ON THE INFLUENCE OF THE TOPOLOGICAL STRUCTURE OF RIEMANNIAN MANIFOLDS UPON THEIR HOLONOMY GROUPS

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1. Introduction Since E. Cartan introduced the notion of holonomy groups, there were published many papers which were concerned holonomy groups of manifolds endowed with various connexions, such as Riemannian manifolds, manifolds with affine, projective or conformal connexions. However, almost all papers, are concerned with holonomy groups defined in a neighbourhood of a point and local properties of the manifolds. Recently, there were published some papers which were concerned holonomy groups defined globally and global properties of the manifold in consideration. The main purposes of these papers seem to be first to study general properties of holonomy groups and secondly to study metrical and topological properties of Riemannian manifolds whose holonomy groups have some prescribed group structures.

When we look over these papers there arises naturally the following question: If we first give topological structure of a differentiable manifold, how far the group structure of holonomy groups are restricted? I shall try the first attack for this problem in the case of Riemannian geometry in this paper. Our problem is then to get systematically a series of theorems of the following kind: Suppose that M^n is a differentiable manifold which has topological properties $A, B, C, \cdots$, then the holonomy group of any "positive definite" Riemannian metric introduced on M^n has the group structure P. Of course there are many Riemannian manifolds which have not properties $A, B, C, \cdots$, but their holonomy groups have the group structure P, because holonomy groups depend largely upon metrics. However, our purpose is to get topological conditions for the manifolds such that they garantee the group structure of the holonomy groups in consideration.

We note that in Riemannian case there are four kinds of holonomy groups: 1) the holonomy group H, 2) the restricted holonomy group H^0, 3) the homogeneous holonomy group h, and 4) the restricted homogeneous holonomy group h^0. The groups H^0 and h^0 of a Riemannian manifold M^n are identical with the holonomy groups $\widetilde{H}(=\widetilde{H}^0)$ and $\widetilde{h}(=\widetilde{h}^0)$of the universal covering manifold $\widetilde{M}^n$ of M^n with the natural Riemannian metric induced from M^n by the covering (Cf. §2). Most of our theorems are concerned with the group h^0 and so the group $\widetilde{h}$ of the covering manifold $\widetilde{M}^n$ of M^n.

At lines 17–18, from bottom, I have roughly said "any Riemannian metric" but I mean by it "any complete Riemannian metric". This may be a strong condition. However, if the manifold M^n is compact, then any Riemannian metric introduced on M^n is complete as is well known.

In § 2, I shall state some preliminary remarks and in § 3 and § 4, I shall state topological conditions for a manifold in order that the holonomy group of any complete Riemannian metric introduced on it is irreducible in the fields of real numbers or complex numbers. Most of the theorems in these paragraphs are either equivalent to known theorems or slight modifications of them, but they are stated in the realm of our idea. In § 5, there are given a series of Lemmas which leads us to the decomposition of the group h^0 into a Kronecker product of matric groups each of which is equivalent to a real full orthogonal group or a full group of unitary symplectic group or one of their simple and absolutely irreducible subgroups. Then there arises naturally the following problem: To find out topological conditions in order that h^0 is not the Kronecker product of two orthogonal groups or of two unitary symplectic groups. This problem is solved in § 6 and §7. Sufficient topological conditions for the simplicity of h^0 follow then immediately from former paragraphs and are stated in § 8.

For the sake of simplicity we assume sufficiently high differentiability for manifolds and Riemannian metrics introduced on the manifolds.

I wish to note that a part of the idea of this paper owes to the late H. Iwamoto and to express my hearty thanks to M. Goto who gave me many kind suggestions for group-theoretical part of this paper.

2. Some preliminary remarks

LEMMA 1. *The restricted holonomy groups H^0 and h^0 of a Riemannian manifold M^n are identical with the holonomy groups $\widetilde{H}$ ($= \widetilde{H}^0$) and $\widetilde{h}$ ($= \widetilde{h}^0$) respectively of the universal covering manifold $\widetilde{M}^n$ of M^n with the natural Riemannian metric induced by the covering on it.*

PROOF. First we denote the base point of holonomy groups in M^n by P and denote the frame with respect to which the holonomy groups are represented by R_0. We denote also the points on M^n which lie over P_0 by $\widetilde{P}_1, \widetilde{P}_2, \cdots$ and take $\widetilde{P}_1$ as the base point and take a frame $\widetilde{R}_1$ which lies over R_0 at P_0 by the covering as the frame of reference when we consider holonomy groups of $\widetilde{M}^n$. Then the restricted holonomy groups H^0 and h^0 of M^n are identical with the holonomy groups $\widetilde{H}$ and $\widetilde{h}$ respectively, because all closed curves on M_n which pass through P_0 and homotopic to zero are images of closed curves on $\widetilde{M}^n$ which pass through $\widetilde{P}_1$, and in addition $\widetilde{M}^n$ has the induced Riemannian metric from M^n.

