ON THE CESÀRO SUMMABILITY OF FOURIER SERIES (II)*)

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1. Introduction. Let $\varphi(t)$ be an even integrable function with period 2π and let

$$\varphi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt, \tag{1.1}$$

$$\varphi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \varphi(u)(t-u)^{\alpha-1}\, du \qquad (\alpha > 0). \tag{1.2}$$

G. H. Hardy and J. E. Littlewood [2] have proved the following theorem:

THEOREM A. *If $\varphi(t)$ satisfies*

$$\int_0^t |\varphi(u)|\, du = o\left(t/\log\frac{1}{t}\right), \qquad \textit{as } t \to 0 \tag{1.3}$$

and

$$\int_0^t |d\{u^\Delta \varphi(u)\}| = O(t), \qquad 0 < t \leqq \eta,\ \Delta > 1, \tag{1.4}$$

then the Fourier series of $\varphi(t)$ converges to zero at $t = 0$.

If we replace the condition (1.3) by

$$\int_0^t |\varphi(u)|\, du = o\left(t/\log\frac{1}{t}\right), \tag{1.5}$$

then the theorem does not hold [8].

Concerning the condition (1.5), S. Izumi and G. Sunouchi [4] have proved the following theorem:

THEOREM B. *If $\beta > 0$ and*

$$\varphi_\beta(t) = o\left(t^\beta/\log\frac{1}{t}\right), \tag{1.6}$$

then the Fourier series of $\varphi(t)$ is summable (C, β) to zero at $t = 0$.

Theorem 1 and 2 are concerning with the condition (1.6) and there are many theorems of analogous type. (See the papers of S. Izumi [3], G. Sunouchi [9], M. Kinukawa [6, 7] and K. Kanno [8].) In this note we shall give some theorems relating closely with these theorems.

THEOREM 1. *If*

$$\varphi_\beta(t) = o\left(t^\beta\Big/\left(\log\frac{1}{t}\right)^{\frac{1}{\gamma}}\right) \qquad (\beta, \gamma > 0), \qquad \textit{as } t \to 0, \tag{1.7}$$

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and

$$\int_0^t \left| d\left\{ \frac{t}{\left(\log \frac{1}{t}\right)^{\Delta}} \varphi(t) \right\} \right| = O(t) \qquad (\Delta > 0,\ 0 < t \leqq \eta), \tag{1.8}$$

then the Fourier series of $\varphi(t)$ *is summable* $\left(C, \dfrac{\Delta\gamma\beta - 1}{1 + \Delta\gamma}\right)$ *to zero at* $t = 0$.

From the Theorem 1, we obtain immediately the following corollary:

COROLLARY. *If* $\varphi_\beta(t) = o\left(t^\beta / \left(\log \frac{1}{t}\right)^{\frac{1}{\Delta\beta}}\right)$, $\qquad (\beta > 0)$ *as* $t \to 0$,

and

$$\int_0^t \left| d\left\{ \frac{t\,\varphi(t)}{\left(\log \frac{1}{t}\right)^{\Delta}} \right\} \right| = O(t) \qquad (\Delta > 0,\ 0 < t \leqq \eta),$$

then the Fourier series of $\varphi(t)$ *converges to zero at* $t = 0$.
This is a dual of F. T. Wang's Theorem [11].

THEOREM 2. *If* (1.7) *and*

$$\varphi(t) = O\left(\left(\log \frac{1}{t}\right)^{\Delta}\right) \qquad (\Delta > 0),\ \textit{as } t \to 0, \tag{1.9}$$

then the Fourier series of $\varphi(t)$ *is summable* $\left(C, \dfrac{\Delta\gamma\beta}{1 + \Delta\gamma}\right)$ *to zero at* $t = 0$.

On the other hand G. Sunouchi [10] has proved the following theorem:

THEOREM C. *If* $\varphi(t) = O(t^{-\delta})$ $(1 > \delta > 0)$, *and* $\varphi_\beta(t) = o(t^\gamma)$, $\gamma > \beta > 0$, *as* $t \to 0$, *then the Fourier series of* $\varphi(t)$ *is summable* (C, α) *to zero at* $t = 0$, *where* $\alpha = \beta\delta/(\gamma + \delta - \beta) + \varepsilon$.

And he remarked that this theorem would be valid without ε. In fact, this is certainly true. That is, we have the following theorem:

THEOREM 3. *Under the assumptions of theorem C, the Fouries series of* $\varphi(t)$ *is summable* (C, α) *to zero at* $t = 0$, *where* $\alpha = \beta\delta/(\gamma + \delta - \beta)$.

2. For the proof of theorems we use frequently Bessel summability instead of Cesàro summability. It is well-known that these two methods of summability are equivalent.

Let $J_\mu(t)$ be the Bessel function of order μ, and let

$$\alpha_\mu(t) = J_\mu(t)/t^\mu \tag{2.1}$$

$$V_{1+\mu}(t) = \alpha_{\mu+\frac{1}{2}}(t), \tag{2.2}$$

then

$$V^{(k)}_{1+\mu}(t) = O(1) \quad \text{as } t \to 0 \quad \text{and} \quad V^{(k)}_{1+\mu}(t) = O(t^{-(\mu+1)}) \quad \text{as } t \to \infty, \tag{2.3}$$

for $k = 0, 1, 2, \ldots\ldots$, where the index k denotes the k-times differentiation.

Moreover we need some lemmas.

LEMMA 1. *Let* $V(x)$ *and* $W(x)$ *satisfy the next condition*:

(i) *$V(x)$ is monotone function and there exists a real number $d>0$ such that $x^dV(x)$ is non-decreasing,*
(ii) *$W(x)$ is non-decreasing,*
(iii) *$W(x)/V(x)=O(1)$ for $x>0$,*
then if $\varphi(x)=O(x^bV(x))$, and $\varphi_a(x)=o(x^cW(x))$ for $x>0$, we have

$$\varphi_{a'}(x)=o\left\{x^{(a-a')b/a+a'c/a}\,(V(x))^{1-\frac{a'}{a}}(W(x))^{\frac{a'}{a}}\right\}\qquad(0<a'<a),$$

where *$V(x)$, $W(x)$ are positive for $x>0$ and $-1<c\leqq a+b$ for $x\to+\infty$, or $V(x), W(x)$ are positive for $0<x\leqq\eta$ and $-1<c\geqq a+b$ for $x\to+0$.*

Proof runs over similarly as a theorem due to G. Sunouchi [10].

