;

(2.2) $$\int_0^1 \| T(\xi) \| \, d\xi < \infty,$$

(2.3) $$\lim_{\lambda\to\infty} \lambda \int_0^\infty e^{-\lambda\xi} T(\xi)x \, d\xi = x$$

for all $x \in X$.

REMARK. From the conditions (a) and (b) one can infer that $\| T(\xi) \|$ is lower semi-continuous and a fortiori is measurable.

DEFINITION 2. The set Σ defined by

(2.4) $$\Sigma \equiv \left\{ x\,;\ \lim_{\xi\to 0} \frac{1}{\xi} \int_0^\xi T(\eta)x d\eta = x \right\}$$

is said to be the (C, 1)-continuity set of $\{T(\xi);\ 0 < \xi < \infty\}$.

DEFINITION 3. The operator A which is defined by

(2.5) $$Ax = \lim_{h\to 0} \frac{1}{h} [T(h) - I]x$$

whenever the limit on the right hand side exists and belongs to Σ, is said to be the *infinitesimal generator* of $\{T(\xi);\ 0 < \xi < \infty\}$ and the set of elements x for which Ax exists will be denoted by $D(A)$.

We prove first the following.

LEMMA. *Let $\{T(\xi);\ 0 < \xi < \infty\}$ be a semi-group of operators satisfying the conditions (a)-(c) and be strongly Abel-ergodic to the identity at zero. If we introduce the new norm by*

(2.6) $$N(x) = \sup_{\xi > 0} \left\| \frac{1}{\xi} \int_0^\xi T(\eta)x d\eta \right\|, \quad x \in \Sigma,$$

then Σ is a Banach space with the norm $N(x)$.

PROOF. By the definition of Σ, there exist a finite positive constant C_x for each $x \in \Sigma$ and a finite positive constant K such that

$$\left\| \frac{1}{\xi} \int_0^\xi T(\eta)x d\eta \right\| \leqq C_x, \quad 0 < \xi \leqq 1,$$

and that

$$\| T(\xi) \| \leqq K, \quad \xi \geqq 1,$$

so that $N(x)$ is finite for each $x \in \Sigma$. It is obvious that Σ is a linear normed space with the norm $N(x)$.

Now, we assume that a sequence $\{x_n\} (\subset \Sigma)$ satisfies $\lim_{n,m\to\infty} N(x_n - x_m) = 0$, then for any $\varepsilon > 0$ there exists a positive integer $N_0 = N_0(\varepsilon)$ such that

$$N(x_n - x_m) = \sup_{\xi > 0} \left\| \frac{1}{\xi} \int_0^\xi T(\eta)(x_n - x_m) d\eta \right\| < \varepsilon$$

for $m > n \geqq N_0$. On the other hand, from the definition of $N(x)$, we have $\| x \| \leqq N(x)$ for each $x \in \Sigma$, hence there exists an element x such that

$$\lim_{n \to \infty} \| x_n - x \| = 0.$$

There fore we have

$$\sup_{\xi > 0} \left\| \frac{1}{\xi} \int_0^\xi T(\eta)(x_n - x) d\eta \right\| < \varepsilon, \qquad n \geqq N_0. \tag{2.7}$$

Since

$$\left\| \frac{1}{\xi} \int_0^\xi T(\eta) x d\eta - x \right\| \leqq \frac{1}{\xi} \left\| \int_0^\xi T(\eta)(x - x_n) d\eta \right\|$$

$$+ \left\| \frac{1}{\xi} \int_0^\xi T(\eta) x_n d\eta - x_n \right\| + \| x_n - x \|$$

$$\leqq \varepsilon + \left\| \frac{1}{\xi} \int_0^\xi T(\eta) x_n d\eta - x_n \right\| + \| x_n - x \|,$$

we have

$$\lim_{\xi \to 0} \frac{1}{\xi} \int_0^\xi T(\eta) x d\eta = x. \tag{2.8}$$

By (2.7) and (2.8) we have $x \in \Sigma$ and $N(x - x_n) \to 0$ as $n \to \infty$. Thus Σ is a Banach space with the norm $N(x)$.

THEOREM 1. *Let $\{T(\xi);\ 0 < \xi < \infty\}$ be a semi-group of operators satisfying the conditions* (a)-(c) *and be strongly Abel-ergodic to the identity at zero. Then*

(i) *for each λ such that $R(\lambda) > 0$, where $R(\lambda)$ denotes the real part of λ, there exists a bounded linear operator $R(\lambda; A)$ from X into Σ satisfying the following conditions*

$$(\lambda - A) R(\lambda; A) x = x, \qquad x \in \Sigma,$$
$$R(\lambda; A)(\lambda - A) x = x, \qquad x \in D(A);$$

(ii) *$D(A)$ is a dense linear subset in X;*

(iii) *there exists a finite positive constant M such that*

$$\| \lambda R(\lambda; A) \| \leqq M, \qquad \lambda \geqq 1;$$

(iv) *there exists a non-negative function $f(\xi, x)$ defined on the product space $< 0, \infty > \times X$ satisfying the properties*

(a′) *for each $x \in X$, $f(\xi, x)$ is a measurable function of ξ,*

(b′) $f(\xi) \equiv \sup_{x \in X} \dfrac{f(\xi, x)}{\| x \|}$ *is integrable on any finite interval* $[0, \varepsilon]$ *and bounded* *measurable on any infinite interval* $[\varepsilon, \infty)$, $\varepsilon > 0$,