LEMMA 2. *If the holonomy group h (h^0) of a Riemannian manifold M^n makes invariant an exterior differential form of degree p with constant coefficients, then the p-form can be extended to a harmonic p-form over M^n (the covering manifold $\widetilde{M}^n$). Especially if M^n ($\widetilde{M}^n$) is compact, then $B_p \neq 0$ ($\widetilde{B}_p \neq 0$), where B_p ($\widetilde{B}_p$) is the p-th Betti number of M^n ($\widetilde{M}^n$).*

PROOF. Let ω^i be the components of the generic infinitesimal vector at the base point P_0 of the holonomy group $h\,(h^0)$ with respect to the frame R_0. Then the p-form invariant under $h\,(h^0)$ can be written as

$$\theta = \frac{1}{p!}\sum c_{i_1 i_2 \cdots i_p}\,\omega^{i_1}\omega^{i_2}\cdots\omega^{i_p}$$

where $c_{i_1\cdots i_p}$'s are constants which are skew symmetric with respect to any two indices and ω's are multiplied exteriorly. We regard θ as a differential form at P_0. First we shall prove the non bracket part of our theorem. Take an arbitrary point P on M^n. We transplant θ from P_0 to P by Levi-Civita's parallelism along a piecewise smooth curve P_0P. θ thus defined at P does not depend upon the choice of curves which bind P_0 to P, because θ is invariant under h. Thus θ can be extended to a differential form defined over the whole M^n. By the construction, it is clear that coefficients of θ define a skew symmetric tensor field whose covariant derivative vanishes. Hence θ, thus extended, is a harmonic p-form. Especially, if M_n is compact, then by Hodge's theory[1] of harmonic integrals, $B_p \neq 0$.

In the next place we shall prove the bracket part. Let us introduce in $\widetilde{M}^n$ the natural Riemannian metric induced from M^n by the covering. Then by Lemma 1, h^0 is identical with the group $\widetilde{h}$. Hence, by the non-bracket part of our theorem $\widetilde{B}_p \neq 0$.

The first and the simplest theorem which follows from our idea is the following

THEOREM 1. *If a differentiable manifold M^n is orientable, the homogeneous holonomy group h of any Riemannian metric introduced on M^n is contained in the group of proper orthogonal transformations $O^+(n)$. The converse is also true.*

The proof is immediate from the definitions of orientability and holonomy groups.

3. Topological conditions for the irreducibility of the group h^0 with respect to the field of real numbers.

THEOREM 2. *Suppose that the universal covering manifold $\widetilde{M}^n$ of a differentiable manifold M^n is not the topological product of two manifolds of dimensions p and q $(p+q=n)$, then the restricted homogeneous holonomy group h^0 of any complete Riemannian metric introduced on M^n can neither fix a real p-dimensional plane nor a real q-dimensional plane.*

PROOF. By Lemma 1 the theorem follows immediately from the following de Rham's theorem[2] to the effect that "If the holonomy group h of a simply connected and complete Riemannian manifold fixes a p-dimensional plane (and

1) W. V. D. HODGE, The Theory and applications of harmonic integrals. Cambridge Univ. Press, 1941.

2) G. DE RHAM, Sur la réductibilité d'un espace de Riemann. Com. Math. Helv., 26 (1952) 328–344.

a q dimensional plane orthogonal to it), then the manifold is a metric product of two Riemannian manifolds of dimensions p and q".

As a corollary of this theorem we get the following

THEOREM 3. *Suppose that the universal covering manifold $\widetilde{M}^n$ of a differentiable manifold M^n is not the topological product of two differentiable manifolds of lower dimensions, then the restricted holonomy group h^0 (and hence h too) of any complete Riemannian metric introduced on M^n is irreducible in the field k of real numbers.*

THEOREM 4. *Suppose that a compact, orientable differentiable manifold M^n has vanishing Betti number $B_p (= B_q, p + q = n, p$ fixed $0 < p < n)$, then the holonomy group h of any Riemannian metric introduced on M^n can not fix an oriented p-dimensional plane (and an oriented q-dimensional plane orthogonal to it).*

PROOF. Suppose that the theorem is not true. Then we can give M^n a Riemannian metric such that h fixes an oriented p-plane. Let us take the orthogonal frame of reference R_0 at the base point P_0 so that the first p vectors $e_1, \cdots, e_p$ span the p-plane, then $e_{p+1}, \cdots, e_n$ span the q-plane orthogonal to it. We denote the generic infinitesimal vector at P_0 by $\omega^1, \cdots, \omega^n$ and consider differential forms[3]

$$\omega^1\omega^2\cdots\omega^p, \quad \omega^{p+1}\cdots\omega^n.$$

These differential forms are evidently invariant under $h \subseteq O^+(p) \dotplus O^+(q)$, where $\dotplus$ means the direct sum. Hence by Lemma 2, $B_p = B_q \neq 0$ contrary to our assumption. Accordingly our assertion is true.

If M^n is simply connected, then our theorem is an immediate consequence of Theorem 2.

THEOREM 5. *Suppose that M^n is a compact, orientable differentiable manifold of even dimension. If the Euler characteristic $\chi(M^n)$ is not equal to zero, then the holonomy group h^0 (and h too) of any Riemannian metric introduced on M^n can not fix a real odd dimensional plane.*

PROOF. We first notice that Willmore's theorem[4] to the effect that "If the holonomy group h of a compact orientable Riemannian manifold of even dimension fixes an odd dimensional plane, then the Euler characteristic of the manifold is equal to zero" holds good too when we replace the group h by h^0 and the theorem becomes sharper. Our theorem follows immediately from this remark.