Let $K_n^{(\alpha)}(t)$ be the n-th Cesàro mean of order α of the series

$$\frac{1}{2}+\sum_{k=1}^{\infty}\cos kt,$$

then we have

LEMMA 2. *If $-1<\alpha\leqq 1$, then*

$$K_n^{(\alpha)}(t)=S_n^{(\alpha)}(t)+R_n^{(\alpha)}(t),$$

and

$$S_n^{(\alpha)}(t)=\frac{\cos(A_nt+A)}{A_n^{(\alpha)}\left(2\sin\frac{t}{2}\right)^{1+\alpha}}.\tag{2.4}$$

where $A_n=n+(\alpha+1)/2,\ A=-(\alpha+1)\pi/2,\ A_n^{(\alpha)}=\binom{n+\alpha}{\alpha}$ *and*

$$|R_n^{(\alpha)}(t)|<M/nt^2,\quad\left|\frac{d}{dt}R_n^{(\alpha)}(t)\right|<M/nt^3+M/nt^4,\tag{2.5}$$

$$\left(\frac{d}{dt}\right)^r S_n^{(\alpha)}(t)=O(n^{r-\alpha}\,t^{-(1+\alpha)}),\qquad \textit{for } nt\geqq 1,\tag{2.6}$$

$$\begin{aligned}\left|\left(\frac{d}{dt}\right)^r K_n^{(\alpha)}(t)\right| &\leqq Mn^{r+1}, && \textit{for } h\geqq 0,\\ &\leqq Mn^{h-\alpha}t^{-(1+\alpha)}, && \textit{for } nt\geqq 1,\ h\geqq 0,\ 0<\alpha\leqq 1.\end{aligned}\tag{2.7}$$

(J. J. Gergen [1] and M. Kinukawa [7])

LEMMA 3. *If $-1<\alpha\leqq 1$ and $\varphi_1(t)=o(t)$, then*

$$\lim_{k\to\infty}\limsup_{n\to\infty}\int_0^{k/n}\varphi(t)\,K_n^{(\alpha)}(t)\,dt=0.$$

LEMMA 4. *If $\varphi_1(t)=O(t)$, then we have*

$$\lim_{n\to\infty}\limsup_{k\to\infty}\int_{k/n}^{x+\xi y}\varphi(t)\,R_n^{(\alpha)}(t)\,dt=0,$$

where ξ is a fixed number and $y=O(k/n)$.

Lemma 3 and 4 are due to J. J. Gergen [1].

3. Proof of Theorem 1. First we consider the case $\alpha = \dfrac{\Delta\gamma\beta - 1}{\Delta\gamma + 1} > 0$. We denote by σ_ω^α the α-th Bessel mean of the Fourier series (1. 1). Neglecting the constant factor,

$$\sigma_\omega^\alpha = \int_0^\infty \omega\varphi(t)\, V_{1+\alpha}(\omega t)\, dt = \left(\int_0^{C\rho(\omega)} + \int_{C\rho(\omega)}^\infty\right) \omega\varphi(t) V_{1+\alpha}(\omega t)\, dt = I + J, \tag{3. 1}$$

say, where C is a fixed large constant and $\rho(\omega) = (\log\omega)^{\frac{\Delta}{\alpha+1}}/\omega$. If we put $\theta(t) = t\varphi(t)/\left(\log\dfrac{1}{t}\right)^\Delta$ and $\Theta(t) = \displaystyle\int_0^t |d\theta(t)|$, then we have by (1. 8)

$$\theta(t) = O(t), \qquad \Theta(t) = O(t). \tag{3. 2}$$

Now we put

$$\Lambda(t) = \int_t^\infty \frac{\left(\log\dfrac{1}{u}\right)^\Delta}{u} V_{1+\alpha}(\omega u)\, du$$

$$= \omega^{-\left(\alpha+\frac{1}{2}\right)} \int_t^\infty \frac{J_{\alpha+\frac{1}{2}}(\omega u)}{u^{\alpha-\frac{1}{2}}} \frac{\left(\log\dfrac{1}{u}\right)^\Delta}{u^2}\, du \qquad \text{for } x \geqq C\rho(\omega).$$

Then, using the formula

$$\int_z^\infty \frac{J_\nu(at)}{t^{\nu-1}}\, dt = J_{\nu-1}(az)/az^{\nu-1} \qquad \text{for } \nu > \frac{1}{2},$$

we get

$$\omega^{\left(\alpha+\frac{1}{2}\right)}\Lambda(t) = \left[-\int_u^\infty \frac{J_{\alpha+\frac{1}{2}}(\omega v)}{v^{\alpha-\frac{1}{2}}}\, dv \cdot \frac{\left(\log\dfrac{1}{u}\right)^\Delta}{u^2} \right]_t^\infty$$

$$+ \int_t^\infty \left\{ \int_u^\infty \frac{J_{\alpha+\frac{1}{2}}(\omega v)}{v^{\alpha-\frac{1}{2}}}\, dv \right\} \left(\frac{\left(\log\dfrac{1}{u}\right)^\Delta}{u^2} \right)' du$$

$$= \left[-\omega^{-1} J_{\alpha-\frac{1}{2}}(\omega u)\, u^{-\left(\alpha+\frac{3}{2}\right)} \left(\log\frac{1}{u}\right)^\Delta \right]_t^\infty$$

$$- \Delta\omega^{-1} \int_t^\infty J_{\alpha-\frac{1}{2}}(\omega u)\, u^{-\left(\alpha-\frac{1}{2}+3\right)} \left(\log\frac{1}{u}\right)^{\Delta-1} du$$

$$- 2\,\omega^{-1} \int_t^\infty J_{\alpha-\frac{1}{2}}(\omega u)\, u^{-\left(\alpha-\frac{1}{2}+3\right)} \left(\log\frac{1}{u}\right)^\Delta du$$

$$= \omega^{-1}(\Lambda_1 + \Lambda_2 + \Lambda_3), \text{ say.}$$

Since $\omega t \geqq C\omega\rho(\omega) > 1$, by (2. 3), we have

$$\Lambda_1 = O\left(\omega^{-\frac{1}{2}} t^{-(\alpha+2)} \left(\log\frac{1}{t}\right)^\Delta\right)$$

and

$$\Lambda_3 = O\left\{\int_t^1 \omega^{-\frac{1}{2}} u^{-(\alpha+3)} \left(\log\frac{1}{u}\right)^\Delta du\right\}$$

$$+ O\left\{\int_1^\infty \omega^{-\frac{1}{2}} u^{-(\alpha+3)} (\log u)^\Delta du\right\}$$

$$= O\left\{\omega^{-\frac{1}{2}} \left(\log\frac{1}{t}\right)^\Delta t^{-(\alpha+2)}\right\} + O(\omega^{-\frac{1}{2}})$$

$$= O\left\{\omega^{-\frac{1}{2}} t^{-(\alpha+2)} \left(\log\frac{1}{t}\right)^\Delta\right\}.$$