(c′) $\sup_{x\in\Sigma}\frac{f(\xi, R(1; A)x)}{\|x\|}$ *is bounded on* $[0, \infty)$,

(d′) *we have, for all* $x \in X$,

$$\|R^{(k)}(\lambda; A)x\| \leqq (-1)^k F^{(k)}(\lambda, x), \quad k = 0, 1, \cdots,$$

where $F(\lambda, x)$ *is defined by*

$$F(\lambda, x) = \int_0^\infty e^{-\lambda\xi} f(\xi, x) d\xi, \quad \lambda > 0,$$

and $R^{(k)}(\lambda; A)$, $F^{(k)}(\lambda, A)$ *denote the k-th derivative of* $R(\lambda; A)$, $F(\lambda, x)$ *respectively;*

(v) Σ *is a Banach space with the norm* $N(x)$, $D(A)$ *is dense in* Σ *with the norm* $N(x)$ *and*

$$N(x) = \sup_{k\geqq 1, \lambda > 0} \left\| \frac{1}{k} \sum_{i=1}^{k} [\lambda R(\lambda: A)]^i x \right\|, \quad x \in \Sigma. \tag{2.9}$$

Proof. For each λ such that $R(\lambda) > 0$ we define $R(\lambda; A)$ by

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda\xi} T(\xi) x d\xi,$$

so that $R(\lambda; A)$ is a bounded linear operator from X into Σ. For $x \in \Sigma$, we have

$$\begin{aligned} &\frac{1}{h}[T(h)R(\lambda; A)x - R(\lambda; A)x] \\ &= \frac{1}{h}(e^{\lambda h} - 1)\int_0^\infty e^{-\lambda\xi} T(\xi) x d\xi - e^{\lambda h}\frac{1}{h}\int_0^h e^{-\lambda\xi} T(\xi) x d\xi \\ &\to \lambda R(\lambda; A)x - x \end{aligned}$$

as $h \to 0$ and $\lambda R(\lambda; A)x - x \in \Sigma$, thus we get the condition (i). Since $\{T(\xi); 0 < \xi < \infty\}$ is strongly *Abel*-ergodic to the identity at zero and $R(\lambda; A)[\Sigma]$[2] $= D(A)$, $D(A)$ is dense in Σ and Σ is dense in X by $R(\lambda; A)[X] \subset \Sigma$. Therefore we get the condition (ii). The condition (iii) is immediately obtained from the strong *Abel*-ergodicity of $T(\xi)$. By the definition of $R(\lambda; A)$

$$\|T(\xi)R(1; A)x\| \leqq e\int_0^\infty e^{-\eta}\|T(\eta)\| d\eta \cdot \|x\|, \quad 0 < \xi \leqq 1,$$

while since $\|T(\xi)\|$ is bounded on $[1, \infty]$ and $\|\lambda R(\lambda; A)\| \leqq M$ for $\lambda \geqq 1$, we have

$$\|T(\xi)R(1; A)x\| \leqq KM\|x\|, \quad \xi \geqq 1.$$

We get also by the definition of $R(\lambda; A)$

2) $R(\lambda; A)[\Sigma]$ denotes the set $\{R(\lambda; A)x ; x \in \Sigma\}$.

$$(2.10)\qquad R^{(k)}(\lambda\,;A)x=(-1)^k\int_0^\infty e^{-\lambda\xi}\xi^k T(\xi)x\,d\xi,\ k=1,2,\dots.$$

Now, if we put

$$f(\xi,x)=\|T(\xi)x\|,$$

then the condition (iv) may be obtained by the relations $f(\xi)=\|T(\xi)\|$,

$$f(\xi,R(1\,;A)x)=\|T(\xi)R(1\,;A)x\|\leqq \max\left(e\int_0^\infty e^{-\eta}\|T(\eta)\|d\eta,KM\right)\|x\|$$

and

$$\|R^{(k)}(\lambda\,;A)x\|\leqq(-1)^kF^{(k)}(\lambda,x).$$

Finally we shall prove the condition (v). By the condition (i) or the definition of $R(\lambda\,;A)$, we get the second resolvent equation

$$R(\lambda\,;A)-R(\mu\,;A)=-(\lambda-\mu)R(\lambda\,;A)R(\mu\,;A),$$

so that we have by (2.10)

$$(2.11)\qquad [R(\lambda\,;A)]^kx=\frac{1}{(k-1)!}\int_0^\infty e^{-\lambda\xi}\xi^{k-1}T(\xi)x\,d\xi,\ k=1,2,\dots.$$