THEOREM 6. *Suppose that M^2 is a differentiable manifold of dimensionality 2 which is not homeomorphic with any one of 5 Euclidean space forms. Then*

3) H. IWAMOTO, On the relation between homological structure of Riemannian spaces and exact differential forms which are invariant under holonomy groups. I. Tôhoku Math. Journ., (2) 3 (1951) 59-70.

4) T. J. WILLMORE, Les plans parallèles dans les espaces riemanniens globaux, C. R. Paris 232 (1951), 298-299.

the holonomy group h of any complete Riemannian metric introduced on M^2 is either $O^+(2)$ or $O(2)$ according as M^2 is orientable or not.

REMARK 1. We do not treat holonomy groups H and H^0 because we are concerned mainly with the case where h^0 is irreducible in the field of real numbers and hence by virtue of our former theorem[5] H^0 (and hence H too) contains all translations of E^n.

4. Topological conditions for absolute irreducibility. We shall now turn to absolute irreducibility (i. e. irreducibility in the field K of complex numbers) of the group h^0.

THEOREM 7. *Suppose that the universal covering manifold $\widetilde{M}^n$ of a differentiable manifold M^n of odd dimensionality is not the topological product of two manifolds of lower dimensionality, then the restricted holonomy group h^0 of any complete Riemannian metric introduced on M^n is absolutely irreducible.*

PROOF. First by Theorem 3 the group h^0 is irreducible in the field k. Now suppose that h_0 is reducible in the field K and denote one of its irreducible invariant plane by E. Then E and its conjugate plane $\bar{E}$ span a real invariant plane, this contradicts to the irreducibility of h^0 with respect to k.

THEOREM 8. *Suppose that the universal covering manifold $\widetilde{M}^n$ of a differentiable manifold M^n of even dimensionality $(n > 2)$ is compact and is not the topological product of two differentiable manifolds of lower dimensions. If either one of the Betti numbers $\widetilde{B}_2, \widetilde{B}_4, \cdots, \widetilde{B}_{n-2}$ is equal to zero or one of the Betti numbers $(\widetilde{B}_1 = 0)$ $\widetilde{B}_3, \cdots, \widetilde{B}_{n-1}$ is odd, or one of the relations $\widetilde{B}_{q-2} < B_q$ $(p < n/2)$ does not hold, then the restricted holonomy group h^0 of any Riemannian metric introduced on M^n is absolutely irreducible.*

PROOF. By virtue of Theorem 3 we first see that the group h^0 is irreducible in the field k. Assume that the group h^0 is reducible in the field K. Now, by Abe-Iwamoto's theory[6] we know that if Γ is a matric group with real coefficients which is irreduble in the field k and reducible in the field K and makes invariant a positive definite quadratic form $G = (g_{ij})$, then Γ makes invariant a skew symmetric tensor S such that $(G^{-1}S)^2 = -1$, so h^0 leaves invariant a 2-form

$$\theta = S_{ij}\, \omega^i \wedge \omega^j.$$

Hence, by Lemma 1 and 2, θ can be extended to a differential form over the

5) S. SASAKI and M. GOTO. Some theorems on holonomy groups of Riemannian manifolds. To be published in Trans. Amer. Math. Soc.

6) M. ABE, Irreduzibiliät und absolute Irreduzilität der Matrizensystems. Proc. Phys. Math. Soc. Japan, 24 (1942), 769–789.

H. IWAMOTO, On the structure of Riemannian space whose holonomy groups fix a null-system. Tôhoku Math. Journ., (2) 1 (1949).

whole $\widetilde{M}^n$ so that it satisfies

$$(G^{-1}S)^2 = -1$$

and its covariant derivative vanishes everywhere on $\widetilde{M}^n$, where G is the fundamental metric tensor of $\widetilde{M}^n$ with the naturally induced metric from M^n. Hence $\widetilde{M}^n$ is nothing but a pseudo-Kählerian manifold. Accordingly, by virtue of the well-known theoerm[7] to the effect that "Let K^n be any compact pseudo-kählerian manifold. Then the even dimensional Betti numbers do not vanish, odd dimensional Betti numbers are even and $B_{p-2} < B_p$ for $p < n/2$", we meet a contradiction to our assumption. Accordingly, the group h^0 must be absolutely irreducible.

REMARK 2. $O^+(2)$ is reducible in the field K and fixes the null system $S_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence, the restricted holonomy group h^0 of any two dimensional Riemannian manifold is always reducible in the field K.

THEOREM 10. *If M^n $(n \geqq 3)$ is a compact differentiable manifold whose universal covering manifold $\widetilde{M}^n$ is a homology-sphere. Then the restricted homogeneous holonomy group h^0 of any Riemannian metric introduced on M^n is absolutely irreducible.*

PROOF. As $\widetilde{M}^n$ is a homology sphere, $\widetilde{M}^n$ is not the topological product of two manifolds of lower dimensions. Hence, we see that our assertion is true by Theorem 8 or 9 according as n is odd or even.

COROLLARY. *Any differentiable manifold M^n $(n > 2)$ whose universal covering manifold is a homology sphere can not be metrized so that it is a pseudo-Kählerian manifold.*

PROOF. The real representation of Kählerian manifold K^n is a $2n$-dimensional Riemannian manifold such that every irreducible part of the holonomy group h^0 with respect to the field k is reducible with respect to the field K. Hence our corollary follows immediately from Theorem 10.