Similarly, for $\Delta \geqq 1$

$$\Lambda_2 = O\left\{\omega^{-\frac{1}{2}} t^{-(\alpha+2)} \left(\log\frac{1}{t}\right)^{\Delta-1}\right\}.$$

For $0 < \Delta < 1$, it is easily to see that $u^{-\epsilon}\left(\log\frac{1}{u}\right)^{\Delta-1}$ has the minimum value at $u = \exp\left(-\frac{1-\Delta}{\varepsilon}\right)$ for $0 < u < 1$ and is monotone decreasing for $0 < u < \exp\left(-\frac{1-\Delta}{\varepsilon}\right)$, where ε is any positive number. And so

$$\Lambda_2 = O\left\{\omega^{-\frac{1}{2}} \int_t^\delta u^{-(\alpha+3-\epsilon)} u^{-\epsilon} \left(\log\frac{1}{u}\right)^{\Delta-1} du\right\}$$

$$+ O\left(\omega^{-\frac{1}{2}} \int_\delta^1 u^{-(\alpha+3)} \left(\log\frac{1}{u}\right)^{\Delta-1} du\right)$$

$$+ O\left\{\omega^{-\frac{1}{2}} \int_1^\infty u^{-(\alpha+3)} (\log u)^{\Delta-1} du\right\}$$

$$= O\left\{\omega^{-\frac{1}{2}} t^{-\epsilon} \left(\log\frac{1}{t}\right)^{\Delta-1} t^{-(\alpha+2-\epsilon)}\right\}$$

$$+ O\left(\omega^{-\frac{1}{2}} t^{-\epsilon} \left(\log\frac{1}{t}\right)^{\Delta-1}\right) + O(\omega^{-\frac{1}{2}})$$

for $0 < \Delta < 1$, where $\delta = e^{-\frac{1-\Delta}{\epsilon}}$.

Summing up the estimations of Λ_1, Λ_2 and Λ_3, we have

$$\Lambda(t) = O(\omega^{-(\alpha+2)} \left(\log\frac{1}{t}\right)^\Delta t^{-(\alpha+2)}) \qquad \text{for} \quad t \geqq C\rho(\omega). \tag{3.3}$$

We first investigate J. Integrating by parts, J becomes

$$\int_{C\rho(\omega)}^\infty \omega\varphi(t) V_{1+\alpha}(\omega t)\,dt = \int_{C\rho(\omega)}^\infty \omega\theta(t) \frac{\left(\log\frac{1}{t}\right)^\Delta}{t} V_{1+\alpha}(\omega t)\,dt$$

$$= -\int_{C\rho(\omega)}^{\infty} \omega\,\theta(t)\,d\Lambda(t) = -\Big[\omega\,\theta(t)\,\Lambda(t)\Big]_{C\rho(\omega)}^{\infty} + \omega\int_{C\rho(\omega)}^{\infty} \Lambda(t)\,d\theta(t) = J_1 + J_2,$$

where

$$J_1 = O\left\{\left[\omega^{-(\alpha+1)}\,t^{-(\alpha+1)}\left(\log\frac{1}{t}\right)^{\Delta}\right]_{C\rho(\omega)}^{\infty}\right\}$$

$$= O\left\{\omega^{-(\alpha+1)}\,C^{-(\alpha+1)}\,\frac{\omega^{\alpha+1}}{(\log\omega)^{\Delta}}\left(\log\frac{\omega}{(C\log\omega)^{\frac{\Delta}{\alpha+1}}}\right)^{\Delta}\right\}$$

$$= O(C^{-(\alpha+1)}) \qquad \text{as } \omega\to\infty,$$

and

$$J_2 = O\left\{\omega\int_{C\rho(\omega)}^{\infty}\omega^{-(\alpha+2)}t^{-(\alpha+2)}\left(\log\frac{1}{t}\right)^{\Delta}|d\theta(t)|\right\}$$

$$= O\left\{\omega^{-(\alpha+1)}\left[t^{-(\alpha+2)}\left(\log\frac{1}{t}\right)^{\Delta}\Theta(t)\right]_{C\rho(\omega)}^{\infty}\right\}$$

$$+ O\left\{\omega^{-(\alpha+1)}\int_{C\omega(\rho)}^{\infty}t^{-(\alpha+3)}\left(\log\frac{1}{t}\right)^{\Delta}\Theta(t)\,dt\right\}$$

$$+ O\left\{\omega^{-(\alpha+1)}\int_{C\rho(\omega)}^{\infty}t^{-(\alpha+3)}\left(\log\frac{1}{t}\right)^{\Delta-1}\Theta(t)\,dt\right\}$$

$$= O\left\{\omega^{-(\alpha+1)}\left[t^{-(\alpha+1)}\left(\log\frac{1}{t}\right)^{\Delta}\right]_{C\rho(\omega)}^{\infty}\right.$$

$$\left.+ O\left\{\omega^{-(\alpha+1)}(C\rho(\omega))^{-(\alpha+1)}\left(\log\frac{1}{C\rho(\omega)}\right)^{\Delta}\right\}\right.$$

$$= O(C^{-(\alpha+1)}) \qquad \text{as } \omega\to\infty,$$

by (3.2) and the similar estimation to those of $\Lambda(t)$.
Thus, if we take C sufficiently large, we get

(3.4) $$J = J_1 + J_2 = o(1) \qquad \text{as } \omega\to\infty.$$

Now there is an integer $k>1$ such that $k-1<\beta\leqq k$. We may suppose that $k-1<\beta<k$, for the case $\beta=k$ can be easily deduced by the following argument. By integration by parts k-times, we have

$$I = \int_0^{C\rho(\omega)} \omega\varphi(t)\,V_{1+\alpha}(\omega t)\,dt$$

$$= \sum_{h=1}^{k}(-1)^{h-1}\Big[\omega^h\varphi_h(t)\,V_{1+\alpha}^{(h-1)}(\omega t)\Big]_0^{C\rho(\omega)}$$

$$+ (-1)^k\omega^{k+1}\int_0^{C\rho(\omega)}\varphi_k(t)\,V_{1+\alpha}^{(k)}(\omega t)\,dt$$

