INFINITE LIE RINGS

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(Received August 1, 1956)

1. Introduction. The recent development of the Lie theory suggests the investigation of Lie rings of operators on a Hilbert space. In this paper we shall treat the more restrictive one, that is, the Lie rings constructed with the operators in a ring of operators (in the sense of von Neumann) on a Hilbert space. The adjective infinite is due to the same reason as and the close connection to the infinite unitary groups and the infinite general linear groups defined by Kadison [6, 7] (they are defined by the set of all unitary operators and all invertible operators in a ring of operators, respectively).

The object of this paper is to introduce the natural definition of the infinite Lie rings as the Lie rings associated to infinite unitary groups and general linear groups, and to investigate the relations between the (closed) normal subgroup structure and the Lie ideal structure.

In § 2 we shall introduce the natural definition of infinite Lie rings following the method of Neumann [10]. He treated the group of matrices on a finite dimensional vector space and the Lie ring associated with it. His results are almost valid in a Hilbert space. In § 3 we shall clarify the fundamental properties of the infinite Lie rings, especially, of its derived rings, which will play the essential rôle in the sequel. § 4 contains the complete determination of the closed Lie ideals in the various cases of factors. In this and next sections we shall assume the underlying Hilbert space to be separable. This assumption is based on the technical reason in the case of factors of the infinite class: that is, we shall essentially use the notion of the trace and the unicity of the closed (two-sided) ideal in infinite factors. In the non-separable case there exist the distinct equivalence classes of infinite projections and the distinct closed ideals. But we may easily see the situation of the non-separable case from the separable case. In § 5 we shall apply the results of § 4 to the determination of all closed normal subgroups of the infinite unitary groups and general linear groups in the various cases of factors. These results are originally due to Kadison [6, 7, 8], but our proof is quite different from his and makes clear the other aspect of the group structure. Moreover, it looks more preferable from the theory of operator rings and permits the unified treatment of the infinite unitary groups and general linear groups.

In the meantime, Herstein [4, 5] published the interesting papers on the Lie ring of an abstract associative ring and his results are closely related to ours. By employing his results some of our proofs in § 3 are considerably simplified.

In this paper we shall make free use of the fundamental results and techniques of rings of operators without further references (refer to [1, 9, 11]), and the common terminologies such as Lie ring, the Lie subring and the Lie ideal should be taken in the usual manner.

Finally, at the preliminary stage of this investigation we are indebted to Professor M. Kondo and Professor M. Orihara for stimulating discussions, to whom we shall want to express our hearty thanks.

2. Definition of infinite Lie rings.*) Let $\mathbf{M}$ be a ring of operators on a Hilbert space. Following Kadison [6, 7] we define the infinite general linear group $\mathbf{G}$ and the infinite unitary group $\mathbf{U}$ by the group of all invertible operators and all unitary operators in $\mathbf{M}$ (with the uniform topology), respectively.

Following the method of von Neumann [10], we denote by the *Lie ring* $\mathbf{L}$ of the *infinite general linear group* $\mathbf{G}$ the set of all operators X, for which there exist a sequence of operators $A_p \in \mathbf{G}$ and a sequence of positive numbers ε_p converging to 0, such that

$$\frac{1}{\varepsilon_p}(A_p - I) \to X \qquad (p \to \infty)$$

in the uniform topology of $\mathbf{M}$. Clearly $X \in \mathbf{M}$. From the results of von Neumann, we know that, for any $X \in \mathbf{L}$ and any sequence of positive numbers η_p converging to 0, there exists a sequence of $A_p \in \mathbf{G}$ such that $\frac{1}{\eta_p}(A_p - I) \to X$. Hence the above relation will be denoted by $\{A_p; \varepsilon_p\} \sim X$, or simply $\{A_p\} \sim X$.

As Neumann has proved, we obtain that if $X,\ Y \in \mathbf{L}$ and let $\{A_p; \varepsilon_p\} \sim X$, $\{B_p; \varepsilon_p\} \sim Y$ then

$$\frac{1}{\frac{1}{\alpha}\varepsilon_p}(A_p - I) \to \alpha X \qquad \text{for any real } \alpha,\ (\alpha \neq 0)$$

$$\frac{1}{\varepsilon_p}(A_pB_p - I) \to X + Y,$$

$$\frac{1}{\varepsilon_p^2}(A_pB_pA_p^{-1}B_p^{-1} - I) \to XY - YX,$$

so that αX, $X + Y$, $[X, Y] = XY - XY \in \mathbf{L}$ (For $\alpha = 0$, $\alpha X \in \mathbf{L}$ is trivial). This shows that $\mathbf{L}$ is really a Lie ring. Moreover, if $\{A_p\} \sim X$, then $\{A_p^*\} \sim X^*$, so that $X \in \mathbf{L}$ implies $X^* \in \mathbf{L}$. Conversely, for any $X \in \mathbf{M}$, we define $\exp X$ by $I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \ldots$, then $\exp X \in \mathbf{G}$ and

$$\frac{1}{t}(\exp tX - I) \to X \qquad (t \to 0).$$

This shows

PROPOSITION 1. *The Lie ring* $\mathbf{L}$ *of the infinite general linear group* $\mathbf{G}$ *is the real linear space of all operators in* $\mathbf{M}$, *with the multiplication* $[X, Y] =$

$XY - YX$.

Next consider the Lie ring $\mathbf{S}$ of the infinite unitary group $\mathbf{U}$. This is defined analogously as above. In this case we see that if $X \in \mathbf{S}$ and let $\{U_p\} \sim X$, then $\{U_p^*\} \sim X^*$, $\{U_p^{-1}\} \sim -X$, that is $X \in \mathbf{S}$ is skew-symmetric (i. e. $X + X^* = 0$). Conversely, if X is a skew-symmetric operator in $\mathbf{M}$, then $\exp X$ is unitary in $\mathbf{U}$ and $\frac{1}{t}(\exp tX - I) \to X$. Hence we obtain

PROPOSITION 2. *The Lie ring* $\mathbf{S}$ *of the infinite unitary group* $\mathbf{U}$ *is the real linear space of all skew-symmetric operators in* $\mathbf{M}$ *with the multipilication* $[X, Y] = XY - YX$. *Further, it is a Lie subring of the Lie ring of the infinite general linear group.*

In this paper, we shall call $\mathbf{L}$ and $\mathbf{S}$ the *infinite Lie ring* $\mathbf{L}$ and the *infinite skew-symmetric Lie ring* $\mathbf{S}$ of a ring of operator $\mathbf{M}$, respectively.

Analogously, we can define the *infinite complex Lie ring* $\widetilde{\mathbf{L}}$ and *complex skew-symmetric Lie ring* $\widetilde{\mathbf{S}}$ as the complex linear space, respectively. But in the complex case, it is obvious that two notions $\widetilde{\mathbf{L}}$ and $\widetilde{\mathbf{S}}$ are coincident, because a skew-symmetric operator S can be written as iH with an hermitian operator H.

Next we shall investigate the relations between subgroups and Lie subrings.

PROPOSITION 3. *For any subgroup* $\mathbf{I}$ *of* $\mathbf{G}$, *there corresponds a Lie subring* $\mathbf{K}$ *of* $\mathbf{L}$. *The uniform closure of* $\mathbf{I}$ *in* $\mathbf{G}$ *has the same Lie subring* $\mathbf{K}$, *and* $\mathbf{K}$ *is uniformly closed in* $\mathbf{L}$. *If* $\mathbf{I}$ *is a normal subgroup then* $\mathbf{K}$ *is a Lie ideal in* $\mathbf{L}$.

PROOF. These propositions are almost evident from the definition. For example, let $A_p \in \bar{\mathbf{I}}$ such that $\{A_p\} \sim X$, then we can choose for each p an $A'_p \in \mathbf{I}$ such that $\|A_p - A'_p\| < \varepsilon_p^2$. Therefore

$$\| \frac{1}{\varepsilon_p}(A_p - I) - \frac{1}{\varepsilon_p}(A'_p - I) \| = \| \frac{1}{\varepsilon_p}(A_p - A'_p)\| < \varepsilon_p,$$

so that $\{A'_p\} \sim X$. This proves the second assertion.