By (2.11),

$$\frac{\lambda^{k+1}}{k!}\int_0^\infty e^{-\lambda\xi}\xi^k\left[\frac{1}{\xi}\int_0^\xi T(\tau)x\,d\tau\right]d\xi=\frac{\lambda^{k+1}}{k!}\int_0^\infty T(\tau)x\left[\int_\tau^\infty e^{-\lambda\xi}\xi^{k-1}d\xi\right]d\tau$$

$$(2.12)\quad =\frac{\lambda^{k+1}}{k!}\int_0^\infty e^{-\lambda\tau}T(\tau)x\left[\frac{1}{\lambda}\tau^{k-1}+\frac{(k-1)}{\lambda^2}\tau^{k-2}+\frac{(k-1)(k-2)}{\lambda^3}\tau^{k-3}\right.$$

$$\left.+\dots+\frac{(k-1)!}{\lambda^k}\right]d\tau=\frac{1}{k}\left[\lambda\int_0^\infty e^{-\lambda\tau}T(\tau)x\,d\tau+\frac{\lambda^2}{1!}\int_0^\infty e^{-\lambda\tau}\tau\, T(\tau)x\,d\tau+\right.$$

$$\left.+\dots\frac{\lambda^k}{(k-1)!}\int_0^\infty e^{-\lambda\tau}\tau^{k-1}T(\tau)x\cdot d\tau\right]=\frac{1}{k}\sum_{i=1}^k[\lambda R(\lambda;A)]^ix.$$

From (2.12) and the definition of $N(x)$, we have

$$\sup_{k\geqq1,\lambda>0}\left\|\frac{1}{k}\sum_{i=1}^k[\lambda R(\lambda\,;A)]^ix\right\|\leqq N(x),\qquad x\in\Sigma.$$

On the other hand, using the well known theorem that if $f(\xi)$ is a bounded continuous function and $k/\lambda\to\eta$ (λ, $k\to\infty$) then

$$\frac{\lambda^{k+1}}{k!}\int_0^\infty e^{-\lambda\xi}\xi^k f(\xi)d\xi\to f(\eta),$$

we have

$$\liminf \left\| \frac{1}{k} \sum_{i=1}^{k} [\lambda R(\lambda; A)]^i x \right\| \geqq \left\| \frac{1}{\eta} \int_0^{\eta} T(\tau) x \, d\tau \right\|$$

for $x \in \Sigma$, so that

$$\sup_{k \geqq 1, \lambda > 0} \left\| \frac{1}{k} \sum_{i=1}^{k} [\lambda R(\lambda; A)]^i x \right\| \geqq \sup_{\eta > 0} \left\| \frac{1}{\eta} \int_0^{\eta} T(\tau) x d\tau \right\| = N(x)$$

for $x \in \Sigma$. Thus we get (2.9).

We shall now prove that $D(A)$ is dense in Σ with the norm $N(x)$. If $x \in \Sigma$, then we can see, from the definition of Σ, that there exists for any positive number ε a positive number $\delta_0 = \delta_0(\varepsilon)$ such that

$$\left\| \frac{1}{\xi} \int_0^{\xi} T(\eta) x d\eta - x \right\| < \varepsilon, \ 0 < \xi \leqq \delta_0.$$

Therefore we have

$$\sup_{\delta_0 \geqq \xi > 0} \left\| \frac{1}{\xi} \int_0^{\xi} T(\eta)[\lambda R(\lambda; A)x - x] d\eta \right\|$$

$$= \sup_{\delta_0 \geqq \xi > 0} \left\| \lambda R(\lambda; A) \left[\frac{1}{\xi} \int_0^{\xi} (T\eta) x d\eta - x \right] - \left[\frac{1}{\xi} \int_0^{\xi} T(\eta) x d\eta - x \right] \right.$$
$$\left. + (\lambda R(\lambda: A)x - x) \right\|$$

$$\leqq \| \lambda R(\lambda; A) \| \sup_{\delta_0 \geqq \xi > 0} \left\| \frac{1}{\xi} \int_0^{\xi} T(\eta) x d\eta - x \right\| + \sup_{\delta_0 \geqq \xi > 0} \left\| \frac{1}{\xi} \int_0^{\xi} \mathrm{T}(\eta) x d\eta - x \right\|$$
$$+ \| \lambda R(\lambda; A)x - x \|$$

$$\leqq (M+1)\varepsilon + \| \lambda R(\lambda; A)x - x \|,$$

while

$$\sup_{\xi \geqq \delta_0} \left\| \frac{1}{\xi} \int_0^{\xi} T(\eta)[\lambda R(\lambda; A)x - x] d\eta \right\|$$

$$\leqq \sup_{\xi \geqq \delta_0} \left\| \frac{1}{\xi} \int^{\delta_0} T(\eta)[\lambda R(\lambda; A)x - x] d\eta \right\| + \sup_{\xi \geqq \delta_0} \left\| \frac{1}{\xi} \int_{\delta_0}^{\xi} T(\eta)[\lambda R(\lambda; A)x - x] d\eta \right\|$$

$$\leqq \left(\frac{1}{\delta_0} \int_0^{\delta_0} \| T(\eta) \| d\eta + \sup_{\xi \geqq \delta_0} \| T(\eta) \| \right) \cdot \| \lambda R(\lambda; A)x - x \|,$$

so that

$$N(\lambda R(\lambda: A)x - x) = \sup_{\xi > 0} \left\| \frac{1}{\xi} \int_0^{\xi} T(\eta)[\lambda R(\lambda; A)x - x] d\eta \right\|$$

$$\leqq (M+1)\varepsilon + \left(1 + \frac{1}{\delta_0}\int_0^{\delta_0} \| T(\eta)\| d\eta + \sup_{\xi \geqq \delta_0} \| T(\eta)\|\right)\cdot \| \lambda R(\lambda;\ A)x - x\|.$$

Since $\lim_{\lambda\to\infty} \| \lambda R(\lambda;\ A)x - x\| = 0$ by our assumptions, we have

$$\limsup_{\lambda\to\infty} N(\lambda R(\lambda;\ A)x - x) \leqq (M+1)\varepsilon.$$

Since ε is arbitrary and $R(\lambda;\ A)x \in D(A)$ for $x \in \Sigma$, we have proved the condition (v), and hence the proof of the theorem is complete.