THEOREM 11. *Suppose that the universal covering manifold $\widetilde{M}^3$ of a differentiable manifold M^3 of three dimension is not the topological product of two differentiable manifolds of lower dimensions. Then the holonomy group h of any complete Riemannian metric on M^3 is $O^+(3)$ or $O(3)$ according as M^3 is orientable or not.*

PROOF. First, by Theorem 7, the holonomy group h^0 is absolutely irreducible. Assume now that the theorem is not true. Then h^0 must be 1 or 2 parametric Lie subgroup of $O^+(3)$ and hence h^0 is a solvable group. Accordingly, h^0 is reducible in the field k or K, contrary to the absolute irreducibility. Therefore h^0 coincides with $O^+(3)$. Hence, by Theorem 1 we see

7) A. LICHNEROWICZ, Généralisation de la géométrie kählérienne globale. Colloque de géom. diff. tenu à Louvain (1951), pp. 99–122.

that our assertion is true.

5. Decomposition of the group h^0 into Kronecker Products.

LEMMA 3. *If the restricted homogeneous holonomy group h^0 is absolutely irreducible, then h^0 is semi-simple.*

PROOF. First, by virtue of Schur's lemma[8] to the effect that "the commutators of an irreducible matric set L in an algebraically closed field are the numerical multiples αE of the unit matrix E", we know that the center of the group h^0 consists only of elements of the form αE. But as h^0 belongs to the orthogonal group α must be $+1$ or -1, the latter occurs only when n is even. Hence the center of the group h^0 is discrete.

On the other hand, by Borel-Lichnerowicz theorem[9], the group h^0 is compact, so we can apply the theorem[10] to the effect that "a compact connected Lie group is semi-simple if and only if its center is discrete" and we know the truth of our assertion.

LEMMA 4. *Let G be a semi-simple and absolutely irreducible Lie group of matrices with complex coefficients. Then G can be written as a Kronecker product of simple and absolutely irreducible matric groups.*

PROOF. Suppose that G is semi-simple but not simple. Then G can be written as

$$G = g_1 g_2,$$

where g_1 and g_2 are semi-simple normal subgroups of G such that 1) for any elements $a_1 \in g_1, a_2 \in g_2$, $a_1 a_2 = a_2 a_1$ and 2) $g_1 \cap g_2 =$ discrete. (If G is compact, 2') $g_1 \cap g_2 =$ finite).

Now, as g_1 is a subgroup of G, g_1 may be reducible in the field K of complex numbers. We decompose each element of g_1 into irreducible components as follows

$$g_1 = A_1 + A_2 + \cdots + A_q.$$

Then we can see that all irreducible representations $A_1, \cdots, A_q$ are equivalent. For, if this is not true, we may assume that

$$A_1 \sim A_2 \sim \cdots \sim A_p, \qquad (p < q)$$

but they are not equivalent to $A_{p+1}, \cdots A_q$. Corresponding to the decomposition of the matrices g_1, we can write matrices g_2 in the form $\| B_{ik} \|$ where B_{ik} are (d_i, d_k) matrices provided that $d_i = \dim A_i$, $d_k = \dim A_k$. Applying 1) we see that

$$A_a B_{a\lambda} = B_{a\lambda} A_\lambda \quad \begin{Bmatrix} a = 1, 2, \cdots, p, \\ \lambda = p+1, \cdots, q \end{Bmatrix}.$$

8) Cf. H. WEYL, Classical groups, p. 83.

9) A. BOREL-A. LICHNEROWICZ, Groupes d'holonomie des variétés riemanniennes, C. R. Paris 234 (1952), 1835.

10) Cf. L. PONTRJAGIN, Topological groups, p. 282.

Hence, by Schur's Lemma[8] to the effect that "If two sets of matrices $C_1(S)$, $C_2(S)$ are irreducible and inequivalent, then there is no matrix B such that

$$C_1(S)B = BC_2(S)$$

holds identically in S, except $B = 0$", we know that $B_{a\lambda} = 0$, and in the same way $B_{\lambda a} = 0$. This means the reducibility of g_2 and hence that of G contrary to our assumption.

As $A_1, \cdots, A_q$ are all equivalent, we can take basis so that

$$g_1 = A_1 \dotplus A_1 \dotplus \cdots \dotplus A_1. \qquad (q \text{ factors})$$

Again applying 1) we have

$$A_1 B_{ab} = B_{ab} A_1$$

which shows, by virtue of Schur's Lemma[8], that

$$B_{ab} = C_{ab} E.$$

Hence we see that

$$G = A_1 \otimes A_2$$

provided that $A_2 = \| C_{ab} \|$ and $\otimes$ means the Kronecker product.

The sets of matrices A_1, A_2 correspond to g_1 and g_2 respectively. If g_1 or g_2 or both are not simple we continue the same process and we know the truth of our Lemma.

LEMMA 5. *If the homogeneous holonomy group h^0 is absolutely irreducible, then h^0 is equivalent to a Kronecker product of several subgroups of real orthogonal groups and several subgroups of unitary symplectic groups, each of which is absolutely irreducible and simple.*

PROOF. By virtue of Frobenius-Schur's theorem[11] to the effect that "Let G be a compact group and $s \to T(s)$ $(s \in G)$ be an absolutely irreducible linear representation with complex coefficients. Then the representation T is equivalent to a subgroup of a real orthogonal group or a subgroup of unitary simpletic group if and only if

$$\int_G Sp\, T(s^2)\, ds > 0 \text{ or } < 0$$

respectively[12]" we see that

$$\int_{h^0} Sp\, A(p^2)\, ds > 0, \qquad s \in h^0 \tag{1}$$

where $A(s^2)$ is the matric corresponding to s^2 of h^0 regarded as a topological group.