Finally, if $\mathbf{I}$ is normal and let $X \in \mathbf{L}$, $Y \in \mathbf{K}$ and let $\{A_p; \varepsilon_p\} \sim X$, $\{B_p; \varepsilon_p\} \sim Y$, $A_p \in \mathbf{G}$, $B_p \in \mathbf{I}$, then

$$\frac{1}{\varepsilon_p^2}(A_pB_pA_p^{-1}B_p^{-1} - I) \to [X, Y],$$

and $A_pB_pA_p^{-1} \in I$ so that $A_pB_pA_p^{-1}B_p^{-1} \in \mathbf{I}$, and this implies $[X, Y] \in \mathbf{K}$.

PROPOSITION 4. *Let* $\mathbf{K}$ *be a Lie subring of* $\mathbf{L}$, *associated to a subgroup* $\mathbf{I}$, *and consider the set* $\mathbf{E} = \{\exp X;\ X \in \mathbf{K}\}$. *Then* $\mathbf{E}$ *is contained in the connected component of the uniform closure of* $\mathbf{I}$; *in particular, if* $\mathbf{E}$ *is a group, then the both are coincident.*

PROOF. The first assertion follows from [10; III § 2], but we shall sketch of Lie ideals in $\mathbf{L}$, especially, the properties of the derived ring $\mathbf{L}^{(1)}$. From it for the completeness. Consider the expression $\log A = -\sum_{n=1}^{\infty}(I-A)^n/n$ for $\| A - I \| < 1$, then it is easily seen that $\exp(\log A) = A$ for $\| A - I \| < 1$ and $\log(\exp A) = A$ for $\|A\| < \log 2$. Further, $\frac{1}{\varepsilon_p}(A_p - I) \to X$ and $\frac{1}{\varepsilon_p}\log A_p \to X$ are equivalent [10; II § 2].

Now let $X \in \mathbf{K}$ and $A_p \in \mathbf{I}$ such that $p(A_p - I) \to X$ $(p \to \infty)$. Then $p \log A_p \to X$, and $\exp(p \log A_p) = (\exp \log A_p)^p = (A_p)^p$ as noted above. Since $\exp X$ is continuous, we obtain $(A_p)^p \to \exp X$. But $(A_p)^p \in \mathbf{I}$, so that $\exp X$ is contained in the uniform closure $\bar{\mathbf{I}}$ of $\mathbf{I}$. Moreover, $\{\exp tX;\ 0 \leqq t \leqq 1\} \subset \bar{\mathbf{I}}$ is a connected arc from the identity I to $\exp X$.

Suppose that $\mathbf{E}$ be a group. Since the identity I is an interior point of $\mathbf{E}$, the group property makes $\mathbf{E}$ to be open and closed in $\bar{\mathbf{I}}$ 1). Hence $\mathbf{E}$ coincides with the connected component of $\bar{\mathbf{I}}$. Thus the proof is completed.

$\mathbf{K}$ may consist of the zero element only. For example, it is the case if $\mathbf{I}$ is discrete. But in our case, we obtain

PROPOSITION 5. *If a closed normal subgroup* $\mathbf{I}$ *has its Lie ideal* $\mathbf{K}$ *in the center, then it is contained in the center. Especially, if* $\mathbf{K} = (0)$, *then* $\mathbf{I}$ *is a discrete subgroup in the center.*

PROOF. If $\mathbf{K}$ is contained in the center, then $\mathbf{E} = \{\exp X;\ X \in \mathbf{K}\}$ is contained in the center and is a group. For, $(\exp X)(\exp Y) = \exp(X+Y)$, because X and Y are commutative. Thus the connected component of $\mathbf{I}$ is contained in the center. Now let A be a noncentral element of $\mathbf{I}$. Then $\{U^{-t}AU^t;\ t \text{ real}, U \in \mathbf{U}\} \subset \mathbf{I}$ are connected arcs about A. Therefore $U^{-t}AU^tA^{-1} \subset \mathbf{I}$ are connected arcs about the identity I, so that they are contained in the center, which contradicts the non-centrality of $\mathbf{A}$.

The second assertion is evident from the above consideration.

These relations between the closed normal subgroups and the closed Lie ideals show that if we determine the closed Lie ideals, then we can use them to determine the closed normal subgroups of $\mathbf{G}$ or $\mathbf{U}$. This will be done in the sequel.

Finally, we note that the above definition and the propositions can be generalized to any operator algebra. If necessary, we shall use this generalized notion of infinite Lie rings.

3. Fundamental properties. Let $\mathbf{L}$ be the real (or complex) infinite Lie ring of operator algebra $\mathbf{M}$, and denote, as usual, $\mathbf{L}^{(1)} = [\mathbf{L}, \mathbf{L}]$, $\mathbf{L}^{(n)} = [\mathbf{L}^{(n-1)}, \mathbf{L}^{(n-1)}]$, then it is well known that each $\mathbf{L}^{(n)}$ is a Lie ideal in $\mathbf{L}$ and $\mathbf{L} \supset \mathbf{L}^{(1)} \supset \mathbf{L}^{(2)} \supset \ldots$ If there exists an integer m such that $\mathbf{L}^{(m)} = (0)$, we call that $\mathbf{L}$ is *solvable*. The object of this § is to investigate the structure

1) This is a well known theorem of Schreier. See, for example, Pontrjagin; Topological Groups, p. 76, Theorem 15.

this we can prove the non-solvability of $\mathbf{L}$ in the case of non-commutative ring of operators.

Following Herstein, a ring $\mathbf{R}$, all of whose elements are nilpotent, is said to be *locally nilpotent* if the subring generated by any finite set of elements of $\mathbf{R}$ is nilpotent, and an ideal of $\mathbf{R}$ is said to be *locally nilpotent* if, as a ring, it is locally nilpotent. Then it is evident that a ring of operators $\mathbf{M}$ has no nonzero locally nilpotent ideal, because any ideal in $\mathbf{M}$ contains a projection. Thus we obtain an operator-theoretic version of Herstein's theorem [4].

THEOREM 1. *Let* $\mathbf{M}$ *be a ring of operators, or more generally, an operator algebra with no non-zero locally nilpotent ideals. Suppose that* $\mathbf{P}$ *is a Lie ideal and also a subring of* $\mathbf{M}$, *then* $\mathbf{P}$ *contains a non-zero ideal of* $\mathbf{M}$ *or* $\mathbf{P}$ *is contained in the center of* $\mathbf{M}$.

As Herstein shows, any simple (abstract) algebra has no non-zero locally nilpotent ideal. Hence we obtain the following

COROLLARY 1.1. *If* $\mathbf{M}$ *is a simple* C^**-algebra,*[2] *then any proper Lie ideal which is, at the same time, a subring of* $\mathbf{M}$ *are only* (0) *or* (αI), *that is, either of sets of reals, purely imaginary or complex numbers.*[3]

PROOF. For it is well known that any simple C^*-algebra has the center of only scalars.

COROLLARY 1.2. *If* $\mathbf{M}$ *is a simple* C^**-algebra, then any Lie ideal* $\mathbf{K}$ *contains* $\mathbf{L}^{(1)}$ *or is a trivial one.*

PROOF. As proved in [4; Lemma 3], put

$$\mathbf{P} = \{A \in \mathbf{M};\ [A,\ \mathbf{M}] \subset \mathbf{K}\}$$

then $\mathbf{P}$ is a Lie ideal and a subring of $\mathbf{M}$. Moreover, $\mathbf{K} \subset \mathbf{P}$.

If $\mathbf{K}$ is non-commutative, by Corollary 1.1, $\mathbf{P} = \mathbf{M} = \mathbf{L}$, so that $\mathbf{L}^{(1)} \subset \mathbf{K}$. If $\mathbf{K}$ is commutative then $\mathbf{K}$ is a trivial one.

Here we note that simple rings of operators are only factors of finite class or countably decomposable (III) case.[4] But if $\mathbf{J}$ is a maximal ideal in a ring of operators, then the quotient ring $\mathbf{M}/\mathbf{J}$ is a simple C^*-algebra.