We shall prove the converse of Theorem 1 which is stated as follows.

THEOREM 2. *Let Σ be a linear subset in X and A be a linear operator on Σ into itself satisfying the conditions* (i)-(iv). *Further we assume that $N(x)$ defined by* (2.9) *is finite valued, that Σ is a Banach space with the norm $N(x)$ and that $D(A)$ is dense in Σ with the norm $N(x)$.*

Then there exists a semi-group of operators $\{T(\xi);\ 0 < \xi < \infty\}$ such that $T(\xi)$ satisfies the conditions (a)-(c) *and is strongly Abel-ergodic to the identity at zero, that A is its inifinitesimai generator and Σ is the* (C, 1)*-continuity set of $\{T(\xi);\ 0 < \xi < \infty\}$ and finally that* (2.6) *is satisfied.*

PROOF. For any positive number λ, we put

$$T_\lambda(\xi) = \exp \xi(-\lambda + \lambda^2 R(\lambda;\ A)) = \exp(-\lambda\xi)\sum_{k=0}^{\infty}\frac{(\xi\lambda)^k}{k!}[\lambda R(\lambda;\ A)]^k. \tag{2.13}$$

By the condition (i) we get the second resolvent equation

$$R(\lambda;\ A) - R(\mu;\ A) = -(\lambda-\mu)R(\lambda;\ A)R(\mu;\ A),$$

so that

$$R^{(k-1)}(\lambda;\ A)x = (-1)^{k-1}(k-1)!\,[R(\lambda;\ A)]^k x,$$

while by the definition of $F(\lambda, x)$

$$(-1)^{k-1}F^{(k-1)}(\lambda, x) = \int_0^\infty e^{-\lambda\xi}\xi^{k-1}f(\xi, x)d\xi,$$

hence we have

$$\| [\lambda R(\lambda;\ A)]^k x\| \leqq \frac{\lambda^k}{(k-1)!}\int_0^\infty e^{-\lambda\xi}\xi^{k-1}f(\xi, x)d\xi,\ k = 1, 2, \ldots.. \tag{2.14}$$

If we denote the upper bound on $\xi \in (0, \infty)$ of $\sup_{x\in X}\frac{f(\xi, R(1;\ A)x)}{\|x\|}$ by M_0, then

$$\| [\lambda R(\lambda;\ A)]^k R(1;\ A)\| \leqq M_0,\ k = 1, 2, \ldots,$$

so that

$$\| T_\lambda(\xi)R(1;\ A)\| \leqq M_0,\ \ \lambda \geqq 1. \tag{2.15}$$

By the conditions (i) and (iii)

$$\| \lambda R(\lambda;\ A)x - x\| = \| R(\lambda;\ A)Ax\| \leqq \frac{M}{\lambda}\| Ax\|,\quad x \in D(A),$$

hence we get by the conditions (ii) and (iii)

$$\lim_{\lambda\to\infty} \| \lambda R(\lambda ; A)x - x \| = 0 \tag{2.16}$$

for all $x \in X$.

Since $R(\lambda ; A)$ commutes with $R(\lambda' ; A)$ for any positive numbers λ, λ' by the second resolvent equation, it follows that

$$T_\lambda(\xi)x - T_{\lambda'}(\xi)x = \int_0^\xi \frac{d}{d\tau}[T_{\lambda'}(\xi-\tau)T_\lambda(\tau)x]d\tau$$

$$= \int_0^\xi T_{\lambda'}(\xi-\tau)T_\lambda(\tau)\{(-\lambda+\lambda^2R(\lambda ; A)) - (-\lambda'+\lambda'^2R(\lambda' ; A))\}x\, d\tau.$$

Then we have by the condition (i)

$$T_\lambda(\xi)[R(1 ; A)]^2x - T_{\lambda'}(\xi)[R(1 ; A)]^2x$$

$$= \int_0^\xi T_{\lambda'}(\xi-\tau)R(1 ; A)T_\lambda(\tau)R(1 ; A)[\lambda R(\lambda ; A)Ax - \lambda' R(\lambda' ; A)Ax]d\tau$$

for $x \in D(A)$, so that by (2.15)

$$\| T_\lambda(\xi)[R(1 ; A)]^2x - T_{\lambda'}(\xi)[R(1 ; A)]^2x \|$$

$$\leqq M_0^2\, \xi \| \lambda R(\lambda ; A)Ax - \lambda' R(\lambda' ; A)Ax \|, \quad x \in D(A).$$