11) G. FROBENIUS–I. SCHUR. Über die reellen Darstellungen der endlichen Gruppe. Sitzungsber. Königlich. Preuss. Akad. Wiss., 1906, 186–208.

12) As G is compact, the set of matrices $T(s)$ is equivalent to a set of unitary matrices $\bar{U}(s)$. Accordingly

On the other hand, as h^0 is compact, each simple group $g_i (i = 1, \cdots, q)$ which appears in the decomposition of h^0

$$h^0 = g_1 g_2 \cdots g_q$$

is compact[13]. (The right hand side is not the direct product, each two of $g_1, \cdots, g_q$ satisfy the conditions 1) and 2′) of Lemma 4). The group g_i corresponds to the set of matrices $\{A_i(S)\}$.

Now the direct product

$$\hat{h} = g_1 \times g_2 \times \cdots \times g_q$$

is a covering group of h^0 of finite sheets, so the integral over $\hat{h}$ analogous to (1) is also finite and positive. Hence we see that

$$\prod_{i=1}^{q} \int_{g_i} Sp\ A_i(s^2) ds > 0. \qquad s \in g_i$$

However, as we noticed in footnote 9) every integral $\int_{g_i} Sp\ A_i(s^2) ds$ is real.

Hence, again by virtue of Frobenius-Schur's theorem that even number of $g_1, \cdots, g_q$ are equivalent to some subgroups of unitary symplectic groups and others are equivalent to some subgroups of real orthogonal groups. Consequently our assertion is true.

Q. E. D.

Now suppose first that one of the matric groups $g_1, \cdots, g_q$ say g_1 is equivalent to a subgroup of an orthogonal group. If h^0 is not simple, then groups $g_2, \cdots, g_q$ exist and $g_2 g_3 \cdots g_q$ is equivalent to a subgroup of an orthogonal group too, as is easily seen by Frobenius-Schur's Lemma. Hence, every matrix of the group h^0 can be written in the following way:

$$A(s) = V \cdot A_1(s) \otimes B(s) \cdot V^{-1},$$

where V is a constant (n, n) matrix and $A_1(s)$, $B(s)$ are matrices which belong to the matric groups $g_1, g_2 g_3 \cdots g_q$ regarded as real orthogonal groups. We do not know at first that whether V is real or not. But even if V is complex, we can easily see that the linear simultaneous equations

$$A(s)V = V \cdot A_1(s) \otimes B(s)$$

$$\int_G Sp\ \overline{T(s^2)}\, ds = \int_G Sp\ \overline{U(s^2)}\, ds = \int_G Sp\ U(s^{-2})\, ds$$

$$= \int_G Sp\ U(s^2)\, ds^{-1} = \int_G Sp\ U(s^2) ds = \int_G Sp\ T(s^2)\, ds,$$

which shows that $\int_G Sp\ T(s^2) ds$ is real. The case $\int_G Sp\ T(s^2) ds = 0$ corresponds to other cases.

13) E CARTAN, La théorie des groupes finis et cotninus et l'Analysis situs. Memorial des Sci. Math., 42 (1930), p. 42.

admit another real non-singular solution V (i. e. det $|V| \neq 0$) if we take account of the fact that the coefficients of the last equations are all real. Hence, if we take suitable real orthogonal frame, our holonomy group becomes of the form $O(l) \otimes O(m)$ or its absolutely irreducible subgroups. Of course $2 < l \leqq m$, for if one of l or m is equal to 2, then $O(l) \otimes O(m)$ is reducible in the field K.

If $q \geqq 3$, then either at least one of $g_1, g_2, \ldots, g_q$ is a simple subgroup of an orthogonal group or $g_1 g_2$ and $g_3 \ldots g_q$ are orthogonal groups. Hence the holonomy group h^0 is again of the form $O(l) \otimes O(m)$ or an absolutely irreducible subgroup of it.

However, if $q = 2$, then there remains the case where both g_1 and g_2 are unitary symplectic groups and hence the holonomy group h^0 is of the form $U Sp(2l) \otimes U Sp(2m)$.

Finally, we notice that if $h^0 \subseteq O(l) \otimes O(m)$ then $h^0 \subseteq O^+(l) \otimes O^+(m)$, because the component of the identity of $O(l) \otimes O(m)$ is $O^+(l) \otimes O^+(m)$. In the same way, if $h^0 \supseteq Sp(2l) \otimes U Sp(2m)$, then $h^0 \subseteq U Sp^+ (2l) \otimes USp^+(2m)$, where we denote by $U Sp^+(2l)$ the subset of $USp(2l)$ such that every element of which is unimodular. In general det $A = +1$ or -1 for $A \in U Sp(2l)$.

We shall now turn to topological conditions in order that the group h^0 is not of the form $O(l) \otimes O(m)$ or $USp(2l) \otimes USp(2m)$.