Now let $\mathbf{M}$ be a ring of operators and consider two cases of finite and infinite class separately.

The following theorem is a generalization of a result of Halmos [3], who treated the total operator ring. But we shall formulate it in the following form:

THEOREM 2. *If* $\mathbf{M}$ *is a ring of operators of infinite class, then* $\mathbf{L}^{(1)} = \mathbf{L}$.

PROOF. We shall proceed as Halmos. First remark that, if P is a pro-

2) By a C^*-algebra we denote the uniformly closed operator algebra.

3) In the sequel, we shall always use this simplified notation of the trivial ideals for the real case, in this sense.

4) A ring of operators is said to be countably decomposable if any collection of mutually orthogonal projections in it is at most countable. See, for example, [12].

jection in $\mathbf{M}$ such that $P \prec I-P$, then there exists a mutually orthogonal system of projections $\{H_i\}$ such that $P=H_1$, $H_i \sim H_j$ and $\sum_{i=2}^{\infty} \oplus H_i \leqq I-P$, because $I-P$ is an infinite projection. Let U_i be the partially isometric operator which gives the equivalence $H_1 \sim H_i$, then $H_i \sim H_j$ is given by $U_j U_i^*$. Denote by x_i the projection of x into H_i and define $y=Ux$ by $\sum_{i=1}^{\infty} U_{i+1}U_i^* x_i$ (that is, $U=0$ on the orthocomplement of $\sum_{i=1}^{\infty} \oplus H_i$), then $U^*y=\sum_{i=1}^{\infty} U_i U_{i+1}^* y_{i+1}$. So that U, $U^* \in \mathbf{M}$ and for $x \in \sum_{i=1}^{\infty} \oplus H_i$, $U^*Ux=x$, $(UU^*x)_i = x_i$ $(i>1)$ but $(UU^*x)_1=0$.

Let A be an hermitian operator in $\mathbf{M}$ and denote by $R(A)$ and $N(A)$ the closure of range and null space of A, then $R(A)$, $N(A)$ belong to $\mathbf{M}$. Let us now consider the positive hermitian operator A^2 in $\mathbf{M}$ such that $R(A) = N(A)^{\perp} \prec N(A)$. Then we can put $R(A)=P$ and consider U, U^* as in the above remark. Define an operator $\widetilde{A}$ in $\mathbf{M}$ by $\widetilde{A}x = \sum_{i=1}^{\infty} U_i A U_i^* x_i$, and put $B=\widetilde{A}U$, then $A^2 = B^*B - BB^* \in \mathbf{L}^{(1)}$. If the given operator is negative, then it is sufficient to consider $-A^2$. The assumption $R(A) \prec N(A)$ can be eliminated by the spectral theory as treated in [3; I, Lemma 3].

To apply the above results to an arbitary operator in $\mathbf{M}$, that is in $\mathbf{L}$, it suffices to note that every operator T in $\mathbf{M}$ is decomposed into the uniquely determined hermitian operators A, B for which $T=A+iB$, and that every hermitian operator is the direct sum of a positive and a negative operators.

Finally, in the real case, if T can be written as $i\,[A, B]$ by the above argument, it is written in the form $T=[iA, B] \in \mathbf{L}^{(1)}$. Thus we obtain the result.

Next consider the ring of operators of finite class and denote by φ the $\natural$-operation defined by Dixmier [1; Theorem 10].

Let

$$\mathbf{N} = \{A \in \mathbf{M};\ \varphi(A) = 0\},$$

then it is clear that $\mathbf{N}$ is a closed Lie ideal, $\mathbf{L}^{(1)} \subset \mathbf{N}$ and that $\mathbf{N}$ contains no projection.

THEOREM 3. *If* $\mathbf{M}$ *is a ring of operators of finite class, then the uniform closure of* $L^{(1)}$ *coincides with* $\mathbf{N}$.

PROOF. From the construction of the $\natural$-operation the following fact is known: Let A be an hermitian operator in $\mathbf{M}$ and ε be any positive number, then there exist unitary operators $U_i \in \mathbf{M}$ and real numbers λ_i $(1 \leqq i \leqq n$, $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1)$ such that $\| \varphi(A) - \sum_{i=1}^{n} \lambda_i U_i A U_i^* \| < \varepsilon$.

Since

$$\sum_{i=1}^{n}\lambda_i U_i A U_i^* - A = \sum_{i=1}^{n}\lambda_i(U_i A U_i^* - A) = \sum_{i=1}^{n}\lambda_i[U_i A, U_i^*] \in [\mathbf{L}, \mathbf{L}],$$

and

$$\|\varphi(A) - A - \sum_{i=1}^{n}\lambda_i[U_i A,\ U_i^*]\| = \|\varphi(A) - \sum_{i=1}^{n}\lambda_i U_i A U_i^*\| < \varepsilon,$$

we have $\varphi(A) - A \in \overline{\mathbf{L}^{(1)}}$.

In particular, let A be hermitian and $A \in \mathbf{N}$, then we obtain $A \in \mathbf{L}^{(1)}$. For arbitary operator A in $\mathbf{N}$, let $A = A_1 + iA_2$ be a decomposition into the hermitian operators A_1, A_2, then A_1, $A_2 \in \mathbf{N}$, so that $A \in \overline{\mathbf{L}^{(1)}}$. This completes the proof.

COROLLARY 3.1. *If $\mathbf{M}$ is a ring of operators of finite class, then for any integer n, the uniform closure of $\mathbf{L}^{(n)}$ equals to $\mathbf{N}$.*

PROOF. Any element $A \in \mathbf{M}$ can be uniquely decomposed into central element Z and $N \in \mathbf{N}$. To see this, it is sufficient to put $N = A - \varphi(A)$, and $Z = \varphi(A)$. That is, $\mathbf{L} = \mathbf{Z} + \mathbf{N}$, where $\mathbf{Z}$ denotes the center of $\mathbf{M}$. So that $[\mathbf{L}, \mathbf{L}] = [\mathbf{N}, \mathbf{N}] = [\overline{\mathbf{L}^{(1)}}, \overline{\mathbf{L}^{(1)}}] \subset \overline{[\mathbf{L}^{(1)}, \mathbf{L}^{(1)}]} = \overline{\mathbf{L}^{(2)}}$, since the ring operations are continuous. Hence $\mathbf{N} \subset \overline{\mathbf{L}^{(2)}}$, but it is clear that $\overline{\mathbf{L}^{(2)}} \subset \mathbf{N}$. Hence we obtain $\mathbf{N} = \overline{\mathbf{L}^{(2)}}$. Thus the proof is completed by mathematical induction.

COROLLARY 3.2. *If $\mathbf{M}$ is a factor of finite class, then $\mathbf{L}^{(1)} = \mathbf{L}^{(2)} = \ldots$*

PROOF. By the above corollary $\mathbf{L}^{(n)} \neq (\alpha I)$. On the other hand, since a factor of finite class is simple, $\mathbf{L}^{(n)} \supset \mathbf{L}^{(1)}$, by Corollary 1.2.

PROPOSITION 6. *Let $\mathbf{M}$ be a ring of operators, and $A \in \mathbf{M}$ be an element which commutes with $\mathbf{L}^{(1)}$. Then A is in the center $\mathbf{Z}$ of $\mathbf{M}$.*

PROOF. Any ring of operators $\mathbf{M}$ has a central decomposition into two rings $\mathbf{M}_1$, $\mathbf{M}_2$, which are of finite and infinite class, resp., and $\mathbf{L}^{(1)} = [\mathbf{L}_1, \mathbf{L}_1] + [\mathbf{L}_2, \mathbf{L}_2]$. Therefore, it is sufficient to consider the respective cases. If $\mathbf{M}$ is of infinite class, then the result follows from Theorem 2. If $\mathbf{M}$ is of finite class, then $\mathbf{M}$ can be decomposed as follows: $\mathbf{M} = \mathbf{Z} + \mathbf{N}$, as noted in the proof of Corollary 3.1. Hence $\mathbf{M}' = \mathbf{Z}' \cap \mathbf{N}'$[5], or $\mathbf{Z} = \mathbf{N}' \cap \mathbf{M}$. From Theorem 3, we know that $\mathbf{L}^{(1)\prime} = \mathbf{N}'$, so that $\mathbf{L}^{(1)\prime} \cap \mathbf{M}' = \mathbf{Z}$. This completes the proof.