(3.5) $$= \sum_{h=1}^{k}(-1)^{h-1}I_h + (-1)^k I_{k+1}, \text{ say.}$$

From (1.8), $\varphi(t) = O\left(\left(\log \frac{1}{t}\right)^{\Delta}\right)$. Then we may put in Lemma 1

$V(t) = \left(\log \frac{1}{t}\right)^{\Delta}$ and $W(t) = \left(\log \frac{1}{t}\right)^{-\frac{1}{\gamma}}$. Hence

(3.6) $$\varphi_h(t) = o\left\{ t^k \left(\log \frac{1}{t}\right)^{\Delta - \frac{h}{\beta}\left(\Delta + \frac{1}{\gamma}\right)} \right\} \qquad \text{for } h = 1, 2, \ldots k-1,$$

and

(3.7) $$\varphi_k(t) = o\left\{ t^k \Big/ \left(\log \frac{1}{t}\right)^{-\frac{1}{\gamma}} \right\}.$$

Therefore,

$$I_h = \Big[\omega^h \varphi_h(t)\, V_{1+\alpha}^{(h-1)}(\omega t) \Big]_0^{C\rho(\omega)}$$

$$= o\left\{ \left[\omega^h t^h \left(\log \frac{1}{t}\right)^{\Delta - \frac{h}{\beta}\left(\Delta + \frac{1}{\gamma}\right)} (\omega t)^{-(1+\alpha)} \right]_0^{C\rho(\omega)} \right\}$$

$$= o\left\{ (\log \quad \omega^{\frac{\Delta}{\alpha+1}(h-(1+\alpha))} \left(\log \frac{\omega}{(\log \omega)^{\frac{\Delta}{\alpha+1}}}\right)^{\Delta - \frac{h}{\beta}\left(\Delta + \frac{1}{\gamma}\right)} \right\}$$

$$= o\left\{ (\log \omega)^{\frac{\Delta}{\alpha+1}(h-(1+\alpha)) + \Delta - \frac{h}{\beta}\left(\Delta + \frac{1}{\gamma}\right)} \right\}.$$

Since $\dfrac{\Delta\gamma + 1}{\gamma} = \dfrac{\Delta(\beta + 1)}{\alpha + 1}$, the exponent of $\log \omega$ is

$$\frac{\Delta h}{\alpha + 1} - \frac{h(\Delta\gamma + 1)}{\beta\gamma} = \frac{\Delta h}{\alpha + 1} - \frac{\Delta h(\beta + 1)}{\beta(\alpha + 1)} = \frac{-\Delta h}{\beta(\alpha + 1)} < 0.$$

Thus, we have

(3.8) $$I_h = o(1) \quad \text{as } \omega \to \infty, \qquad \text{for } h = 1, 2, \ldots, k-1.$$

Here the terms I_h, $h = 2, 3, \ldots, k-1$, appear for $\alpha > 2$.

$$I_k = \Big[\omega^k \varphi_k(t)\, V_{1+\alpha}^{(k-1)}(\omega t) \Big]_0^{C\rho(\omega)}$$

$$= o\left\{ \omega^{k-(1+\alpha)} \left[t^{k-(1+\alpha)} \left(\log \frac{1}{t}\right)^{-\frac{1}{\gamma}} \right]_0^{C\rho(\omega)} \right\}$$

$$= o\left\{ (\log \omega)^{\frac{\Delta(k-(1+\alpha))}{1+\alpha}} \left(\log \frac{\omega}{(\log \omega)^{\frac{\Delta}{\alpha+1}}}\right)^{-\frac{1}{\gamma}} \right\}$$

(3.9) $$= o\left\{ (\log \omega)^{\frac{k\Delta}{1+\alpha} - \frac{\Delta\gamma+1}{\gamma}} \right\}$$

$$= o\left\{ (\log \omega)^{\frac{(k-(\beta+1))\Delta}{\alpha+1}} \right\} = o(1), \qquad \text{as } \omega \to \infty.$$

Concerning I_{k+1}, we split up four parts,

$$I_{k+1} = \omega^{k+1} \int_0^{C\rho(\omega)} V_{1+\alpha}^{(k)}(\omega t)\, dt \int_0^t \varphi_\beta(u)\, (t-u)^{k-\beta-1}\, du$$

$$= \int_0^{\omega^{-1}} du \int_u^{u+\omega^{-1}} dt + \int_{\omega^{-1}}^{C\rho(\omega)} du \int_u^{u+\omega^{-1}} dt$$

$$+ \int_0^{C\rho(\omega)-\omega^{-1}} du \int_{u+\omega^{-1}}^{C\rho(\omega)} dt - \int_0^{C\rho(\omega)} du \int_{C\rho(\omega)}^{u+\omega^{-1}} dt$$

$$(3.10) \qquad = K_1 + K_2 + K_3 - K_4, \text{ say.}$$

Since $V_{1+\alpha}^{(k)}(t) = O(1)$ for $0 \leqq t \leqq 1$,

$$K_1 = \omega^{k+1} \int_0^{\omega^{-1}} \varphi_\beta(u)\, du \int_u^{u+\omega^{-1}} V_{1+\alpha}^{(k)}(\omega t)\, (t-u)^{k-\beta-1}\, dt$$

$$= o\left\{ \omega^{k+1} \int_0^{\omega^{-1}} \frac{u^\beta}{\left(\log \frac{1}{u}\right)^{\frac{1}{\gamma}}} \Big[(t-u)^{k-\beta} \Big]_u^{u+\omega^{-1}} du \right\}$$

$$(3.11) \qquad = o\left\{ \omega^{\beta+1} \int_0^{\omega^{-1}} \frac{u^\beta}{\left(\log \frac{1}{u}\right)^{\gamma}} du \right\} = o\left((\log \omega)^{-1}\right) = o(1), \quad \text{as } \omega \to \infty.$$

$$K_2 = \omega^{k+1} \int_{\omega^{-1}}^{C\rho(\omega)} \varphi_\beta(u)\, du \int_u^{u+\omega^{-1}} V_{1+\alpha}^{(k)}(\omega t)\, (t-u)^{k-\beta-1}\, dt$$

$$= o\left\{ \omega^{k+1} \int_{\omega^{-1}}^{C\rho(\omega)} \frac{u^\beta}{\left(\log \frac{1}{u}\right)^{\frac{1}{\gamma}}} du \int_u^{u+\omega^{-1}} (\omega t)^{-(1+\alpha)} (t-u)^{k-\beta-1}\, dt \right\}$$