5. Topological conditions in order that h^0 is not of the form $O(l) \otimes O(m)$.

We first note that if the dimension n of a differentiable manifold M^n is a prime number, the holonomy group h^0 of any Riemannian metric introduced on M^n can not be of the form $O(l) \otimes O(m)$, for if h^0 is of the form $O(l) \otimes O(m)$, then $n = lm$ contrary to our assumption.

THEOREM 11. *Suppose that the universal covering manifold $\widetilde{M}^n$ of a differentiable manifold M^n is compact. We assume that n is not a prime number. If $\widetilde{B}_{2p} = 0$ ($p \geqq 2$ even), then the holonomy group h^0 of any Riemannian metric introduced on M^n can not be the Kronecker product $O(l) \otimes O(m)$ or a subgroup of it provided that $n = lm > 2p$, $2 \leqq l$ and, $p \leqq m$.*

PROOF. Suppose that the theorem is not true. Then we can endow to M^n a Riemannian metric so that the holonomy group h^0 with respect to the metric is equivalent with $O(l) \otimes O(m)$ or a subgroup of it.

Let $e_{i\alpha}(i, j, k = 1, \ldots, l;\ \alpha, \beta, \gamma = 1, \ldots, m)$ be vectors which constitute an orthogonal frame with respect to the metric such that the holonomy group h^0 referred to the frame R is of the form $O(l) \otimes O(m)$. Denote the base point of the holonomy group h^0 by P_0 and the components of the generic infinitisimal vector at P_0 with respect to the frame R by $\omega_{i\alpha}$. $\omega_{i\alpha}$'s are $lm (= n)$ linearly independent differential forms of the first degree at P_0.

Now we shall first prove the case $p = 2$. We put

$$\Phi_4 = \sum \omega_{i\alpha} \omega_{j\alpha} \omega_{j\beta} \omega_{i\beta}$$

$$= S(\omega\omega^*\omega\omega^*)$$

where S is a symbol which means summation with respect to the first index and the last index in the bracket which follow S and $\omega\omega^*\omega\omega^*$ is the matrix multiplication. In each monomial of Φ_4, the components of ω are multiplied exteriorly, so Φ_4 is a differential form of the fourth degree at P_0. Φ_4 is not identically zero for $2 \leqq l \leqq m$, because the form

$$\omega_{1\alpha}\omega_{2\alpha}\omega_{2\beta}\omega_{1\beta}$$

($\alpha \neq \beta$ and we do not sum on α and β) in Φ_4 do not cancel with each other as is easily seen.

Now by h^0 ω transforms as follows:

$$\omega' = A\,\omega\,B^*, \tag{2}$$

where $A \in O(l)$ and $B \in O(m)$. As $A^*A = 1$, $B^*B = 1$, we can easily see that Φ_4 is invariant under the holonomy group h^0. Hence, by virtue of Lemma 2, $\widetilde{B}_4 \neq 0$, contrary to our assumption. Accordingly, h^0 can not be of the form $O(l) \otimes O(m)$.

Next, we shall prove the general case. As the case $p = 2$ we put

$$\Phi_{2p} = S(\underbrace{\omega\omega^*\omega\omega^*\ldots\omega\omega^*}_{2p}),$$

then Φ_{2p} is a differential form of $2p$-th degree at P_0 which is invariant under the holonomy group h^0. This form is not identically zero if $2 \leqq l$, $p \leqq m$, for in such case all the terms in Φ_{2p} which reduce to

$$\omega_{1\alpha_1}\omega_{2\alpha_1}\omega_{2\alpha_2}\omega_{1\alpha_2}\omega_{1\alpha_3}\omega_{2\alpha_3}\omega_{2\alpha_4}\omega_{1\alpha_4}\ldots\omega_{2\alpha_p}\omega_{1\alpha_p}$$

do not cancel with each other, provided that $\alpha_1, \alpha_2, \ldots, \alpha_p$ are all different fixed suffixes and we do not take sum in the last monomial. Accordingly, by virtue of Lemma 2, $\widetilde{B}_{2p} \neq 0$ contrary to our assumption $\widetilde{B}_{2p} = 0$. Hence, the group can not be of the form $O(l) \otimes O(m)$ or a subgroup of it.

REMARK 3. Readers may wonder that if the theorem can be applicable to the case $p = 4$, $l = m = 3$ or not. But, as we can easily verify it, Φ_8 vanishes identically in this case. For $p = 2$ and 4, the restrictions $2 \leqq l \leqq m$ and $2 \leqq l$, $4 \leqq m$ are the best possible. However, for general p there may be l and m which do not satisfy $2 \leqq l$, $p \geqq m$, but satisfy $3 \leqq l; m < p$ ($lm > 2p$) and the criterion of our theorem still holds good. But, unfortunately I can not state now the exact conditions for l and m for such cases. The difficulty lies only in algebraic points.

REMARK 4. If p is odd, then we can easily see $\Phi_{2p} = 0$. Hence, even if $\widetilde{B}_{2p} = 0$ (p odd), we can not get any result by this method.