REMARK. The result of this type has been obtained for arbitrary semi-simple ring by Kaplansky.[6] It is well known that rings of operators are semi-simple.

THEOREM 4. *Every solvable infinite Lie ring is commutative.*

5) By $\mathbf{P}'$ we denote the set of all operators which commute with $\mathbf{P}$.

6) I. Kaplansky, Semi-automorphisms of rings, Duke Math. Journal, 59(1947)521–525.

PROOF. As noted in the proof of the above Proposition, it is sufficient to consider the two cases of finite class and infinite class separately. However, the result is evident from Theorem 2 and Corollary 3.1.

COROLLARY 4.1. *Every non-commutative skew-symmetric Lie ring is not solvable.*

PROOF. As remarked in § 2, the complexification of $\mathbf{S}$ is the complex infinite Lie ring $\widetilde{\mathbf{L}}$, and the notion of solvability coincides in $\mathbf{S}$ and in its complexification $\widetilde{\mathbf{L}}$. Hence this corollary follows immediately from the above theorem.

4. Determination of closed Lie ideals in a factor. In this and the next § we shall treat only a factor on a separable Hilbert space. The assumption of the separability is based on the technical reason in infinite factors as mentioned in introduction. However, the situation in a factor of finite class remains to the same and the proof given here remains valid in a non-separable case, also in a factor of infinite class we may easily see the situation from the separable case.

THEOREM 5. *If $\mathbf{M}$ is a factor of finite class and by φ denote the trace on $\mathbf{M}$, then the closed Lie ideals in the infinite Lie ring are*

(i) (0), (αI), $\mathbf{N}$ *for complex case,*

(ii) (0), (αI), $\mathbf{N}$, $\mathbf{N} \cup \mathbf{S}$, $\mathbf{N} \cup \mathbf{H}$ *for real case;*

the closed Lie ideals in the infinite skew-symmetric Lie ring $\mathbf{S}$ *are the intersection of* $\mathbf{S}$ *with those of* (i). *Where* $\mathbf{N} = \{A;\ \varphi(A) = 0\}$, $\mathbf{S}$ *the set of all skew-symmetric operators,* $\mathbf{H}$ *the set of all hermitian operators.*

PROOF. It is well known that the $\natural$-operation is reduced to the trace in the case of factor. So Theorem 3 is applicable and the minimality of $\mathbf{N}$ follows immediately from Corollary 1.2, and Theorem 3.

First consider the complex case. Suppose that there exists a Lie ideal $\mathbf{K}$ containing $\mathbf{N}$, and let $X \in \mathbf{K}$ such that $\varphi(X) \neq 0$, then $\mathbf{M}$ is represented as $(\alpha X) + \mathbf{N} \subset \mathbf{K}$. Hence we obtain $\mathbf{K} = \mathbf{M}$.

In the real case, it is sufficient to remark that any hermitian operator has a real trace and that any skew-symmetric operator S can be written as iH with a hermitian operator H.

Finally in the skew-symmetric case, the result follows by the complexification.

THEOREM 6. *If $\mathbf{M}$ is a factor of case* (III), *then closed Lie ideals in the infinite Lie ring are only trivial, that is,* (0) *or* (αI) *for both real and complex cases. So that the closed Lie ideals in the infinite skew-symmetric Lie ring are also trivial.*

PROOF. Evidently from Corollary 1.2 and Theorem 2.

Finally let us consider a factor of case (I_∞) or (II_∞). In these cases $\mathbf{M}$

is not simple and any ideal in $\mathbf{M}$ is also Lie ideal in $\mathbf{L}$. Let $\mathbf{F}$ be the set of all operators of finite rank, that is, all operators which are contained in some finite projections. Then it is well known that $\mathbf{F}$ is an ideal in $\mathbf{M}$ and that its uniform closure $\mathbf{J}$ is the maximal ideal and the unique closed one. (See, for example, [12].) It is also known that, in these cases, there exists the unique trace φ (up to a constant factor) with the following properties, (cf. [11; Chap. I]);

(i) If $P \in \mathbf{M}$ is a finite projection, then $\varphi(P) = D(P)$, where $D(P)$ denotes the relative dimension of P.

(ii) Let $A \in \mathbf{F}$ be contained in a finite projection P, and consider the induced ring $\mathbf{M}_P$ by P, then $\mathbf{M}_P$ is a factor of finite class with the identity P. Let φ_P be the trace defined in $\mathbf{M}_P$, then

$$\varphi(A) = D(P)\varphi_P(A). \qquad \text{(This is the definition of } \varphi(A).)$$

(iii) Let an hermitian positive $H \in \mathbf{M}$ is the sum of strongly convergent hermitian positive $H_i \in \mathbf{M}$, then

$$\varphi(H) = \sum \varphi(H_i).^{7)}$$

By $\mathbf{T}$ denote the set of all linear combinations of hermitian positive operators of finite trace, then $\mathbf{T}$ is an ideal in $\mathbf{M}$ and satisfies

$$\varphi(AX) = \varphi(XA) \text{ for } A \in \mathbf{T},\ X \in \mathbf{M}.^{7)}$$

Clearly $\mathbf{F} \subset \mathbf{T} \subset \mathbf{J}$. Now put $\mathbf{N}_0 = \{A \in \mathbf{T};\ \varphi(A) = 0\}$, then it is evident that $\mathbf{N}_0$ and its uniform closure $\mathbf{N}$ are Lie ideals in $\mathbf{L}$. Further we obtain

PROPOSITION 7. *If* $\mathbf{M}$ *is a factor of case* (I_∞) *or* (II_∞), *then* $\mathbf{N}$ *coincides with the uniform closure of* $[\mathbf{J}, \mathbf{J}]$ *or* $[\mathbf{J}, \mathbf{L}]$. *Any closed Lie ideal in* $\mathbf{L}$ *contains* $\mathbf{N}$ *or is trivial.*

PROOF. We shall first show that $\mathbf{F} \cap \mathbf{N}$ is uniformly dense in $\mathbf{N}_0$, so that $\mathbf{N}$. For any hermitian operator $A \in \mathbf{N}_0$, denote by A^+, A^- the positive and negative parts of A, resp., then $\varphi(A) = 0$ implies $\varphi(A^+) - \varphi(A^-) = 0$. By the spectral theory, for any preassigned positive number ε, there exist sequences of positive numbers $\{\lambda_i\}$, $\{\mu_j\}$ such that

$$0 \leqq A^+ - \sum \lambda_i E(\Delta_i) < \varepsilon I, \quad 0 \leqq A^- - \sum \mu_j F(\Delta_j) < \varepsilon I.$$

By the property (iii) of the trace, we can assume that

$$\varphi(A^+) - \sum \lambda_i \varphi(E(\Delta_i)) < \varepsilon, \ \ \varphi(A^-) - \sum \mu_j \varphi(F(\Delta_j)) < \varepsilon.$$

Put

$$\gamma = \sum \lambda_i \varphi(E(\Delta_i)) - \sum \mu_j \varphi(F(\Delta_j))$$

then $|\gamma| < 2\varepsilon$. But take a projection E such that $\varphi(E) = 1$, and consider

$$B = \sum \lambda_i E(\Delta_i) - \sum \mu_j F(\Delta_j) - \gamma E,$$

then $\varphi(B) = 0$ and $\| A - B \| < 4\varepsilon$. However, we know that from A^+, $A^- \in \mathbf{J}$ the spectral projections $E(\Delta_i)$, $F(\Delta_j)$ is finite (See [12]) so that $B \in \mathbf{F} \cap \mathbf{N}$.

7) See §2 of R. Godement, Théorie des caractères, I. Ann of Math., 59 (1954), 47-62.