From the above inequality and (2.16), $\lim_{\lambda\to\infty} T_\lambda(\xi)x$ exists for each $x \in [R(1 ; A)]^2[D(A)]$. The condition (ii) implies that $R(1 ; A)[D(A)]$ is dense in $R(1 ; A)[X]$, and, since $D(A) \subset R(1 ; A)[X]$, $R(1 ; A)[X]$ is dense in X, hence $R(1 ; A)[D(A)]$ is dense in X. Accordingly, if $x \in X$, there exists a sequence $\{x_n\}$ $(\subset R(1 ; A)[D(A)])$ such that $x_n \to x$ as $n \to \infty$. If we put $y = R(1 ; A)x$, then

$$\| T_\lambda(\xi)y - T_{\lambda'}(\xi)y \| = \| T_\lambda(\xi)R(1 ; A)x - T_{\lambda'}(\xi)R(1 ; A)x \|$$

$$\leqq \| T_\lambda(\xi)R(1 ; A)x - T_\lambda(\xi)R(1 ; A)x_n \| + \| T_\lambda(\xi)R(1 ; A)x_n - T_{\lambda'}(\xi)R(1 ; A)x_n \|$$

$$+ \| T_{\lambda'}(\xi)R(1 ; A)x_n - T_{\lambda'}(\xi)R(1 : A)x \|$$

$$\leqq 2M_0 \| x - x_n \| + \| T_{\lambda'}(\xi)R(1 ; A)x_n - T_{\lambda'}(\xi)R(1 ; A)x_n \|,$$

where the second term of the right hand side tends to zero as $\lambda \to \infty$, $\lambda' \to \infty$ since $R(1 ; A)x_n \in [R(1;A)]^2[D(A)]$ and the first term tends also to zero with $1/n$, so that $\lim_{\lambda\to\infty} T_\lambda(\xi)x$ exists for all $x \in R(1 ; A)[X]$. Hence we may define $T(\xi)$, $0 < \xi < \infty$, by

$$T(\xi)x = \lim_{\lambda\to\infty} T_\lambda(\xi)x \tag{2.17}$$

for all $x \in R(1 ; A)[X]$.

If we denote $\sup_{\xi \geqq \eta > 0} f(\xi)$ by M_η, then, for any fixed numbers δ' and δ where $0 < \delta' < \delta$, we have

$$\frac{\lambda^k}{(k-1)!}\int_{\delta'}^\infty e^{-\lambda\xi}\xi^{k-1}f(\xi, x)d\xi \leqq \frac{\lambda^k}{(k-1)!}\int_{\delta'}^\infty e^{-\lambda\xi}\xi^{k-1}f(\xi)\,d\xi\cdot \| x \| \leqq M_{\delta'} \| x \|$$

for all $x \in X$, and if $k \geqq \lambda\delta$,

$$\frac{\lambda^k}{(k-1)!}\int_0^{\delta'} e^{-\lambda\xi}\xi^{k-1}f(\xi, R(1\,;\,A)x)d\xi \leqq \frac{M_0}{(k-1)!}\int_0^{\lambda\delta'} e^{-\xi}\xi^{k-1}d\xi\cdot\|x\|$$

$$< \frac{M_0 q}{(1-q)^2 k}\|x\|$$

for all $x \in X$, where $q = \delta'/\delta$[3]. Accordingly, for any positive number ε, there exists a positive number $\lambda_0(\varepsilon)$ such that

$$\|[\lambda R(\lambda\,;\,A)]^k R(1\,;\,A)x\| \leqq M_{\delta'}\|R(1\,;\,A)x\| + \varepsilon\|x\|$$

for $k \geqq \lambda\delta$, $\lambda > \lambda_0(\varepsilon)$ and $x \in X$.

We now put $N = [\lambda q\xi]$[4] for any fixed numbers q', q where $0 < q' < q < 1$, then we have by the above inequality with $\delta = q\xi$, $\delta' = q'\xi$

$$\left\| e_{-\lambda} \sum_{k=N+1}^{\infty} \frac{(\xi\lambda)^k}{k!}[\lambda R(\lambda\,;\,A)^k]R(1\,;\,A)x\right\| \leqq M_{q'\xi}\|R(1\,;\,A)x\| + \varepsilon\|x\|,$$

while

$$\left\| e^{-\lambda\xi}\sum_{k=0}^{N}\frac{(\xi\lambda)^k}{k!}[\lambda R(\lambda\,;\,A)]^k R(1\,;\,A)\,x\right\| \leqq M_0\|x\|\cdot e^{-\lambda\xi}\sum_{k=0}^{N}\frac{(\lambda\xi)^k}{k!}$$

$$\leqq \frac{M_0\|x\|}{\lambda\xi(1-q)^2}.$$

by Hille's lemma ([1], Lemma 9.3.2), and whence

$$\|T_\lambda(\xi)R(1;A)x\| \leqq M_{q'\xi}\|R(1\,;\,A)x\| + \left(\varepsilon + \frac{M_0}{\lambda\xi(1-q)^2}\right)\cdot\|x\|$$

for all $x \in X$. Passing to the limit with λ we get

$$\|T(\xi)x\| \leqq M_{q'\xi}\|x\| \tag{2.18}$$

for $x \in R(1\,;\,A)[X]$. Hence $T(\xi)$ is a bounded linear operator defined on the dense set $R(1\,;\,A)[X]$ in X, so that $T(\xi)$ can be extended to a bounded linear operator on X. We denote again such an extension by $T(\xi)$. Then we have by (2.18)

$$\|T(\xi)\| \leqq M_{q'\xi}, \tag{2.19}$$

and $T(\xi)$ satisfies the condition (c) by the definition of M_η.