$$= o\left\{ \omega^{k-\alpha} \int_{\omega^{-1}}^{C\rho(\omega)} \frac{u^{\beta-(1+\alpha)}}{\left(\log \frac{1}{u}\right)^{\frac{1}{\gamma}}} \Big[(t-u)^{k-\beta} \Big]_u^{u+\omega^{-1}} du \right\}$$

$$= o\left\{ \omega^{k-\alpha-(k-\beta)} \left(\log \frac{\omega}{\left(\log \omega\right)^{\frac{\Delta}{\alpha+1}}} \right)^{-\frac{1}{\gamma}} \Big[u^{\beta-\alpha} \Big]_{\omega^{-1}}^{C\rho(\omega)} \right\}$$

$$(3.12) \qquad = o\left\{ (\log \omega)^{-\frac{1}{\gamma} + \frac{\Delta(\beta-\alpha)}{\alpha+1}} \right\} = o(1), \quad \text{as } \omega \to \infty.$$

Concerning K_3, if we use integration by parts in the inner integral, then

$$K_3 = \omega^{k+1} \int_0^{C\rho(\omega)-\omega^{-1}} \varphi_\beta(u)\, du \int_{u+\omega^{-1}}^{C\rho(\omega)} V_{1+\alpha}^{(k)}(\omega t)(t-u)^{k-\beta-1}\, dt$$

$$= \omega^{k+1} \int_0^{C\rho(\omega)-\omega^{-1}} \varphi_\beta(u)\, du \left\{ \Big[\omega^{-1} V_{1+\alpha}^{(k-1)}(\omega t)(t-u)^{k-\beta-1} \Big]_{u+\omega^{-1}}^{C\rho(\omega)} \right.$$

$$\left. - (k-\beta-1) \int_{u+\omega^{-1}}^{C\rho(\omega)} w^{-1} V_{1+\alpha}^{(k-1)}(\omega t)(t-u)^{k-\beta-2}\, dt \right\}$$

$$= M_1 - (k-\beta-1) M_2.$$

$$M_1 = O\left\{ \omega^k(\log\omega)^{-\Delta}\int_0^{C\rho(\omega)} \varphi_\beta(u)\,(C\rho(\omega)-u)^{k-\beta-1}\,du \right.$$

$$\left. + \omega^{\beta-\alpha}\int_0^{C\rho(\omega)} \varphi_\beta(u)(u-\omega^{-1})^{-(1+\alpha)}\,du \right\}$$

$$= o\left\{ \omega^k(\log\omega)^{-\Delta}\int_0^{C\rho(\omega)} \frac{u^\beta}{\left(\log\dfrac{1}{u}\right)^{\frac{1}{\gamma}}}(C\rho(\omega)-u)^{k-\beta-1}\,du \right.$$

$$\left. + \omega^{\beta-\alpha}\int_0^{C\rho(\omega)} \frac{u^\beta}{\left(\log\dfrac{1}{u}\right)^{\frac{1}{\gamma}}}(u-\omega^{-1})^{-(1+\alpha)}\,du \right\}$$

$$= o\left\{ \omega^k(\log\omega)^{-\Delta}\left(\log\frac{\omega}{(\log\omega)^{\frac{\Delta}{\alpha+1}}}\right)^{-\frac{1}{\gamma}}\left(\frac{(\log\omega)^{\frac{\Delta}{\alpha+1}}}{\omega}\right)^k \right\}$$

$$+ o\left\{ \omega^{\beta-\alpha}\left(\log\frac{\omega}{(\log\omega)^{\frac{\Delta}{\alpha+1}}}\right)^{-\frac{1}{\gamma}}\left(\frac{(\log\omega)^{\frac{\Delta}{\alpha+1}}}{\omega}\right)^{\beta-\alpha} \right\}$$

$$= o\left\{(\log\omega)^{-\frac{\Delta\gamma+1}{\gamma}+\frac{\Delta k}{\alpha+1}}\right\} + o\left\{(\log\omega)^{-\frac{1}{\gamma}+\frac{\Delta(\beta-\alpha)}{\alpha+1}}\right\}$$

$$= o\left\{ (\log\omega)^{\frac{k-(\beta+1)}{\alpha+1}} \right\} + o(1) = o(1), \qquad \text{as } \omega\to\infty.$$

$$M_2 = o\left\{ \omega^k \int_0^{C\rho(\omega)} \frac{u^\beta}{\left(\log\dfrac{1}{u}\right)^{\frac{1}{\gamma}}}du \int_{u+\omega^{-1}}^{C\rho(\omega)} \omega^{-(1+\alpha)}\,t^{-(1+\alpha)}\,(t-u)^{k-\beta-2}\,dt \right\}$$

$$= o\left\{ \omega^{k-(1+\alpha)}\int_0^{C\rho(\omega)} u^{\beta-(1+\alpha)}\frac{1}{\left(\log\dfrac{1}{u}\right)^{\frac{1}{\gamma}}}\Big[(t-u)^{k-\beta-1}\Big]_{u+\omega^{-1}}^{C\rho(\omega)}\,du \right\}$$

$$= o\left\{ \omega^{\beta-\alpha}\int_0^{C\rho(\omega)} u^{\beta-(1+\alpha)}\left(\log\frac{1}{u}\right)^{-\frac{1}{\gamma}} du \right\}$$

$$= o\left\{ \omega^{\beta-\alpha}(\log\omega)^{-\frac{1}{\gamma}}\,\omega^{-(\beta-\alpha)}\,(\log\omega)^{\frac{\Delta(\beta-\alpha)}{\alpha+1}} \right\} = o(1), \qquad \text{as } \omega\to\infty.$$