Thus $\mathbf{F} \cap \mathbf{N}$ is uniformly dense in $\mathbf{N}$.

Now, from the property (ii) and Theorem 3, we see $\mathbf{F} \cap \mathbf{N} \subset \overline{[\mathbf{F}, \mathbf{F}]}$, so that $\mathbf{N} \subset \overline{[\mathbf{F}, \mathbf{F}]} \subset [\mathbf{J}, \mathbf{J}] \subset [\mathbf{J}, \mathbf{L}]$. On the other hand $\mathbf{N} \supset [\mathbf{F}, \mathbf{L}]$ is clear from the fact $\mathbf{F} \subset \mathbf{T}$. So $\mathbf{N} \supset \overline{[\mathbf{F}, \mathbf{L}]} \supset [\mathbf{F}, \mathbf{L}] = [\mathbf{J}, \mathbf{L}] \supset [\mathbf{J}, \mathbf{J}]$ because the ring operations are continuous. Therefore $\mathbf{N} = \overline{[\mathbf{J}, \mathbf{J}]} = \overline{[\mathbf{J}, \mathbf{L}]}$.

Finally, let $\mathbf{K}$ be any closed Lie ideal, and put $\mathbf{P} = \{A \in \mathbf{M};\ [A, \mathbf{M}] \subset \mathbf{K}\}$, then $\mathbf{P}$ is a Lie ideal and subring of $\mathbf{M}$, $\mathbf{K} \subset \mathbf{P}$, and further uniformly closed. According to Theorem 1, $\mathbf{P}$ contains an ideal of $\mathbf{M}$ or is contained in the center. In the latter case $\mathbf{P}$ is trivial, so that $\mathbf{K}$ is also trivial. In the first case, it is sufficient to suppose that the ideal contained in $\mathbf{P}$ is uniformly closed. By a theorem in [12], we obtain $\mathbf{P} \supset \mathbf{J}$, so that $[\mathbf{J}, \mathbf{L}] \subset \mathbf{K}$. By the above proved, this implies $\mathbf{N} \subset \mathbf{K}$. Thus the proof is completed.

PROPOSITION 8. *If* $\mathbf{M}$ *is a factor of case* (I_∞) *or* (II_∞), *then any proper closed Lie ideal* $\mathbf{K}$ *containing* $\mathbf{J}$ *is* $\mathbf{J} + (\alpha I)$.

PROOF. Since $\mathbf{J}$ is a closed ideal (and Lie ideal) in $\mathbf{M}$ ($\mathbf{L}$) consider the C^*-algebra $\mathbf{M}/\mathbf{J}$ and its Lie ring, then the latter is identified with the Lie quotient ring $\mathbf{L}/\mathbf{J}$. Now $\mathbf{M}/\mathbf{J}$ is simple and $\mathbf{K}/\mathbf{J}$ is a Lie ideal in $\mathbf{L}/\mathbf{J}$. Hence, by Corollary 1.2, $\mathbf{K}/\mathbf{J}$ is (0), (αI) or contains $[\mathbf{L}/\mathbf{J},\ \mathbf{L}/\mathbf{J}] = \mathbf{L}^{(1)}/\mathbf{J}$. In the first case, $\mathbf{K} \subset \mathbf{J}$ and this contradicts the assumption. In the second case we obtain $\mathbf{K} = (\alpha I) + \mathbf{J}$. In the final case $\mathbf{K}/\mathbf{J} \supset \mathbf{L}^{(1)}/\mathbf{J}$, so that $\mathbf{K} \supset \mathbf{L}^{(1)} = \mathbf{L}$, by Theorem 2.

THEOREM 7. *If* $\mathbf{M}$ *is a factor of case* (I_∞) *or* (II_∞), *then the closed Lie ideals in the infinite Lie ring are*

(i) (0), (αI), $\mathbf{N}$, $\mathbf{J}$, $\mathbf{J} + (\alpha I)$ *for complex case,*

(ii) (0), (αI), $\mathbf{N}$, $\mathbf{J}$, $\mathbf{N} \cup (\mathbf{J} \cap \mathbf{S})$, $\mathbf{N} \cap (\mathbf{J} \cup \mathbf{H})$,
$\mathbf{N} \cup (\mathbf{J} \cap \mathbf{S}) + (\rho I)$, $\mathbf{N} \cup (\mathbf{J} \cap \mathbf{H}) + (r I)$, $\mathbf{J} + (\alpha I)$, *for real case*:

and the closed Lie ideals in the infinite skew-symmetric Lie ring $\mathbf{S}$ *are the intersection of* $\mathbf{S}$ *with those of* (i). *Where* $\mathbf{N}$ *is the uniform closure of* $\mathbf{N}_0 = \{A \in \mathbf{T};\ \varphi(A) = 0\}$, $\mathbf{J}$ *the maximal ideal,* $\mathbf{H}$ *and* $\mathbf{S}$ *denote the set of all hermitian and skew-symmetric operators in* $\mathbf{M}$, *and* (ρ) *and* (r) *the set of all purely imaginary and all real numbers, respectively.*

PROOF. First consider the complex case. The set cited above are evidently closed Lie ideals in $\mathbf{L}$. Remembering the above two propositions, it is sufficient to prove that there is no closed Lie ideal which contains $\mathbf{N}$ but does not contain $\mathbf{J}$. Suppose contrary and let $\mathbf{K}$ be such a Lie ideal. If $\mathbf{K}$ contains an operator A of finite, non-zero trace, then any element of $\mathbf{F}$ can be written as the sum $(\alpha A) + \mathbf{N} \subset \mathbf{K}$. Hence its uniform closure $\mathbf{J}$ is contained in $\mathbf{K}$, which contradicts the assumption. Next if $\mathbf{K}$ contains an operator (αI) not contained in $\mathbf{J}$, consider as before, the Lie ideal $\mathbf{K}/\mathbf{J}$ in $\mathbf{L}/\mathbf{J}$. Then $\mathbf{K}/\mathbf{J}$ is a non trivial Lie ideal in the simple C^*-algebra $\mathbf{M}/\mathbf{J} = \mathbf{L}/\mathbf{J}$, so that $\mathbf{K}/\mathbf{J} \supset \mathbf{L}^{(1)}/\mathbf{J}$. Put now $\mathbf{P} = \{A \in \mathbf{M};\ [A, \mathbf{M}] \subset \mathbf{K}\}$, then $\mathbf{P}$ is a Lie ideal and subring of $\mathbf{M}$ and $\mathbf{K} \subset \mathbf{P}$. Since $\mathbf{K}$ is closed, $\mathbf{P}$ is also closed, so that $\mathbf{P}$

contains $\mathbf{J}$ from Theorem 1. Therefore, $\mathbf{L}^{(1)}/\mathbf{J} \subset \mathbf{K}/\mathbf{J} \subset \mathbf{P}/\mathbf{J}$, so $\mathbf{P} \supset \mathbf{L}^{(1)} = \mathbf{L}$. From this we obtain $\mathbf{K} \supset \mathbf{L}^{(1)} = \mathbf{L}$. Finally, if $\mathbf{K}$ contains an αI, let us consider the induced ring $\mathbf{M}_P$ by a finite projection P. It is well known that $\mathbf{M}_P$ is a finite factor with the identity P. The corresponding Lie ideal $\mathbf{K}_P$ in $\mathbf{L}_P$ contains $\mathbf{N}_P$ and $(\alpha I)_P = (\alpha P)$. Therefore $\mathbf{K}_P = \mathbf{L}_P$ from Theorem 5. Since P is an arbitrary finite projection in $\mathbf{M}$, we obtain $\mathbf{K} = \mathbf{J} + (\alpha I)$, which is also a contradiction.

For the real case we note that $[\mathbf{L}, \mathbf{J}] \subset \mathbf{N}$, so that the sets cited above are evidently closed Lie ideals in $\mathbf{L}$. However, considering the above result and the result of the finite class, we see these are all.

The skew-symmetric case follows by considering the complexification as usual.

Thus we have determined the all closed Lie ideals in various cases of factors completely. Of course, our treatment is topological and the purely algebraic questions remain untouched, as remarked by Kadison.

