A GENERALIZATION OF HAAR FUNCTIONS

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1. The well-known system of Rademacher functions was generalized to a system of functions whose values are ω^k, $k = 0, 1, 2, \ldots, \alpha - 1$, where α is a natural number and ω is one of the primitive α-th roots of 1, by P. Lévy [3], and the latter was made to be a complete orthonormal system, i. e. the W_α system of generalized Walsh functions by H. E. Chrestenson [1]. These systems are known to preserve some essential properties of original ones. In this direction, it would be of some interest to ask if analogous generalization is possible for the Haar system [2], which is very close to the Walsh system, and we shall show it is possible, preserving some of the convergence characters of the Haar system.

2. We shall define two systems of orthogonal functions, with period 1, the latter we shall call the generalized Haar system or simply the H_α system; the denomination will be seen natural, as the argument proceeds.

We put

$$\varphi_0(x) = \varphi_{0,0}(x) = 1 \qquad (0 \leqq x < 1)$$

$$\varphi_1(x) = \varphi_{1,0}(x) = \omega^k \qquad (k/\alpha \leqq x < (k+1)/\alpha,\ k = 0, 1, \ldots, \alpha - 1)$$

and generally for $\lambda \geqq 2$,

$$\varphi_{\lambda,\mu}(x) = \begin{cases} \omega^k & (\mu/\alpha^{\lambda-1} + k/\alpha^\lambda \leqq x < \mu/\alpha^{\lambda-1} + (k+1)/\alpha^\lambda,\ k = 0, 1, \ldots, \alpha - 1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu = 0, 1, \ldots, \alpha^{\lambda-1} - 1;\ \lambda = 2, 3, \ldots,$$

$$\chi_0(x) = \varphi_0(x)$$

$$\chi_{\lambda,\mu}^{(\nu)}(x) = \Big(\varphi_{\lambda,\mu}(x)\Big)^\nu \cdot \alpha^{(\lambda-1)/2} \qquad \begin{pmatrix} \nu = 1, \ldots, \alpha - 1; & \\ \mu = 0, 1 \ldots, \alpha^{\lambda-1} - 1; & \lambda = 1, 2, \ldots \end{pmatrix}$$

and we arrange $\chi_{\lambda,\mu}^{(\nu)}$ lexicographically with respect to λ, μ, ν.

It is clear that the system $\{\chi_n\} = \chi_{\mu,\lambda}^{(\nu)}$ thus defined forms an orthonormal system, i. e.

$$\int_0^1 \chi_m(x)\overline{\chi}_n(x)\,dx = \begin{cases} 1 & (m = n) \\ 0 & (m \neq n). \end{cases}$$

Moreover, it is also a complete system, for, $f \in L(0, 1)$ and

$$\int_0^1 f(x)\overline{\chi}_n(x)\,dx = 0 \qquad (n = 0, 1. 2, \ldots)$$

imply in particular

$$\int_0^1 f(x)\,dx = 0, \quad \int_0^1 f(x)\Big(\varphi_{1,0}(x)\Big)^\nu dx = 0 \qquad (\nu = 1, \dots, \alpha - 1)$$

and, denoting $\int_0^x f(t)\,dt$ by $F(x)$, we obtain

$$F(0) = F(1) = 0$$

$$\sum_{k=1}^{\alpha-1} \omega^{-\nu k}\left(F\Big(\frac{k+1}{\alpha}\Big) - F\Big(\frac{k}{\alpha}\Big)\right) = 0 \qquad (\nu = 1, \dots, \alpha - 1).$$

The determinant of these linear equations with respect to $F(0)$, $F(1)$, ..., $F((k+1)/\alpha) - F(k/\alpha)$ $(k = 1, \dots, \alpha - 1)$ is the simplest alternating function of ω^{k}'s, and ω being a primitive α-th roots of 1, this determinant has a value other than 0. So we have

$$F(0) = F(1) = F((k+1)/\alpha) - F(k/\alpha) = 0 \qquad (k = 1, \dots, \alpha - 1)$$

i. e.

$$F(k/\alpha) = 0 \qquad (k = 0, 1, \dots, \alpha).$$

Similarly, starting from

$$\int_0^1 f(x)\Big(\overline{\varphi}_{2,\mu}(x)\Big)^\nu dx = \int_{\mu/\alpha}^{(\mu+1)/\alpha} f(x)\Big(\overline{\varphi}_{2,\mu}(x)\Big)^\nu dx = 0$$

we see

$$F(\mu/\alpha + k/\alpha^2) = 0 \qquad (k = 0, 1, \dots, \alpha;\ \mu = 0, 1, \dots, \alpha - 1).$$

The same argument shows, in general, that

$$F(k/\alpha^\lambda) = 0 \qquad (k = 0, 1, \dots, \alpha^\lambda - 1;\ \lambda = 0, 1, 2, \dots)$$

and since $F(x)$ is (absolutely) continuous, it follows that $F(x)$ vanishes identically for $0 \leqq x < 1$, from which follows at once

$$f(x) = \frac{d}{dx} F(x) = 0 \qquad \text{a. e.}$$

It should be noticed that every W_α (resp. H_α) function is a finite linear combination of H_α (resp. W_α) functions, and so, the completeness of one of the two systems is deduced from the same property of the other.