It follows from (2.15) and (2.17) that

$$\lim_{\lambda\to\infty} T_\lambda(\xi)T_\lambda(\eta)x = T(\xi)T(\eta)x,\ \ x \in [R(1\,;\,A)]^2[X],$$

and that $[R(1\,;\,A)]^2[X]$ is dense in X, so that $\{T(\xi)\,;\, 0 < \xi < \infty\}$ satisfies the condition (a).

From $T_\lambda(\xi)x - x = \int_0^\xi \frac{d}{d\tau}[T_\lambda(\tau)x]d\tau$ we have

3) For the proof of this inequality see W. Feller ([7], (3.22)).

4) $[\lambda q\xi]$ denotes the integral part of $\lambda q\xi$.

$$T_\lambda(\xi)R(1\,;\,A)x - R(1\,;\,A)x = \int_0^\xi T_\lambda(\zeta)R(1\,;\,A)[\lambda R(\lambda\,;\,A)Ax]d\zeta$$

for $x \in D(A)$. Passing to the limit with λ one obtains

$$T(\xi)R(1\,;\,A)x - R(1\,;\,A)x = \int_0^\xi T(\zeta)R(1\,;\,A)Ax\,d\zeta \tag{2.20}$$

for $x \in D(A)$. We have

$$\lim_{\xi\to 0} T(\zeta)R(1\,;\,A)x = R(1\,;\,A)x, \quad x \in X, \tag{2.21}$$

according to $\| T(\xi)R(1\,;\,A)\| \leqq M_0$ and the condition (ii). Then $T(\xi)$ is strongly continuous in $< 0, \infty >$ and a fortiori is strongly measurable. (We note that the strong measurability of $T(\xi)$ is obvious from the construction of $T(\xi)$, and then $T(\xi)$ is also strongly continuous in $< 0, \infty >$). Thus $T(\xi)$ satisfies the condition (b).

By (2.13)

$$\int_0^1 \| T_\lambda(\xi)\| d\xi$$

$$\leqq \frac{1}{\lambda}(1 - e^{-\lambda}) + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!}\frac{\lambda^k}{(k-1)!}\int_0^\infty e^{-\lambda\eta}\eta^{k-1}f(\eta)d\eta \int_0^1 e^{-\lambda\xi}\xi^k d\xi,$$

and

$$\int_0^\infty e^{-\lambda\eta}\eta^{k-1}f(\eta)d\eta = \int_0^1 e^{-\lambda\eta}\eta^{k-1}f(\eta)d\eta + \int_1^\infty e^{-\lambda\eta}\eta^{k-1}f(\eta)d\eta$$

$$\leqq \int_0^1 e^{-\lambda\eta}\eta^{k-1}f(\eta)d\eta + M_1\frac{(k-1)!}{\lambda^k}$$

$$\int_0^1 e^{-\lambda\xi}\xi^k\,d\xi \leqq \int_0^\infty e^{-\lambda\xi}\xi^k\,d\xi = \frac{k!}{\lambda^{k+1}},$$

so that

$$\int_0^1 \| T_\lambda(\xi)\| d\xi \leqq \frac{1}{\lambda}(1 - e^{-\lambda}) + \int_0^1 f(\eta)d\eta + M_1.$$

By the definition of $T(\xi)$

$$\| T(\xi)\| \leqq \liminf_{\lambda\to\infty} \| T_\lambda(\xi)\|,$$

hence we have by Fatou's theorem

$$\int_0^1 \| T(\xi)\| d\xi \leqq \liminf_{\lambda\to\infty} \int_0^1 \| T_\lambda(\xi)\| d\xi \leqq \int_0^1 f(\eta)d\eta + M_1 < \infty. \tag{2.22}$$

Accordingly, if we define $R^1(\lambda\,;\,A^*)$, for each λ such that $R(\lambda) > 0$, by

$$(2.23) \qquad R^*(\lambda;\ A^*)x = \int_0^\infty e^{-\lambda\xi} T(\xi)x\, d\xi$$

for all $x \in X$, and if we denote the (C, 1)-continuity set of $\{T(\xi);\ 0 < \xi < \infty\}$ by Σ^* and its infinitesimal generator by A^*, then, for each λ such that $R(\lambda) > 0$, we have the following relation similarly as in the proof of Theorem 1(i):

$$(2.24) \qquad \begin{cases} (\lambda - A^*)R^*(\lambda;\ A)x = x, & x \in \Sigma^*, \\ R^*(\lambda;\ A)(\lambda - A^*)x = x, & x \in D(A^*). \end{cases}$$

where $D(A^*)$ denotes the domain of A^*.

From (2.20) and (2.21)

$$\lim_{\xi \to 0} \frac{1}{\xi} [T(\xi)R(1;\ A)x - R(1;\ A)x] = R(1;\ A)Ax = AR(1;\ A)x$$

for $x \in D(A)$, and furthermore $R(1; A)Ax \in \Sigma^*$, hence we have $R(1;\ A)[D(A)]D(A^*)$ and

$$(2.25) \qquad A^*R(1;\ A)x = AR(1;\ A)x, \quad x \in D(A).$$

Since $R^*(\lambda;\ A^*) = R(\lambda;\ A)$ for $x \in D(A)$ by the condition (i) and (2.25), we get

$$(2.26) \qquad R^*(\lambda;\ A^*) = R(\lambda;\ A)$$

for each λ such that $R(\lambda) > 0$. It follows hence from (2.16), (2.22) and (2.26) that $T(\xi)$ is strongly *Abel*-ergodic to the identity at zero.