REMARK 5. The key point of the last theorem is the discovery of invariants under all transformations of the form (2). It is equivalent with the elimination of A and B from (2). From this point of view we become aware that if $l = m$, then

$$\det|\omega_{i\alpha}| = \sum \varepsilon_{i_1 i_2 \ldots i_l}\, \omega_{i_1 1}\omega_{i_2 2}\ldots\omega_{i_l l} \tag{3}$$

may be an invariant under the group h^0 of the form $O(l) \otimes O(l)$, where

$\varepsilon_{i_1 i_2 \dots i_l}$'s are equal to 1, -1 or 0, according as $i_1, i_2, \dots, i_l$ are all different and even or odd permutation of $1, 2, \dots, l$ or at least two of them coincide with each other. However, contrary to our expectation det $|\omega_{i\alpha}|$ is not invariant under the transformations of the form (2), on account of the fact that in each monimial of the right hand side of (3) factors ω are not multiplied ordinarily but are multiplied exteriorly. So, we shall modify the determinant so that the modified one is an invariant under all transformations of the form (2). To do this we put

$$\tau = \omega\omega^*.$$

Hereafter we do not assume that $l = m$. τ transforms under (2) by

$$\tau' = A\tau A^* \tag{4}$$

As τ_{ij}'s are differential forms of the second degree any two τ_{ij}'s in each monomial of

$$\Delta_{2l} \equiv \det|\tau_{ij}| = \sum \varepsilon_{i_1 i_2 \dots i_l} \tau_{i_1 1} \tau_{i_2 2} \dots \tau_{i_l l} \tag{5}$$

are interchangeable without changing the sign of the term, so we can apply ordinary algebraic rule to (5) and know that det $|\tau_{ij}|$ is invariant under all transformations which belong to $O(l) \otimes O(m)$. However, as Δ_{2l} is a skew-symmetric determinant, Δ_{2l} is not identically zero only when l is even and $m \geqq 2$. In such case, Φ_{2l} is not identically zero, the criterion which uses Δ_{2l} is not so powerful than the criterion which uses Φ_{2l}.

6. Topological conditions in order that h^0 is not of the form $USp(2l) \otimes USp(2m)$.

We first note that if the dimension n of a differentiable manifold M^n is not of the form $4lm$ the holonomy group h^0 of any Riemannian metric introduced on M^n can not be of the form $USp(2l) \otimes USp(2m)$, for if h^0 is of fhe form $USp(2l) \otimes USp(2m)$, then $n = 4lm$ contrary to our assumption.

THEOREM 12. *Suppose that the universal covering manifold $\widetilde{M}^n$ of a differentiable manifold M^n is compact. We assume that n is an integer which can be written in the form $4lm$ $(1 \leqq l \leqq m)$. If $\widetilde{B}_{2p} = 0$ $(p \geqq 2$, even), then the holonomy group h^0 of any Riemannian metric introduced on M^n can not be equivalent to the Kronecker product $USp(2l) \otimes USp(2m)$ or a subgroup of it provided that $n = 4lm > 2p$ and (i)$1 \leqq l, p-1 \leqq m$ or (ii)$p-2 \leqq l \leqq m$.*

PROOF. Suppose that the theorem is not true. Then we can endow to M^n a Riemannian metric so that the holonomy group h^0 with respect to the metric is equivalent with $USp(2l) \otimes USp(2m)$ or a subgroup of it. Let $e_{i\alpha}$ $(i, j, k = 1, \dots 2l; \alpha, \beta, \gamma = 1, \dots 2m)$ be vectors of the frame R such that the holonomy group h^0 with respect to the frame R is of the form $USp(2l) \otimes USp(2m)$. Denote the base point of the holonomy group h^0 by P_0 and the components of the generic infinitesimal vector at P_0 with respect to the frame R by $\omega_{i\alpha}$. $\omega_{i\alpha}$'s are $4lm (= n)$ linearly independent differential forms of the first degree with complex coefficients at P_0. We denote by

$$I: I_{ij} = \begin{cases} 1 & \text{when } i = j + l, \\ -1 & \text{when } i = j - l, \\ 0 & \text{otherwise,} \end{cases}$$

$$J: J_{\alpha\beta} = \begin{cases} 1 & \text{when } \alpha = \beta + m, \\ -1 & \text{when } \alpha = \beta - m, \\ 0 & \text{otherwise,} \end{cases}$$

the coefficients of the skew-symmetric 2-forms which are invariant under $USp(2l)$ and $US_p(2m)$ respectively.

First let us prove the case $p = 2$. We define a differential form Ψ_4 by

$$\Psi_4 = S(I\omega J\omega^* I\omega J\omega^*)$$

and show that Ψ_4 is invariant under the group h^0. By an arbitrary transformation of h^0, ω is transformed as follows:

$$\omega' = A\,\omega\, B^*$$

where $A \in USp(2l)$ and $B \in USp(2m)$. As

$$A^*\bar{A} = 1, \quad A^* IA = I,$$

$$B^*\bar{B} = 1, \quad B^* JB = J,$$

we see easily that

$$\bar{A}I = IA, \quad \bar{B}J = JB.$$

Hence we get

$$\begin{aligned} \Psi_4' &= S(I\omega' J\omega'^* I\omega' J\omega'^*) \\ &= S(IA\,\omega\, B^* JB\,\omega^* A^* IA\,\omega\, B^* JB\,\omega^* A^*) \\ &= S(I\omega\, J\,\omega^*\, I\,\omega\, J\,\omega^*) = \Psi_4, \end{aligned}$$

which shows that Ψ_4 is invariant under h^0.