5. **Closed normal subgroups in factors.** Let us now apply the results of § 4 to determine the closed normal subgroups of the infinite unitary group $\mathbf{U}$ and the general linear group $\mathbf{G}$ in a given factor $\mathbf{M}$, according to Propositions 3 and 4. Since we have known all closed Lie ideals in the infinite Lie rings of factors, it is sufficient to characterize what kinds of Lie ideals are associated to the closed normal subgroups (or, more precisely to the connected components of the closed normal subgroups).

The following proposition makes clear the structure of $\mathbf{G}$ and $\mathbf{U}$.

PROPOSITION 9. *By* $\mathbf{U}$ *and* $\mathbf{G}$ *denote the infinite unitary group and the general linear group in a ring of operators. Then* $\mathbf{U}$ *is the exponential image of the elements of the infinite skew-symmetric Lie ring* $\mathbf{S}$ *and* $\mathbf{G}$ *is generated by the exponential image of the elements of the infinite Lie ring* $\mathbf{L}$, *and* $\mathbf{U}$ *and* $\mathbf{G}$ *are connected.*

PROOF. For any unitary operator U or positive hermitian operator H, consider $\frac{1}{t}(U^t - I)$ or $\frac{1}{t}(H^t - I)$, (for $t \to 0$), then these converge to $\log U$ and $\log H$, by the spectral theory. Further it is evident that $\log U \in \mathbf{S}$, $\log H \in \mathbf{L}$, and $U = \exp(\log U)$, $H = \exp(\log H)$.

Hence $\mathbf{U}$ is the exponential image of $\mathbf{S}$. Since any $U, V \in \mathbf{U}$ are connected by the continuous arc $\{\exp(\alpha \log U + (1-\alpha)\log V);\ 0 \leqq \alpha \leqq 1\}$, $\mathbf{U}$ is connected.

Regarding to $\mathbf{G}$, consider the polar decomposition $A = UH$, where U is unitary and H is positive hermitian, then $A = \exp(\log U)\exp(\log H)$, so that $\mathbf{G}$ is generated by the exponential image of $\mathbf{L}$. Now $A_1, A_2 \in \mathbf{G}$ are connected by the continuous arc $\{\exp(\alpha \log \mathbf{U}_1 + (1-\alpha)\log U_2)\exp(\alpha \log H_1 + (1-\alpha)\log H_2)\}$ where $A_1 = U_1H_1$ and $A_2 = U_2H_2$ are the polar decompositions of A_1 and A_2. Hence $\mathbf{G}$ is also connected.

PROPOSITION 10. *Any closed normal subgroup of* $\mathbf{U}$ (*or* $\mathbf{G}$), *which has* $\mathbf{S}$

(or $\mathbf{L}$*) as the corresponding Lie ideal, is* $\mathbf{U}$ *(or* $\mathbf{G}$*) itself.*

PROOF. This follows from Propositions 4 and 9.

Let us first investigate the factors of finite class.

According to Proposition 3 and Theorem 5, the Lie ideal corresponding to a closed normal subgroup of the infinite unitary group $\mathbf{U}$ in a factor of finite class is one of (0), (ρI), or $\mathbf{N} \cap \mathbf{S}$. But by the Proposition 5 the first two cases correspond to normal subgroups contained in the center. Moreover, we obtain

THEOREM 8. *If* $\mathbf{M}$ *is a factor of case* (II_1), *then the only proper, closed normal subgroups of the infinite unitary group* $\mathbf{U}$ *of* $\mathbf{M}$ *are the subgroups of the center.*[8] (This is [6; Theorem 2].)

PROOF. By [6: Lemma 4], if a closed normal subgroup $\mathbf{V}$ is not a subgroup of the center $\{\lambda I;\ |\lambda| = I\}$, it is sufficient to suppose that $\mathbf{V}$ contains (λI). Then the corresponding Lie ideal $\mathbf{K}$ contains the identity I, so that $\mathbf{K} = \mathbf{S}$, and the result follows from the Proposition 10.

Now consider the infinite general linear group $\mathbf{G}$.

LEMMA. *If* $\mathbf{M}$ *is a factor of case* (II_1), *then a closed normal subgroup* $\mathbf{I}$ *of* $\mathbf{G}$, *not contained in the center, contains the infinite unitary group* $\mathbf{U}$. ([7; Lemma 4])

PROOF. By the non-centrality of $\mathbf{I}$ and Proposition 5, it is sufficient to suppose that the corresponding Lie ideal $\mathbf{K}$ contains $\mathbf{N}$. So that $\mathbf{K} \cap \mathbf{S} \supset \mathbf{N} \cap \mathbf{S}$. Let $\mathbf{X} \in \mathbf{K} \cap \mathbf{S}$, then by Proposition 4 $\exp X \in \mathbf{I} \cap \mathbf{U}$ and $\mathbf{K} \cap \mathbf{S}$ corresponds to the non-central normal subgroup of $\mathbf{U}$, contained in $\mathbf{I} \cap \mathbf{U}$. Hence we obtain, by the above theorem, $\mathbf{I} \cap \mathbf{U} = \mathbf{U}$, that is, $\mathbf{I} \supset \mathbf{U}$.

Now we shall use the notion of the determinant on $\mathbf{G}$ defined by Fuglede and Kadison [2]. Let $A \in \mathbf{G}$ and let $A = UH$ be the polar decomposition. Then the determinant of A is defined by

$$\Delta(A) = \Delta(H) = \exp(\varphi(\log H)) = \exp\left(\int \log \lambda d\varphi(E_\lambda)\right),$$

where $\int \lambda\, dE_\lambda$ is the spectral representation of H. It is known that Δ is a uniformly continuous representation of $\mathbf{G}$ onto the real positives, and the kernel $\mathbf{G}_1$ is evidently a non-central closed normal subgroup. Further we obtain the Theorem 1 of [7], that is,

THEOREM 9. *If* $\mathbf{M}$ *is a factor of finite class, then the Lie ideal in the infinite Lie ring* $\mathbf{L}$, *corresponding to proper closed normal subgroups (not contained in the center), is unique. This consists of* $\mathbf{N} \cup \mathbf{S}$ *in the case* (II_1) *and* $\mathbf{N}$ *in the case* (I_n) *and is associated to* $\mathbf{G}_1$, *which is the kernel of the determinantial representation* Δ.

Moreover, any non-central closed normal subgroup of $\mathbf{G}$ *contains* $\mathbf{G}_1$ *as*

8) In a factor of case (I_n), it is well known that the theorem is false. There exists the normal subgroup of all operators of determinant 1.

the connected component.

PROOF. This theorem is well known in the case (I_n), so we shall consider the case (II_1). By the above lemma, any closed Lie ideal $\mathbf{K}$ corresponding to a non-central closed normal subgroup $\mathbf{I}$ contains $\mathbf{N} \cup \mathbf{S}$. But, if $\mathbf{K}$ contains $\mathbf{N} \cup \mathbf{S}$ properly, then $\mathbf{K}$ contains some hermitian operator of non-zero trace, so that $\mathbf{K} = \mathbf{L}$. This shows that $\mathbf{I} = \mathbf{G}$, which is a contradiction. It follows that $\mathbf{K}$ is unique and $\mathbf{K} = \mathbf{N} \cup \mathbf{S}$.

Next we shall prove that $\mathbf{N} \cup \mathbf{S}$ is associated to $\mathbf{G}_1$. Since $\mathbf{G}_1$ is the set of all $A \in \mathbf{G}$ with the determinant 1, that is, $A = UH$ with $\varphi(\log H) = 0$. This implies that the set $\mathbf{E}$ generated by $\{\exp X;\ X \in \mathbf{N} \cup \mathbf{S}\}$ contains $\mathbf{G}_1$. On the other hand, since $\mathbf{G}_1$ is non-central, by Proposition 5, $\mathbf{N} \cup \mathbf{S}$ corresponds to $\mathbf{G}_1$, so that $\{\exp X;\ X \in \mathbf{N} \cup \mathbf{S}\}$ is contained in $\mathbf{G}_1$. Thus we see that $\mathbf{N} \cup \mathbf{S}$ is associated to $\mathbf{G}_1$ and $\mathbf{G}_1 = \mathbf{E}$.