Further we obtain similarly as in the proof of Theorem 1(v) that Σ^* is a Banach space with the norm $N^*(x)$ defined by $N^*(x) = \sup_{\xi > 0} \left\| \frac{1}{\xi} \int_0^\xi T(\eta)x d\eta \right\|$, $D(A^*)$ is dense in Σ^* with the norm $N^*(x)$ and that

$$N^*(x) = \sup_{k \geqq 1, \lambda > 0} \left\| \frac{1}{k} \sum_{i=1}^{k} [\lambda R^*(\lambda;\ A^*)]\, x \right\|, \quad x \in \Sigma^*.$$

Accordingly, by (2.26),

$$(2.27) \qquad N^*(x) = \sup_{k \geqq 1, \lambda > 0} \left\| \frac{1}{k} \sum_{i=1}^{k} [\lambda\, R(\lambda;\ A)]\, x \right\|, \quad x \in \Sigma^*.$$

Now

$$D(A) \subset \Sigma^* \cap \Sigma$$

by the condition (i), (2.26) and $R^*(1;\ A^*)[X] \subset \Sigma^*$,

$$D(A^*) \subset \Sigma^* \cap \Sigma$$

by (2.24), (2.26) and $R(1;\ A)[X] \subset \Sigma$, and further $N(x) = N^*(x)$ for $x \in \Sigma^* \cap \Sigma$. Since $D(A)$ is dense in Σ with the norm $N(x)$ and $D(A^*)$ is dense in Σ^* with the norm $N^*(x)$, we get

$$\Sigma^* = \Sigma.$$

Finally we obtain from (2.24), the condition (i) and the strong *Abel*-ergodicity of $T(\xi)$ that

$$D(A^*) = D(A), \quad A = A^*.$$

Thus it follows that the given operator A is the infinitesimal generator of $\{T(\xi), 0 < \xi < \infty\}$, that Σ is the (C, 1)-continuity set of $\{T(\xi); 0 < \xi < \infty\}$ and that (2.6) is satisfied. This completes the proof.

3. Semi-group of operators strongly (C,1)-ergodic at zero.

DEFINITION 4. $T(\xi)$ is said to be *strongly* (C, 1)-*ergodic to the identity at zero* if it satisfies (2.2) and the following condition

$$\lim_{\xi \to 0} \frac{1}{\xi} \int_0^\xi T(\eta) x d\eta = x \tag{3.1}$$

for all $x \in X$.

In this case the (C, 1)-continuity set of $\{T(\xi); 0 < \xi < \infty\}$ coincides with the whole space X, so that our definition of the infinitesimal generator (see Definition 3) becomes the ordinary one, further the norm $N(x)$ defined by (2.6) is equivalent to the original one.

In fact, by (3.1) and the condition (c), there exists a finite positive constant M such that

$$\sup_{\xi > 0} \left\| \frac{1}{\xi} \int_0^\xi T(\eta) x \, d\eta \right\| \leqq M \| x \|$$

for all $x \in X$, while by (3.1)

$$\| x \| \leqq \sup_{\xi > 0} \left\| \frac{1}{\xi} \int_0^\xi T(\eta) x d\eta \right\|,$$

so that we have

$$\| x \| \leqq \sup_{\xi > 0} \left\| \frac{1}{\xi} \int_0^\xi T(\eta) x d\eta \right\| = N(x) \leqq M \| x \| \tag{3.2}$$

for all $x \in X$.

We denote by A the infinitesimal generator of $\{T(\xi); 0 < \xi < \infty\}$ and by $D(A)$ the domain of A.

THEOREM 3. *Let* $\{T(\xi); 0 < \xi < \infty\}$ *be a semi-group of operators satisfying the assumptions* (a)-(c) *and be strongly* (C, 1)-*ergodic to the identity at zero. Then*

(i′) *A is a closed linear operator and its spectrum is located in* $R(\lambda) \leqq 0$;

(ii′) *$D(A)$ is a dense linear subset in X*;

(iii′) *there exists a finite positive constant M such that*

$$\sup_{k \geqq 1, \lambda > 0} \left\| \frac{1}{k} \sum_{i=1}^{k} [\lambda R(\lambda; A)]^i x \right\| \leqq M \| x \|$$

for all $x \in X$;

(iv′) *the condition* (iv) *is satisfied.*

PROOF. Since $\frac{d}{d\xi}T(\xi)x = T(\xi)Ax$ for $x \in D(A)$, we have

$$\frac{1}{\xi}[T(\xi)x - x] = \frac{1}{\xi}\int_0^\xi T(\eta)Axd\eta, \quad x \in D(A). \tag{3.3}$$

Suppose that $\{x_n\}$ is a sequence in $D(A)$ and that $x_n \to x$, $Ax_n \to y$. Formula (3.3) holds for $x = x_n$ so that

$$\frac{1}{\xi}[T(\xi)x_n - x_n] = \frac{1}{\xi}\int_0^\xi T(\eta)Ax_n d\eta.$$

Passing to the limit with n one obtains

$$\frac{1}{\xi}[T(\xi)x - x] = \frac{1}{\xi}\int_0^\xi T(\eta)yd\eta.$$

Because of (3.1) the right hand side tends to y when $\xi \to 0$. Hence Ax exists and equals to y, so that A is a closed linear operator.