Thus, we have

(3.13) $$K_3 = o(1), \qquad \text{as } \omega\to\infty.$$

$$K_4 = \omega^{k+1}\int_{C\rho(\omega)-\omega^{-1}}^{C\rho(\omega)} \varphi_\beta(u)\,du \int_{C\rho(\omega)}^{u+\omega^{-1}} V_{1+\alpha}^{(k)}(\omega t)(t-u)^{k-\beta-1}\,dt$$

$$= O\left\{ \omega^{k+1-(1+\alpha)}\int_{C\rho(\omega)-\omega^{-1}}^{C\rho(\omega)} \varphi_\beta(u)\,du \int_{C\rho((\omega)}^{u+\omega^{-1}} t^{-(1+\alpha)}\,(t-u)^{k-\beta-1}\,dt \right\}$$

$$= o\left\{ \omega^{k-\alpha}\int_{C\rho(\omega)-\omega^{-1}}^{C\rho(\omega)} u^\beta\left(\log\frac{1}{u}\right)^{-\frac{1}{\gamma}} du\,(C\rho(\omega))^{-(1+\alpha)}\int_{C\rho(\omega)}^{u+\omega^{-1}} (t-u)^{k-\beta-1}\,dt \right\}$$

$$= o\left\{ \omega^{k+1}(\log \omega)^{-\Delta} \int_{C\rho(\omega)-\omega^{-1}}^{C\rho(\omega)} u^{\beta}\left(\log \frac{1}{u}\right)^{-\frac{1}{\gamma}} \left[(t-u)^{k-\beta} \right]_{C\rho(\omega)}^{u+\omega^{-1}} du \right\}$$

$$= o\left\{ \omega^{\beta+1}(\log\omega)^{-\Delta} \int_{C\rho(\omega)-\omega^{-1}}^{C\rho(\omega)} u^{\beta}\left(\log \frac{1}{u}\right)^{-\frac{1}{\gamma}} du \right\}$$

$$= o\left\{ \omega^{\beta+1}(\log \omega)^{-\Delta}(\log \omega)^{-\frac{1}{\gamma}} \left(\frac{(\log \omega)^{\frac{\Delta}{\alpha+1}}}{\omega} \right)^{\beta+1} \right\}$$

$$(3.14) \qquad = o\left\{ (\log \omega)^{-\frac{\Delta\gamma+1}{\gamma}+\frac{\Delta(\beta+1)}{\alpha+1}} \right\} = o(1), \qquad \text{as } \omega \to \infty.$$

Summing up (3.10), (3.11). (3.12). (3.13) and (3.14), we get

$$(3.15) \qquad I_{k+1} = o(1), \qquad \text{as } \omega \to \infty.$$

From (3.1), (3.4), (3.5), (3.8), (3.9) and (3.15) we have

$$(3.16) \qquad \sigma_{\omega}^{\alpha} = o(1) \qquad \text{as } \omega \to \infty, \quad \text{for } \alpha > 0.$$

Next, we consider the case $-1 < \alpha \leqq 0$.
If we denote by σ_n^{α} the n-th Cesàro mean of order α of the Fourier series of $\varphi(t)$ at $t = 0$, and

$$\pi\sigma_n^{\alpha} = \int_0^{\pi} \varphi(t)\, K_n^{\alpha}(t)\, dt$$

$$(3.17) \qquad = \int_0^{k/n} \varphi(t)\, K_n^{\alpha}(t)\, dt + \int_{k/n}^{\pi} \varphi(t)\, R_n^{\alpha}(t)\, dt + \int_{k/n}^{\pi} \varphi(t)\, S_n^{\alpha}(t)\, dt$$

$$= I_1 + I_2 + I_3,$$

say. By (3.6)

$$(3.18) \qquad \varphi_1(t) = o\left\{ t\left(\log \frac{1}{t}\right)^{\Delta-\frac{1}{\beta}\left(\Delta+\frac{1}{\gamma}\right)} \right\} = o(t),$$

$$\text{for} \qquad \frac{1}{\beta}\left(\Delta\beta - \frac{\Delta\gamma+1}{\gamma}\right) = \frac{\Delta(\beta\alpha-1)}{\beta(\alpha+1)} < 0.$$

Hence, by Lemma 3 and Lemma 4, we get

$$(3.19) \qquad I_1 = o(1), \qquad I_2 = o(1), \qquad \text{as } n \to \infty.$$

Therefore, it is sufficient to show that $I_3 = o(1)$. Let

$$\rho(n) = \frac{(\log n)^{\frac{\Delta}{\alpha+1}}}{n},$$

$$(3.20) \qquad I_3 = \left\{ \int_{k/n}^{k\rho(n)} + \int_{k\rho(n)}^{\pi} \right\} \varphi(t)\, S_n^{\alpha}(t)\, dt = J_1 + J_2,$$

say. If we put

$$\Lambda(t) = \int_t^{\pi} \frac{\left(\log \frac{1}{u}\right)^{\Delta}}{u} \frac{\cos(A_n u + A)}{\left(2\sin \frac{u}{2}\right)^{\alpha+1}}\, du,$$

then

$$\Lambda(t) = O\Big(\Big(\log \frac{1}{t}\Big)^{\Delta} / nt^{\alpha+2}\Big).$$

Integrating by parts we have

$$J_2 = -\frac{1}{A_n^\alpha}\int_{k\rho(n)}^{\pi} \theta(t)\,d\Lambda(t) = -\frac{1}{A_n^\alpha}\Big\{\Big[\theta(t)\,\Lambda(t)\Big]_{k\rho(n)}^{\pi} - \int_{k\rho(n)}^{\tau} \Lambda(t)\,d\theta(t)\Big\} = K_1 + K_2,$$

say. The calculations of K_1 and K_2 are similar to those of J_1 and J_2 in the former half of this theorem. Thus, if we take k sufficiently large we obtain

$$J_2 = o(1), \qquad \text{as } n \to \infty. \tag{3.21}$$

In the estimation of J_1, we may suppose that $m-1<\beta<m$, where $m\ (>1)$ is an integer. Integrating by parts m-times we have

$$J_1 = \int_{k/n}^{k\rho(n)} \varphi(t)\,S_n^\alpha(t)\,dt = \Big[\sum_{h=1}^{m}(-1)^{h-1}\varphi_h(t)\Big(\frac{d}{dt}\Big)^{h-1}S_n^\alpha(t)\Big]_{k/n}^{k\rho(n)}$$

$$+(-1)^m\int_{k/n}^{k\rho(n)} \varphi_m(t)\Big(\frac{d}{dt}\Big)^m S_n^\alpha(t)\,dt = \sum_{h=1}^{m}(-1)^{h-1}L_h + (-1)^m L_{m+1}, \tag{3.22}$$

say. Using Lemma 4 and (3.6), we get

$$L_h = o\left\{\Big[\,t^h\Big(\log\frac{1}{t}\Big)^{\Delta-\frac{h}{\beta}\left(\Delta+\frac{1}{\gamma}\right)} n^{h-(1+\alpha)}t^{-(1+\alpha)}\Big]_{k/n}^{k\rho(n)}\right\}$$

$$= o(n^{-\frac{\Delta h}{\beta(\alpha+1)}} + o((\log n)^{\Delta-\frac{h}{\beta}\left(\Delta+\frac{1}{\gamma}\right)}) = o(1) \qquad \text{as } n\to\infty, \tag{3.23}$$

for $h = 1, 2, \ldots, m-1$.