Finally, as we first remarked, the Lie ideal $\mathbf{N} \cup \mathbf{S}$ corresponds to any closed normal subgroup $\mathbf{I}$, so that $\{\exp X;\ X \in \mathbf{N} \cup \mathbf{S}\} \subset \mathbf{I}$. But the group $\mathbf{G}_1$ is generated by $\{\exp X;\ X \in \mathbf{N} \cup \mathbf{S}\}$ hence $\mathbf{I}$ contains $\mathbf{G}_1$ as the connected component (Proposition 4). This completes the proof.

REMARK. $\mathbf{G}_1$ is characterized as the topological commutator subgroup of $\mathbf{G}$. (Cf. [7; p. 90])

THEOREM 10. *If* $\mathbf{M}$ *is a factor of case* (III), *then both its infinite unitary group and the general linear group have no proper, non-central, closed normal subgroup.* ([6; Theorem 2] and [7; Theorem 3]).

PROOF. The theorem follows from Theorem 6.

Let us now investigate a factor of case (I_∞) or (II_∞). First consider the infinite unitary group $\mathbf{U}$. The Lie ideal corresponding to the non-central closed normal subgroup is one of $\mathbf{N} \cap \mathbf{S}$, $\mathbf{J} \cap \mathbf{S}$, or $(\mathbf{J} \cap \mathbf{S}) + (\rho\mathbf{I})$. However,

LEMMA. *In a factor of case* (I_∞) *or* (II_∞), *there exists no non-central closed normal subgroup of* $\mathbf{U}$ *corresponding to* $\mathbf{N} \cap \mathbf{S}$.

PROOF. Suppose that there exists a non-central closed normal subgroup $\mathbf{V}$ which corresponds to $\mathbf{N} \cap \mathbf{S}$. From [6; Lemma 4] it is sufficient to assume that $\mathbf{V}$ contains the center $\{\lambda I\}$. Then the corresponding Lie ideal contains (ρI). This is a contradiction.

Now denote by

$\mathbf{U}_{f0}^{(1)}$: the subgroup consisting of those operators in $\mathbf{U}$ which act as the identity on the orthocomplement of a subspace of finite relative dimension,

$\mathbf{U}_f^{(1)}$: its uniform closure in $\mathbf{U}$,

$\mathbf{U}_{f0}$: the subgroup consisting of those operators in $\mathbf{U}$ which act as λI on the orthocomplement of a subspace of finite relative dimension,

$\mathbf{U}_f$: its uniform closure in $\mathbf{U}$.

It will be easily proved that these are normal subgroups of $\mathbf{U}$. Also clearly $\mathbf{U}_f^{(1)}$ and $\mathbf{U}_f$ are connected and $\mathbf{U}_f$ is the direct product of $\mathbf{U}_f^{(1)}$ and (λI).

PROPOSITION 11. *$\mathbf{U}_f^{(1)}$ is the smallest (non-central) closed normal subgroup in $\mathbf{U}$ and is associated to the Lie ideal $\mathbf{J} \cap \mathbf{S}$.*

PROOF. If a unitary operator U is in $\mathbf{U}_{f0}^{(1)}$, then there exists a finite projection $P \in \mathbf{M}$ such that $U = UP + (I - P)$. Now let $U_p \in \mathbf{U}_{f0}^{(1)}$, such that $\frac{1}{\varepsilon_p}(U_p - I) \to X$, then

$$\frac{1}{\varepsilon_p}(U_p - I) = \frac{1}{\varepsilon_p}(U_pP_p + (I - P_p) - I) = $$
$$= \frac{1}{\varepsilon_p}(U_p - I)P_p \to X,$$

and $\frac{1}{\varepsilon_p}(U_p - I)P_p \in \mathbf{F}$, hence $X \in \mathbf{J} \cap \mathbf{S}$.

Conversely, let $X = XP$ be an element of $\mathbf{F} \cap \mathbf{S}$, with a finite projection P, then $\exp X = \exp XP = (\exp X)P + (I - P) \in \mathbf{U}_{f0}^{(1)}$. Considering the structure of $\mathbf{U}$ (Proposition 9), for any X, $Y \in \mathbf{F} \cap \mathbf{S}$, there exists a $Z \in \mathbf{S}$ such that $\exp Z = (\exp X)(\exp Y)$, but $\exp X, \exp Y \in \mathbf{U}_{f0}^{(1)}$, so that $\exp Z \in \mathbf{U}_{f0}^{(1)}$ or $Z \in \mathbf{F} \cap \mathbf{S}$. Therefore $\{\exp X;\ X \in \mathbf{F} \cap \mathbf{S}\}$ is a group, so that the Lie ideal $\mathbf{J} \cap \mathbf{S}$ is associated to $U_f^{(1)}$ by Propositions 3 and 4. This fact and the above lemma implies also that $\mathbf{U}_f^{(1)}$ is the smallest in the closed normal subgroups corresponding to $\mathbf{J} \cap \mathbf{S}$.

PROPOSITION 12. *$\mathbf{U}_f$ is the maximal closed normal subgroup in $\mathbf{U}$ and is associated to the closed Lie ideal $(\mathbf{J} + (\alpha I)) \cap \mathbf{S}$.*

PROOF. By the same consideration as in the above proposition, we see that if $U_p \in \mathbf{U}_{f0}$ such that $\frac{1}{\varepsilon_p}(U_p - I) \to X$, and if U_p is of the form $U_pP_p + \lambda_p(I - P_p)$ with a finite projection P_p, then

$$\frac{1}{\varepsilon_p}(U_p - I) = \frac{1}{\varepsilon_p}((U_p - \lambda_p)P_p + (\lambda_p - 1)I) \to X$$

and the middle term is contained in $\mathbf{F} + (\alpha I)$, so that $X \in (\mathbf{J} + (\alpha I)) \cap \mathbf{S}$. The converse relation may be proved analogously. Thus we see that $\mathbf{U}_f$ is the smallest closed subgroup corresponding to $(\mathbf{J} + (\alpha I)) \cap \mathbf{S}$.

Let us now suppose that there exists a proper closed normal subgroup $\mathbf{V}$ containing $\mathbf{U}_f$, and consider the C^*-algebra $\mathbf{M}/\mathbf{J}$. Then $\mathbf{V}/\mathbf{J}$ is a normal subgroup of $\mathbf{U}/\mathbf{J}$, the unitary group of $\mathbf{M}/\mathbf{J}$ and $\mathbf{V}/\mathbf{J}$ has its Lie ideal (λI) in $\mathbf{S}/\mathbf{J}$, for the mapping $\mathbf{M} \to \mathbf{M}/\mathbf{J}$ is uniformly continuous. According to Proposition 5[9], this implies $\mathbf{V}/\mathbf{J}$ is contained in (λI) or $\mathbf{V} \subset \mathbf{J} + (\lambda I)$. Since $\mathbf{U}_f = \mathbf{U} \cap (\mathbf{J} + (\alpha I))$, $\mathbf{V} \subset \mathbf{U}_f$, that is, the maximality of $\mathbf{U}_f$ is obtained, and the proof is completed.

9) The C^*-algebra $\mathbf{M}/\mathbf{J}$ has sufficiently many unitary or hermitian operators as the images of those operators in $\mathbf{M}$.

Combining the above two propositions we obtain the Theorem 4 of [6], that is,

THEOREM 11. *If* $\mathbf{M}$ *is a factor of case* (I_∞) *or* (II_∞) *and* $\mathbf{U}$ *is its unitary group, then the only proper closed normal subgroups of* $\mathbf{U}$ *are the closed normal subgroups of* $\mathbf{U}_f$. *The closed normal subgroups of* $\mathbf{U}_f$ *are those generated by* $\mathbf{U}_f^{(1)}$ *and the finite subgroups of* (λI), *and those finite subgroups themselves.*

The final step is to investigate the infinite general linear group $\mathbf{G}$ of a factor of case (I_∞) or (II_∞). The treatment is analogous as the infinite unitary group.