We note next that the (C, 1)-ergodicity implies the *Abel*-ergodicity and that $\Sigma = X$, then we get the conclusions (i)-(iv) by (3.2) and Theorem 1.

The converse of this theorem is stated as follows.

THEOREM 4. *Let A be a closed linear operator on X into itself satisfying the conditions* (i′)-(iv′). *Then there exists a semi-group of operators* $\{T(\xi);\ 0 < \xi < \infty\}$ such that $T(\xi)$ *satisfies the conditions* (a)-(c) *and is strongly* (C, 1)-*ergodic to the identity at zero and that A is its infinitesimal generator.*

PROOF. If we denote the resolvent of A by $R(\lambda;\ A)$ for each λ such that $R(\lambda) > 0$, we can derive the first resolvent equation by the assumption (i). In virtue of the assumption (iii′) we get $\|\lambda R(\lambda;\ A)\| \leqq M$, so that one obtains similarly as (2.16) the following relation

$$\lim_{\lambda\to\infty}\|\lambda R(\lambda;\ A)x - x\| = 0$$

for all $x \in X$. From this we obtain

$$\|x\| \leqq \sup_{k\geqq 1,\ \lambda>0}\left\|\frac{1}{k}\sum_{i=1}^{k}[\lambda R(\lambda;\ A)]^i x\right\| \leqq M\|x\|$$

for all $x \in X$, and therefore if we take the whole space X as Σ, our assumptions imply those of Theorem 2. Thus there exists a semi-group of operators $\{T(\xi);\ 0 < \xi < \infty\}$ such that $T(\xi)$ satisfies the conditions (a)-(c) and is strongly *Abel*-ergodic to the identity at zero, that the whole space X is the (C, 1)-continuity set of $\{T(\xi);\ 0 < \xi < \infty\}$ and that A is its infinitesimal generator. Hence it follows that $T(\xi)$ is strongly (C, 1)-ergodic to the identity at zero. This completes the proof.

From Theorems 3 and 4 we get the following corollary.

COROLLARY. *A necessary and sufficient condition that a closed linear operator*

A becomes the infinitesimal generator of a semi-group of operators $\{T(\xi); 0 < \xi < \infty\}$ satisfying the conditions (a), (c) *and* $\lim_{\xi\to 0} T(\xi)x = x$ *for all $x \in X$, is that*

(i″) *the conditions* (i′) *and* (i″) *are satisfied*;

(ii″) *there exists a finite positive constant M such that*

$$\|[\lambda R(\lambda; A)]^k\| \leqq M$$

for $\lambda > 0$ and $k = 1, 2, \ldots\ldots$

PROOF. Since $T(\xi)$ is strongly continuous at $\xi = 0$, where $T(0) = I$ (= the identity), there exists a finite positive constant M such that $\|T(\xi)\| \leqq M$ for $0 \leqq \xi < \infty$. Therefore we get by (2.11)

$$(3.4) \qquad \|[\lambda R(\lambda; A)]x^k\| \leqq \frac{M\lambda^k}{(k-1)!}\int_0^\infty e^{-\lambda\xi}\xi^{k-1}\,d\xi \cdot \|x\| = M\|x\|$$

for all $x \in X$ and $k = 1, 2, \ldots\ldots$ Thus the necessity of the conditions is established by Theorem 3 and (3.4).

If we put $f(\xi, x) = M\|x\|$, then the conditions (i″) and (ii″) imply the assumptions of Theorem 4, while we get $\|T(\xi)\| \leqq M$ for $0 \leqq \xi < \infty$ from the condition (ii″) and the construction of $T(\xi)$. (see (2.13) and (2.17)). Thus the sufficiency of the conditions is established by use of Theorem 4.

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[After this paper was written up, the author found the abstract of Phillips' paper [8], in which he writes that the necessary and sufficient conditions that a closed linear operator be the c. i. g. (the smallest closed extension of the infinitesimal generator) of a semi-group of operators which is strongly Abel (or Cesàro) ergodic (summable) to the identity at zero are obtained. But the detail seems not yet to be published].

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ADDED IN PROOF. (June 5, 1954). R. S. Phillips' paper (An inversion formula for Laplace transforms and semi-groups of linear operators, Ann. of Math., vol. 59(1954)) has appeared. Under the condition $\int_0^\infty \| T(\xi) \| d\xi < \infty$ instead of our condition (c), he has obtained a necessary and sufficient condition in order that a closed linear operator be the complete infinitesimal generator (the smallest closed extension of the infinitesimal generator) of a semi-group of operators strongly Abel ergodic to the identity at zero. But our results (Theorems 1 and 2) are the necessary and sufficient condition in order that a linear operator (not necessary closed) be the infinitesimal generator (in the sense of Def. 3) of a semi-group of operators strongly Abel ergodic to the identity at zero. Our results in the Cesàro case are essentially identical to the Phillips'.