3. Fourier expansion. Let $f(x) \in L(0, 1)$ and put

$$c_n = \int_0^1 f(x)\overline{\chi}_n(x)\,dx. \tag{1}$$

We shall call the (formal) series

$$f(x) \sim c_0\chi_0(x) + c_1\chi_1(x) + \dots + c_n\chi_n(x) + \dots \tag{2}$$

the H_α Fourier series of $f(x)$; the sign $\sim$ means that the coefficients c_n are defined by the formula (1).

Let $s_n(x) = s_n(x; f)$ be the sum of the n first terms of the series (2):

$$s_n(x) = \sum_{i=0}^{n-1} c_i \chi_i(x) = \sum_{i=0}^{n-1} \chi_i(x) \int_0^1 f(t)\overline{\chi}_i(t)\,dt$$
$$= \int_0^1 f(t) K_n(x, t)\,dt$$

where

$$K_n(x, t) = \sum_{i=0}^{n-1} \chi_i(x)\overline{\chi}_i(t).$$

The kernel $K_n(x, t)$ is a quasi-positive kernel, i. e. it satisfies

1° $$\int_0^1 |K_n(x, t)|\,dt \leqq C \qquad (n = 1, 2, \ldots; 0 \leqq x < 1)$$

for some C (independent of n and x),

2° $$\int_0^1 K_n(x, t)\,dt = 1 \qquad (n = 1, 2, \ldots; 0 \leqq x < 1)$$

3° $$\lim_{n\to\infty} \max_{|x-t| \geqq \delta} |K_n(x, t)| = 0 \qquad \text{for every (small but fixed) } \delta > 0.$$

2° is a direct consequence of the orthonormality of the H_α system. For the proofs of 1° and 3°, it will be convenient, after Haar [3], to illustrate the behaviour of $K_n(x, t)$.

We note, first of all, if $\alpha^{\lambda-1} < n \leqq \alpha^\lambda$, then the last of χ's in K_n (i. e. χ_{n-1}) is the $\chi_{\lambda,\mu}^{(\nu)}$, where

$$n = 1 + (\alpha - 1)\sum_{l=1}^{\lambda-1} \alpha^{l-1} + \mu(\alpha - 1) + \nu = \alpha^{\lambda-1} + \mu(\alpha - 1) + \nu$$

with

$$0 \leqq \mu \leqq \alpha^{\lambda-1}, \quad 1 \leqq \nu \leqq \alpha - 1.$$

Next, it is easily seen that, if $\varphi_{\lambda,\mu}(x)\overline{\varphi}_{\lambda,\mu}(t) \neq 0$,

$$\sum_{\nu=1}^{\alpha-1} \left(\varphi_{\lambda,\mu}(x)\overline{\varphi}_{\lambda,\mu}(t) \right)^\nu = \begin{cases} \alpha - 1 \\ -1 \end{cases}$$

in accordance with the fact that t is, or is not in the interval

$$I_\lambda(x): k/\alpha^\lambda \leqq x < (k + 1)/\alpha^\lambda \qquad 0 \leqq k \leqq \alpha^\lambda - 1$$

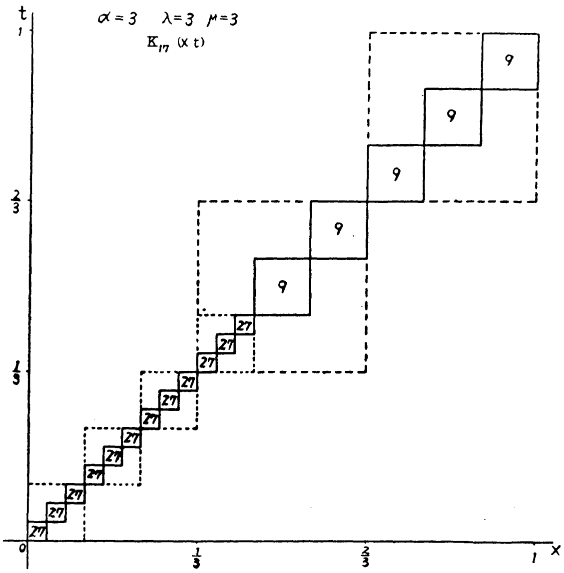
and consequentely

$$\sum_{\nu=1}^{\alpha-1} \chi_{\lambda,\mu}^{(\nu)}(x)\overline{\chi}_{\lambda,\mu}^{(\nu)}(t) = \begin{cases} \alpha^\lambda - \alpha^{\lambda-1} & (t \in I_\lambda(x)) \\ -\alpha^{\lambda-1} & (t \in I_{\lambda-1}(x) - I_\lambda(x)). \end{cases}$$