We can verify that Ψ_4 is not identically zero. The simplest way is to notice that the term

$$\omega_{11}\,\omega_{12}\,\omega_{\bar{1}\bar{1}}\,\omega_{\bar{1}\bar{2}}$$

does not vanish, where we have put $\bar{i} = l + i$, $\bar{\alpha} = m + \alpha$. Hence at least one of the two real differential froms $\Re\Psi_4$ and $\Im\Psi_4$, say $\Re\Psi_4$, is not identically zero and is invariant under h^0. Hence, by virtue of Lemma 2, $\widetilde{B_4} \neq 0$, contrary to our assumption. Accordingly, h^0 can not be of the form $USp(2l) \otimes USp(2m)$.

Next, we shall prove the general case. We put

$$\Psi_{2p} = S(\underbrace{I\,\omega\, J\,\omega^*\, I\,\omega\, J\,\omega^* \ldots I\,\omega\, J\,\omega^*}_{2p}),$$

then Ψ_{2p} is a differntial form of $2p$-th degree at P_0, which is invariant under the holonomy group h^0, where P_0 is the base point of the holonomy group h^0. We can verify that Ψ_{2p} is not identically zero.

For, if $1 \leqq l$, $p - 1 \leqq m$, then the term

$$\omega_{1\alpha_1}\,\omega_{\bar{1}\bar{\alpha}_1}\,\omega_{1\alpha_2}\,\omega_{\bar{1}\bar{\alpha}_2} \ldots \omega_{1\alpha_{p-1}}\,\omega_{\bar{1}\bar{\alpha}_{p-1}}\,\omega_{1\bar{\alpha}_1}\,\omega_{\bar{1}\alpha_1}$$

does not vanish, and if $p - 2 \leqq l \leqq m$, the term

$$\omega_{1\alpha_1}\,\omega_{\bar{2}\bar{\alpha}_1}\,\omega_{2\alpha_2}\,\omega_{\bar{3}\bar{\alpha}_2}\cdots\omega_{\overline{p-2}\,\bar{\alpha}_{p-3}}\,\omega_{p-2\,\alpha_{p-2}}\,\omega_{\overline{p-2}\,\bar{\alpha}_{p-2}}\,\omega_{p-2\,\alpha_1}\,\omega_{1\bar{\alpha}_1}\,\omega_{\bar{1}\alpha_1}\,\omega_{\bar{1}\alpha_1}$$

does not vanish, where we assume that $\alpha_1, \alpha_2, \ldots \alpha_{p-2}, \alpha_{p-1}$ are all different fixed integers and we do not take sum about α. Hence at least one of the two differential forms $\Re\Psi_{2p}$ and $\Im\Psi_{2p}$ is not identically zero and is invariant under h^0. Hence, by virtue of Lemma 2, we see $\widetilde{B}_{2p} \neq 0$, contrary to our assumption. Accordingly, the group h^0 can not be of the form $USp(2l) \otimes USp(2m)$.

REMARK 6. As in §5 readers may wonder that if the restrictions (i) and (ii) are the best possible or not. We can say that for $p = 2$ these restrictions become trivial and for $p = 4$ they are the best possible. However, we guess that they are not the best possible for general p, in other words, there will exist a set of values of l and m which do not satisfy (i) and (ii), but the vanishing of the Betti number $\widetilde{B}_{2p}$ still implies that the group h^0 can not be of the form $USp(2l) \otimes USp(2m)$.

REMARK 7. If p is odd, we can easily see that $\Psi_{2p} = 0$. Hence from the vanishing of $\widetilde{B}_{2p}$ (p odd) we can not get any result by our method.

7. Topological conditions for simplicity of the group h^0.

THEOREM 13. *Suppose that the universal covering manifold $\widetilde{M}^n$ of a differentiable manifold M^n is compact, is not the topological product of two manifolds of lower dimensions and satisfies one of the two following conditions*: (i) *n is an odd number*, (ii) *n is an even number and M^n satisfies topological conditions stated in Theorem* 8. *If one of the following two conditions is satisfied, then the holonomy group h^0 of any Riemannian metric introduced on M^n is simple* : (a) *n is a prime number*, (b) *n is not a prime number, but either $n > 4$ and $\widetilde{B}_4 = 0$ or $n > 8$ and $\widetilde{B}_8 = 0$.*

PROOF. First, by Theorem 3, 7 and 8, we see that the group h^0 is absolutely irreducible. Then by Theorem 11 and 12, we see that h^0 can neither be of the form $O(l) \otimes O(m)$ nor of the form $USp(2l) \otimes USp(2m)$. Hence by virtue of §4 we know that h^0 is simple.

COROLLARY. *The holonomy group $h^0 (= h)$ of any Riemannian metric introduced on a homology sphere S^n is simple if $n \geqq 2$ and $n \neq 4$.*

When $n = 4$, the holonomy group $h^0 (= h)$ of any Riemannian metric introduced on S^4 is absolutely irreducible as is seen by Theorem 9. However, we can not apply Theorem 12 and 13 and hence 14. But, h^0 can not be $O(2) \otimes O(2)$, because $O(2) \otimes O(2)$ is reducible in the field K of complex numbers. However $O^+(4)$ itself is equivalent to $USp(2) \otimes USp(2)$ and of course $O^+(4)$ can be the group h^0, h^0 is semi-simple and not simple in general.

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