Similarly,

$$L_m = \Big[\varphi_m(t)\Big(\frac{d}{dt}\Big)^{m-1}S_n^\alpha(t)\Big]_{k/n}^{k\rho(n)} = o((\log n)^{\frac{\Delta(k-\beta-1)}{\alpha+1}}) = o(1), \text{ as } n\to\infty. \tag{3.24}$$

Concerning L_{m+1},

$$L_{m+1} = \int_{k/n}^{k\rho(n)}\Big(\frac{d}{dt}\Big)^m S_n^\alpha(t)\,dt\int_0^t \varphi_\beta(t)\,(t-u)^{m-\beta-1}\,du$$

$$= \int_0^{k/n}\varphi_\beta(u)\,du\int_{k/n}^{u+k/n}\Big(\frac{d}{dt}\Big)^m S_n^\alpha(t)\,(t-u)^{m-\beta-1}\,dt$$

$$+\int_{k/}^{k\rho(n)}du\int_u^{u+k/n}dt + \int_0^{k\rho(n)-k/n}du\int_{u+k/n}^{k\rho(n)}dt - \int_{k\rho(n)-k/n}^{k\rho(n)}du\int_{k\rho(n)}^{u+k/n}dt$$

$$= M_1 + M_2 + M_3 + M_4, \text{ say.}$$

The methods of the estimations of M_ν ($\nu = 1, 2, 3, 4$) are similar to those of the former half of his theorem. Thus, we get

$$L_{m+1} = o(1), \qquad \text{as } n\to\infty. \tag{3.25}$$

Summing up (3.17), (3.19), (3.21), (3.22), (3.23), (3.24) and (3.25), we get

$$(3.26) \qquad \sigma_n^\alpha = o(1), \qquad \text{as } n \to \infty \quad \text{for } -1 < \alpha \leqq 0.$$

From (3.16) and (3.26), the theorem is proved completely.

4. Proof of Theorem 2. We denote by σ_ω^α the α-th Bessel mean of the Fourier series (1.1), where $\alpha = \dfrac{\Delta\gamma\beta}{\Delta\gamma + 1} > 0$. Then

$$(4.1) \quad \sigma_\omega^\alpha = \int_0^\infty \omega\,\varphi(t) V_{1+\alpha}(\omega t)\,dt = \left(\int_0^{C\rho(\omega)} + \int_{C\rho(\omega)}^\infty\right) \omega\,\varphi(t) V_{1+\alpha}(\omega t)\,dt$$
$$= I + J,$$

say, where C is a fixed large constant and $\rho(\omega) = \dfrac{(\log \omega)^{\frac{\Delta}{\alpha}}}{\omega}$.

By the assumption (1.9),

$$I = O\left\{\int_{C\rho(\omega)}^\infty \omega(\omega t)^{-(1+\alpha)}\left(\log\frac{1}{t}\right)^\Delta dt\right\} = O\left\{\omega^{-\alpha}(C\rho(\omega))^{-\alpha}\left(\log\frac{1}{C\rho(\omega)}\right)^\Delta\right\}$$
$$= O\left\{\omega^{-\alpha}C^{-\alpha}\omega^\alpha\left(\log\omega\right)^{-\Delta}\left(\log\frac{\omega}{C(\log\omega)^{\frac{\Delta}{\alpha}}}\right)\right\} = O(C^{-\alpha}).$$

Thus, if we take C sufficiently large, we have

$$(4.2) \qquad I = o(1), \qquad \text{as } \omega \to \infty.$$

The estimation of J is similar to those of Theorem 1. So we have

$$(4.3) \qquad J = o(1), \qquad \text{as } \omega \to \infty.$$

From (4.1), (4.2) and (4.3), we have

$$\sigma_\omega^\alpha = o(1), \qquad \text{as } \omega \to \infty,$$

which is the required.

5. Proof of Theorem 3. We use Bessel summability and denote by σ_ω^α the Bessel mean of Fourier series (1.1), where $\alpha = \dfrac{\beta\delta}{\gamma + \delta - \beta}$. Then,

$$(5.1) \quad \int_0^\infty \omega\varphi(t) V_{1+\alpha}(\omega t)\,dt = \left(\int_0^{C\omega^{-\rho}} + \int_{C\omega^{-\rho}}^\infty\right)\omega\varphi(t) V_{1+\alpha}(\omega t)\,dt = I + J,$$

say, where $\rho = \dfrac{\beta}{\gamma+\delta} = \dfrac{\alpha}{\alpha+\delta}$.

By the assumption $\varphi(t) = O(t^{-\delta})$ and (2.3), we have.

$$J = O\left\{\int_{C\omega^{-\rho}}^\infty \omega t^{-\delta}(\omega t)^{-(1+\alpha)}\,dt\right\} = O\left\{\omega^{-\alpha}\left[t^{-(\alpha+\delta)}\right]_{C\omega^{-\rho}}^\infty\right\}$$
$$= O(C^{-(\alpha+\delta)}\omega^{-\alpha+\rho(\alpha+\delta)}) = O(C^{-(\alpha+\delta)}).$$

Therefore, if we take C sufficiently large, we get

$$(5.2) \qquad J = o(1), \qquad \text{as } \omega \to \infty.$$

Now, there is an integer $k > 1$ such that $k - 1 < \beta \leqq k$. We may suppose that $k - 1 < \beta < k$. By integration by parts k-times, we have

$$I = \sum_{h=1}^{k}(-1)^{h-1}\Big[\omega^h \varphi_h(t)\, V_{1+\alpha}^{(h-1)}(\omega t)\Big]_0^{C\omega^{-\rho}} + (-1)^k \omega^{k+1}\int_0^{C\omega^{-\rho}} \varphi_k(t)\, V_{1+\alpha}^{(k)}(\omega t)\, dt$$

$$= \sum_{h=1}^{k}(-1)^{h-1} I_h + (-1)^k I_{k+1}, \text{ say.} \tag{5.3}$$

In Lemma 1, we may put $V(t) = W(t) = 1$, $b = -\delta$, $a = \beta$ and $c = \gamma$. Hence we get

$$\varphi_h(t) = o(t^{-\delta(\beta-h)/\beta+h\gamma/\beta}) \qquad \text{for } h = 1, 2, \ldots, k-1.$$