Let us first introduce two closed normal subgroups following to Kadison [7]. Denote by

$\mathbf{G}_{f0}$: the subgroup of operators each of which acts as a complex multiple of the identity on the orthocomplement of a subspace of finite relative dimension,

$\mathbf{G}_f$: its uniform closure in $\mathbf{G}$,

$\mathbf{G}_{f0}^{(1)}$: the subgroup of those operators as mentioned above, for which this scalar is 1,

$\mathbf{G}_f^{(1)}$: its uniform closure in $\mathbf{G}$.

Evidently $\mathbf{G}_f^{(1)}$ and $\mathbf{G}_f$ are connected, and $\mathbf{G}_f$ is the direct product of $\mathbf{G}_f^{(1)}$ and (αI).

PROPOSITION 13. $\mathbf{G}_f^{(1)}$ *is the smallest (non-central) closed normal subgroup in* $\mathbf{G}$ *of a factor of case* (I_∞) *or* (II_∞) *and is associated to the Lie ideal* $\mathbf{J}$.

PROOF. As in the proof of Proposition 11, let $A_p \in G_{f0}^{(1)}$ such that $\frac{1}{\varepsilon_p}(A_p - I) \to X$, and of the form $A_p = A_pP_p + (I - P_p)$ with a finite projection P_p, then

$$\frac{1}{\varepsilon_p}(A_p - 1) = \frac{1}{\varepsilon_p}(A_pP_p + (I - P_p) - I)$$
$$= \frac{1}{\varepsilon_p}(A_p - I)P_p \to X,$$

and $\frac{1}{\varepsilon_p}(A_p - I)P_p \in \mathbf{F}$, so that $X \in \mathbf{J}$. Further we note that if $A \in \mathbf{G}_{f0}^{(1)}$ and let $A = UH$ be the polar decomposition, then $U, H \in \mathbf{G}_{f0}^{(1)}$. For, $H^2 = A^*A \in \mathbf{G}_{f0}^{(1)}$, so that $H \in \mathbf{G}_{f0}^{(1)}$ by the spectral theory. This implies $U = AH^{-1} \in \mathbf{G}_{f0}^{(1)}$. Hence $\log U$, $\log H \in \mathbf{F}$ by the spectral theory. Conversely, consider $\exp(\log U)$, $\exp(\log H)$ such that $\log U, \log H \in \mathbf{F}$, then these are contained in $\mathbf{G}_{f0}^{(1)}$ and generate $\mathbf{G}_{f0}^{(1)}$, as we can see from Proposition 9. Hence we obtain that $\mathbf{G}_f^{(1)}$ and $\mathbf{J}$ are correspondent according to Propositions 3 and 4.

The minimality of $\mathbf{G}_f^{(1)}$ will be obtained if the following lemma is proved.

LEMMA. *In a factor of case* (I_∞) *or* (II_∞), *there is no closed normal subgroup* *corresponding to any one of the Lie ideals* $\mathbf{N}$, $\mathbf{N} \cup (\mathbf{J} \cap \mathbf{H})$, $\mathbf{N} \cup (\mathbf{J} \cap \mathbf{H}) + (rI)$, $\mathbf{N} \cup (\mathbf{J} \cap \mathbf{S})$, *or* $\mathbf{N} \cup (\mathbf{J} \cap \mathbf{S}) + (\rho I)$.

PROOF. Let $\mathbf{K}$ denotes any one of the above Lie ideals, and suppose that there exists a closed normal subgroup $\mathbf{I}$ corresponding to $\mathbf{K}$. Consider the finite factor $\mathbf{M}_P$, together with $\mathbf{I}_P$ and $\mathbf{K}_P$, induced by a finite projection P, then Theorem 9 is applicable. Since $\mathbf{M}_P$ is a factor of case (I_n) or (II_1) according to (I_∞) or (II_∞) of $\mathbf{M}$ ([9; Lemma 11.3.7.]), so we must treat the two cases separately.

First consider the (I_∞) case. If $\mathbf{K} = \mathbf{N}$, then by Theorem 9, $\mathbf{I}_p$ contains all operators of determinant 1 on $\mathbf{G}_P$. This will be considered later.

Otherwise we obtain $\mathbf{I}_P = \mathbf{G}_P$. Since P is an arbitrary finite projection, it follows that $\mathbf{I} \supset \mathbf{G}_f$, which implies the contradiction to the above Proposition.

Next, the (II_∞) case. If $\mathbf{K}$ is either of $\mathbf{N}$, $\mathbf{N} \cup (\mathbf{J} \cap \mathbf{H})$, $\mathbf{N} \cup (\mathbf{J} \cap \mathbf{H}) + (rI)$, then $\mathbf{K}_P$ does not contain the skew-symmetric operators, so that $\mathbf{I}_P \subset (\alpha I)_P$ from the lemma of Theorem 9. Since the finite projection P is arbitrarily chosen, we obtain $\mathbf{I} \subset (\alpha I)$ which contradicts the assumption. Finally $\mathbf{K}$ is either of $\mathbf{N} \cup (\mathbf{J} \cap \mathbf{S})$ or $\mathbf{N} \cup (\mathbf{J} \cap \mathbf{S}) + (\rho I)$, then $\mathbf{I}_P$ contains the unitary group $\mathbf{U}_P$ and all those operators which have the determinant 1 on $\mathbf{M}_P$ by Theorem 9.

Therefore it is sufficient to consider the case that $\mathbf{I}_P$ contains all operators of determinant 1 on $\mathbf{M}_P$. By the same argument as the first paragraph of the proof of [7; Lemma 6] we can conclude that $\mathbf{I}_P = \mathbf{G}_P$. For, let a positive number γ be given and n so large that $|1 - \gamma^{-1/n}| < \varepsilon$, where ε is a preassigned positive number. Let $P_1, \ldots, P_n$ be n orthogonal projections in $I - P$, each equivalent to P (this choice is possible, since $I - P$ is infinite). Then the operator

$$B = \gamma P + \gamma^{-1/n}(P_1 + \ldots + P_n) + I - (P_1 + \ldots + P_n)$$

is in $\mathbf{I}$, because it has determinant 1 on the finite factor induced by the finite projection $(P_1 + \ldots + P_n)$. Moreover, $\| B - \gamma P - (I - P)\| < \varepsilon$. Since $\mathbf{I}$ is uniformly closed, $\gamma P - (I - P)$ is in $\mathbf{I}$ for each positive scalar γ, so that $\mathbf{I}_P$ contains $\mathbf{G}_P$ by Theorem 9. Since P is an arbitrarily chosen, finite projection, $\mathbf{I}$ contains $\mathbf{G}_f$. But then we have contradictions to the above Proposition.

PROPOSITION 14. *In a factor of case* (I_∞) *or* (II_∞), *the closed normal subgroup* $\mathbf{G}_f$ *is maximal in* $\mathbf{G}$ *and is associated to the closed Lie ideal* $\mathbf{J} + (\alpha I)$, *where* (α) *are complex.*

PROOF. By the same argument as in the proof of Propositions 11 and 12, we see that $\mathbf{G}_f$ is the smallest closed normal subgroup corresponding to the Lie ideal $\mathbf{J} + (\alpha I)$.

The maximality of $\mathbf{G}_f$ follows from the same reasons as in $\mathbf{U}_f$, if we note that $\mathbf{G}_f = \mathbf{G} \cap (\mathbf{J} + (\alpha I))$ and take the quotient modulo $\mathbf{J}$. Hence the proof is completed.

Thus, combining the above two propositions, we obtain Theorem 1 of

[8].

THEOREM 12. *Let* $\mathbf{M}$ *be a factor of case* (I_∞) *or* (II_∞), *and* $\mathbf{G}$ *be its general linear group. Then each proper closed normal subgroup in* $\mathbf{G}$ *is the direct product of* $\mathbf{G}_f^{(1)}$ *and some closed subgroup of complex scalars* (αI), *or such subgroup itself.*

In this way we complete the determination of all closed normal subgroups of the infinite general linear groups and unitary groups of the various cases of factors, which was originally due to Kadison.

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