From this equality it follows that

(3) $$K_{\alpha^{\lambda-1}+(\mu+1)(\alpha-1)}(x, t) = \begin{cases} \alpha^\lambda; & k/\alpha^\lambda \leqq x, t < (k+1)/\alpha^\lambda, k = 0, 1, \ldots, (\mu + 1)\alpha - 1 \\ \alpha^{\lambda-1}, & m/\alpha^{\lambda-1} \leqq x, t < (m+1)/\alpha^{\lambda-1}, m = \mu + 1, \ldots \alpha^{\lambda-1} - 1 \\ 0, & \text{elsewhere.} \end{cases}$$

The circumstance is illustrated in the figure for the case $\alpha = 3, \lambda = 3, \mu = 3$.



In particular, for $\mu = \alpha^{\lambda-1} - 1$,

$$K_{\alpha^\lambda}(x, t) = \begin{cases} \alpha^\lambda & (t \in I_\lambda(x)) \\ 0 & (t \bar{\in} I_\lambda(x)). \end{cases} \tag{4}$$

Now it would be clear that K_n satisfies the condition 3°, for, the $K_{\alpha^\lambda}(x, t)$ does vanish except for $t \in I_\lambda(x)$, and so do all the χ_n's, $\alpha^\lambda \leqq n \leqq \alpha^{\lambda+1} - 1$.

The inequality

$$\int_0^1 |K_n(x, t)|\, dt \leqq \int_0^1 K_{\alpha^{\lambda-1}+\mu(\alpha-1)}(x, t)\, dt + \sum_{i=1}^{\nu} \int_0^1 |\chi_{\alpha^{\lambda-1}+\mu(\alpha-1)+i}(x)\overline{\chi}_{\alpha^{\lambda-1}+\mu(\alpha-1)+i}(t)|\, dt$$
$$= 1 + \nu \leqq 1 + (\alpha - 1) = \alpha$$

shows that $K_n(x, t)$ satisfies also 1°, with $C = \alpha$.

Thus we have proved the following proposition:

THEOREM. *The H_α functions form a complete orthonormal system; for any integrable function $f(x)$, the H_α Fourier series of $f(x)$ converges at every point of continuity to $f(x)$; in particular, if $f(x)$ is continuous at every point of an interval $[a, b] \subset [0, 1)$, its H_α Fourier series converges to $f(x)$ uniformly in $[a, b]$.*

N. B. 1. A moment's consideration shows that the constant C in 1° can be replaced by $1 + \alpha/2$ if α is even, and by $1 + (\alpha - 1)/2$ if α is odd.

2. We have incidentally shown (see the formula (4)) that if $K_n^*(x, t)$ denotes the kernel of W_α system, then

$$K_{\alpha\lambda}(x, t) = K^*_{\alpha\lambda}(x, t)$$

and, for every integrable f, we have

$$s_{\alpha\lambda}(x; f) = \int_0^1 f(t) K_{\alpha\lambda}(x, t) dt = \alpha^\lambda \int_{I_\lambda(x)} f(t) dt \to f(x) \qquad \text{a. e.}$$

as $\lambda \to \infty$.

3. From the formulas (3) and (4) it is easily seen that, for every λ, μ (which may be measurable functions of x),

$$K_{\alpha\lambda-1+(\mu+1)(\alpha-1)}(x, t) \qquad \text{either} = K_{\alpha\lambda}(x, t) \text{ or } K_{\alpha\lambda-1}(x, t).$$

And the argument in proving 3° shows

$$\begin{aligned} |s_{n(x)}(x)| &= \left| \int_0^1 f(t) K_n(x, t)\, dt \right| \\ &\leqq \int_0^1 |f(t)| \, |K_{\alpha l}(x, t)| \, dt + \int_0^1 |f(t)| \sum_{j=1}^{\nu} |\chi^{(j)}_{\lambda,\mu}(x) \overline{\chi}^{(j)}_{\lambda,\mu}(t)| \, dt \\ &\leqq 2\Phi(x) + 2\Phi(x) = 4\Phi(x) \qquad (l = \lambda \text{ or } \lambda - 1), \end{aligned}$$

where $\Phi(x)$ is the "maximal average" of $|f(t)|$ at $t = x$:

$$\Phi(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)| \, dt.$$

Thus the well-known maximal theorem of Hardy and Littlewood gives

$$\left(\int_0^1 |s_{n(x)}(x)|^r \, dx \right)^{1/r} \leqq A_r \int_0^1 |f(x)| \, dx \qquad (0 < r < 1).$$

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