And

$$\varphi_k(t) = o(t^{\gamma-\beta+1}).$$

Therefore,

$$I_h = \Big[\omega^h t^{\{-\delta(\beta-h)/\beta+h\gamma\}/\beta}(\omega t)^{-(1+\alpha)}\Big]_0^{C\omega^{-\rho}}$$

$$= o\left\{\omega^{h-(1+\alpha)}\omega^{-\rho\{-\delta(\beta-h)+h\gamma\}/\beta+\rho(1+\alpha)}\, C^{-(\alpha+1)-\{\delta(\beta-h)+h\gamma\}/\beta}\right\}.$$

Since $\rho = \dfrac{\beta}{\gamma+\delta} = \dfrac{\alpha}{\alpha+\delta}$, the exponent of ω of the last formula is

$$h-(1+\alpha) - \frac{\rho}{\beta}\{-\delta(\beta-h)+h\gamma-\beta(1+\alpha)\}$$

$$= h-(1+\alpha) - \frac{\rho}{\beta}\{-\beta(1+\alpha+\delta)+h(\gamma+\delta)\}$$

$$= -(\alpha+1) + \frac{\alpha}{\alpha+\delta}(1+\alpha+\delta) = -\frac{\delta}{\alpha+\delta} < 0.$$

Thus, we have

$$I_h = o(1), \qquad \text{as } \omega \to \infty \quad (h = 1, 2, \ldots, k-1). \tag{5.4}$$

Concerning I_k,

$$I_k = \Big[\omega^k t^{\gamma-\beta+k}(\omega t)^{-(1+\alpha)}\Big]_0^{C\omega^{-\rho}} = O\{\omega^{k-(1+\alpha)-\rho(\gamma-\beta+k-(1+\alpha))}\, C^{\gamma-\beta+k-(1+\alpha)}\}.$$

The exponent of ω is

$$k(1-\rho) - (1+\alpha)(1-\rho) - \rho(\gamma-\beta)$$

$$= \frac{k(\gamma+\delta-\beta)}{\gamma+\delta} - \frac{\gamma+\delta-\beta+\beta\delta}{\gamma+\delta} - \frac{\beta(\gamma-\beta)}{\gamma+\delta}$$

$$= \frac{\gamma+\delta-\beta}{\gamma+\delta}(k-1-\beta) < 0.$$

Therefore,

$$I_k = o(1), \qquad \text{as } \omega \to \infty. \tag{5.5}$$

Concerning I_{k+1}, we split up four parts,

$$I_{k+1} = \int_0^{C\omega^{-\rho}} \omega^{k+1}\varphi_\beta(u)\, du \int_u^{C\omega^{-\rho}} V_{1+\alpha}^{(k)}(\omega t)(t-u)^{k-\beta-1}\, dt$$

$$= \int_0^{\omega^{-1}} du \int_u^{u+\omega^{-1}} dt + \int_{\omega^{-1}}^{C\omega^{-\rho}} du \int_u^{u+\omega^{-1}} dt$$

$$+\int_0^{C\omega^{-\rho}-\omega^{-1}} du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} dt - \int_0^{C\omega^{-\rho}} du \int_{C\omega^{-\rho}}^{u+\omega^{-1}} dt$$

$$= K_1 + K_2 + K_3 + K_4, \tag{5.6}$$

say. Estimations of them are similar to those of the theorem of the author [5]. And so, leaving out the detailed calculations, we have

$$K_1 = o(\omega^{-(\gamma-\beta)}) = o(1), \qquad \text{for } \gamma > \beta,$$

$$K_2 = o(\omega^{\beta-\alpha-\rho(\gamma-\alpha)}), \qquad \text{for } \gamma - \alpha = \frac{(\gamma-\beta)(\gamma+\delta)}{\gamma+\delta-\beta} > 0,$$

$$K_3 = o(\omega^{k+(1+\alpha)(\rho-1)-\rho(\gamma+k-\beta)}) + o(\omega^{\beta-\alpha-\rho(\gamma-\alpha)}),$$

and

$$K_4 = o(\omega^{\beta-\alpha-\rho(\gamma-\alpha)}).$$

Since $\beta - \alpha - \rho(\gamma - \alpha) = \beta - \alpha(1-\rho) - \rho\gamma = \beta - \dfrac{\alpha\delta}{\alpha+\delta} - \dfrac{\alpha\gamma}{\alpha+\delta} = \beta - \rho(\gamma+\delta) = 0$ and $k + (1+\alpha)(\rho-1) - \rho(\gamma+k-\beta) = \dfrac{\gamma+\delta-\beta}{\gamma+\delta}(k-1-\beta) < 0,$

$$K_i = o(1), \qquad \text{as } \omega \to \infty \ (i = 1, 2, 3, 4). \tag{5.7}$$

Summing up (5.1), (5.2), (5.3), (5.4), (5.5), (5.6) and (5.7), we obtain

$$\sigma_\omega^\alpha = o(1), \qquad \text{as } \omega \to \infty,$$

which completes the proof of theorem 3.

REFERENCES

[1] J. J. GERGEN, Convergence and Summability criteria for Fourier series, Quart. Jour. Math., 1(1930), 252-275.

[2] G. H. HARDY and J. E. LITTLEWOOD, Some new convergence criteria for Fourier series, Annali di Pisa (2), 3(1934), 43-62.

[3] S. IZUMI, Some trigonometrical series VIII, Tôhoku Math. Jour., 5(1954), 296-301.

[4] S. IZUMI and G. SUNOUCHI, Theorems concerning Cesàro summability, Tôhoku Math. Jour., 1(1951), 313-326.

[5] K. KANNO, On the Cesàro summability of Fourier series, Tôhoku Math. Jour., 7(1955), (to appear), 110-119.

[6] S. KINUKAWA, On the Cesàro summability of Fourier series, Tôhoku Math. Jour., 6(1954), 109-120.

[7] S. KINUKAWA, On the Cesàro summability of Fourier series (II), Tôhoku Math. Jour., 7(1955), 252-264.

[8] W. C. RANDELS, Three examples in the theory of Fourier series, Annals of Math., 36(1935), 838-858.

[9] G. SUNOUCHI, A new convergence criteria for Fourier series, Tôhoku Math. Jour., 5(1954), 238-242.

[10] G. SUNOUCHI, Convexity theorems and Cesàro summability, Journ. of Math. Tokyo, 1(1953), 104-109.

[11] F. T. WANG, On Riesz summability of Fourier series (III), Proc. London Math. Soc., 51(1950), 215-231.

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