A NOTE ON EILENBERG-MACLANE INVARIANT

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Introduction. It was proved in [3] that if X is arcwise connected and $\pi_i(X) = 0$ for $i < n, n < i < q$, then $H_i(X, G) \cong H_i(K, G)$ for $i < q$, and $H_q(X, G)/\Sigma_q(X, G) \cong H_q(K, G)$, where $K = K(\pi_m(X, m))$, and $\Sigma_q(X, G)$ is the spherical subgroup of the q-th homology group $H_q(X, G)$. In other words under the above conditions, the group π_n determines in a purely algebraic fashion the homology structure of X in dimension $< q$. The group π_n also partially determines the q-dimensional homology group of X. In [3] Eilenberg-MacLane invariant $\mathbf{k}^{q+1}$ determines fully the structure of X in the dimension $\leqq q$.

A. L. Blakers introduced the notions of group system and set system in [2]. It was proved that if in the set system $\mathfrak{S} = \{X_i\}$ the natural homomorphisms $\pi_i(X_{i-1}) \to \pi_i(X_i)$ for all $i < q$ $(q > 0)$ are trivial, then the chain transformation κ induces isomorphism $\kappa_* : H_i(S(\mathfrak{S})) \cong H_i(K(\Pi(\mathfrak{S}))$ for all $i < q$, and for $i = q$, the induced homomorphism $\kappa_* : H_q(K(\mathfrak{S})) \to H_q(K(\Pi(\mathfrak{S}))$ is onto.

In § 2 we give a generalization of Eilenberg-MacLane invariant $\mathbf{k}^{q+1}(\Phi)$; this invariant is a cohomology class of a suitable algebraic cohomology group $H^{q+1}(K(\Pi(\mathfrak{S}), \pi_q(X_q))$ of the group $K(\Pi(\mathfrak{S}))$, with coefficients in $\pi_q(X_q)$.

It is shown that this invariant $\mathbf{k}^{q+1}(\Phi)$ fully determines the structure $S(\mathfrak{S})$ in the dimension $\leqq q$, and we have the following:

THEOREM. *If the natural homomorphisms $\pi_i(X_{i-1}) \to \pi_i(X_i)$ for $i < q, q > 0$ are trivial, then*

$$H^i(S(\mathfrak{S}), G) \cong H^i(K(\Pi(\mathfrak{S}), G) \quad \textit{for } i < q$$
$$H^q(S(\mathfrak{S}), G) \cong H^q(K^*, G),$$

where K^* is the new complex which we will define in § 3.

The main purpose of the present paper is to show the second part of the above theorem.

In § 4 we state algebraic considerations.

1. Preliminaries. We shall use notations and terminologies in [2] and [3].

Let X be an arcwise connected topological space with a point x_0 which will be used as base point for all of the homotopy groups considered in the sequel. Let a sequence $\mathfrak{S} = \{X_i\}$, $i = 0, 1, \dots.$ be a set system in X (cf. [2]). With the system we associate the groups $\pi_i(\mathfrak{S}) = \pi_i(X_i, X_{i-1})$, $i = 1, 2, \dots.$ with x_0 as base point. ($\pi_1(\mathfrak{S}) = \pi_1(X_1, X_0) = \pi_1(X)$.) We consider operator homomorphisms $\Delta_i : \pi_i(\mathfrak{S}) \to \pi_{i-1}(\mathfrak{S})$, for $i = 2, 3\dots.$

(1.1) *For each set system $\mathfrak{S}$; the groups $\pi_i(\mathfrak{S})$ and homomorphisms Δ_i form* *a group system.* $\Pi(\mathfrak{S})$ *is called the group system associated with the set system* $\mathfrak{S}$. (*See* [2], § 10).

We write $[n]$ for the naturally ordered set of integers $\{0, 1, \ldots . n\}$. Let $\alpha: [m] \to [n]$ be a monotonic map, such that $\alpha(i) \leqq \alpha(j)$ for $i < j$. The map α is called degenerate if $\alpha(i) = \alpha(j)$ for some $i < j$. We introduce the special monotonic maps $\varepsilon_n: [n] \to [n]$ defined as the identy map and $\varepsilon_n^i: [n-1] \to [n]$. If $i = 0, \ldots . n$, the map ε_n^i (i fixed) is defined as the monotonic map of $[n-1]$ onto the ordered set $\{0, 1, \ldots., i-1, i+1, n\}$.

(1.2) $\qquad \varepsilon_n^i \varepsilon_{n-1}^j = \varepsilon_n^j \varepsilon_{n-1}^{i-1} \qquad 0 \leqq j < i \leqq n.$ (See [1] § 1)

$\alpha^{(i)}: [m-1] \to [n]$ for $\alpha: [m] \to [n]$ are defined by $\alpha^{(i)} = \alpha \varepsilon_m^i$. If $0 \leqq i_1 < \ldots . < i_r \leqq m$, then we define $\alpha^{(i_1, i_2, \ldots, i_r)}$ inductively by $\alpha^{(i_1, i_2, \ldots, i_r)} = [\alpha^{(i_2, \ldots, i_r)}]^{(i_1)}$. Let $0 \leqq j_0 < j_1 < \ldots . \leqq m$ be the set complementary to $i_1, \ldots . i_r$ in $[m]$, then we also write

$$\alpha^{(i_1 i_2, \ldots i_r)} = \alpha_{(j_0 \ldots j_{m-r})}.$$

Let $\mathfrak{G} = \{G_i, \psi_i\}$ be a group system. Blakers introduced the semi-simplicial complex $K(\mathfrak{G})$ on the group system. We shall recall the definition and some results. An n-cell of $K(\mathfrak{G})$ is defined to be a sequence of functions $\Phi = (\varphi_1, \varphi_2, \ldots .)$, where φ_i (i fixed) is a function of a variable α, α being the map from $[i]$ to $[n]$ with values in G_i subject to the following conditions:

($\Phi 1$): If $\alpha: [i] \to [n]$ is degenerate, then $\varphi_i(\alpha) = 0$ for $n > 2$,
$\varphi_i(\alpha) = 1$ for $n = 1$ or 2,

($\Phi 2$): $\psi_2 \varphi_2(\alpha) = \varphi_1(\alpha^{(2)}) \varphi_1(\alpha^{(0)}) [\varphi_1(\alpha^{(1)})]^{-1}$, where α is a map $[2] \to [n]$,

($\Phi 3$): $\psi_3 \varphi_3(\alpha) = [\varphi_1(\alpha_{(0,1)}) \varphi_2(\alpha^{(0)})] \varphi_2(\alpha^{(2)}) [\varphi_2(\alpha^{(1)})]^{-1} [\varphi_2(\alpha^{(3)})]^{-1}$, where α is a map $[3] \to [n]$,

(Φi): $\psi_i(\varphi_i)(\alpha) = \varphi_1(\alpha_{(0,1)}) \varphi_{i-1}(\alpha^{(0)}) + \sum_{j=1}^{i} (-1)^j \varphi_{i-1}(\alpha^{(j)})$, where α is a map $[i] \to [n]$.

We define the i-face ($i = 0, 1$) of an 1-cell Φ_1 to be the unique 0-cell Φ_0. The j-face $\Phi_n^{(j)}$ of an n-cell Φ_n ($j = 0, 1, \ldots, n$) is defined by $\Phi^{(j)} = (\varphi^{(j)}, \varphi_2^{(j)}, \ldots .)$ such that $\varphi_i^{(j)}(\alpha) = \varphi_i(\varepsilon_n^j \alpha)$ where α is the map $[i] \to [n-1]$. $\varphi^{(j)}$ satisfy the conditions ($\Phi 1$), ($\Phi 2$), ($\Phi 3$), (Φi).

With these definitions $K(\mathfrak{G})$ is obviously a semi-simplicial complex. We shall intoduce the symbol $\Phi^{(i_1, \ldots i_r)} = \Phi_{(j_0, \ldots j_{m-r})}$ for $0 \leqq i_1 < i_2 < i_3 < \ldots . \leqq n$.

Let $\Phi = (\varphi_1, \varphi_2 \ldots .)$ be an n-cell, then $\varphi_{n+1} \ldots .$ is the trivial functions. Since φ_n has the only non-degenerate map $\varepsilon_n: [n] \to [n]$, φ_n is determined by $\varphi_n(\varepsilon_n) \in G_n$. We define $\gamma(\Phi)$ by $\gamma(\Phi) = \varphi_n(\varepsilon_n)$.

Let G be an abelian group with G_1 as operator, and we assume that $\varphi_2 G_2$ is trivial on G. Then we can construct a local system of abelian groups in K, and we denote by $H^i(K(\mathfrak{G}), G)$ the cohomology group of K with coefficients in this local group. Let $S(X)$ be the total singular complex. We denote by $S(\mathfrak{S})$ the subcomplex of $S(X)$ consisting of all singular simplexes T such that $T: \Delta_q \to X_i \subset X$ for $i \geqq 0$. A subcomplex $M(\mathfrak{S})$ of $S(\mathfrak{S})$ will be called minimal provided: (i) For each $q \geqq 0$ the collapsed q-simplex $T: \Delta_q \to x_0$ is in $M(\mathfrak{S})$

and (ii) For each $T \in S(\mathfrak{S}), M(\mathfrak{S})$ contains a unique sigular simplex $T' \in S(\mathfrak{S})$ compatible with and homotopic to T.

We shall consider the prism $\Pi_q = \Delta_{q-1} \times I, q > 0$ where Δ_{q-1} is the $(q-1)$-simplex used to define singular $(q-1)$-simplexes. The maps $e^i_{q-1}: \Delta_{q-2} \to \Delta_{q-1}, i = 0, \ldots, q-1$ define maps $p^i_q: \Pi_{q-1} \to \Pi_q$ by setting $p^i_q(x, t) = (e^i_{q-1}(t), t)$.

We further have the maps $b^t_q: \Delta_{q-1} \to \Pi_q$ $(0 \leqq t \leqq 1)$ defined by $b^t_q(x) = (x, t)$. P: $\Pi_q \to X$ is a singular q-prism in X, and $P^{(i)} = Pp^i_q: \Pi_{q-1} \to X$ is the i-th face of $P, i = 0, \ldots, q-1$. The singular $(q-1)$-simplexes $P(t) = pb^t_q$: $\Delta_{q-1} \to X$, $0 \leqq t \leqq 1$ will be considered.

(1.3) *For any q-simplex in $S(\mathfrak{S})$, there is a singular $(q+1)$-prism P_T in X subject to the following conditions*: (i) $P_T(i) = P_T^{(i)}$, (ii) $P_T(0) = T$, (iii) $P_T(1) \in M(\mathfrak{S})$, (iv) *If $T \in M(\mathfrak{S})$ then $P_T(t) = T$ for all $t \in I$*, (v) $P_T(t)(\Delta_{q,i}) \subset X_i$. (*cf.* [1] §5 (5.1))

If we denote $\varphi_t T = P_T(t)$ $(0 \leqq t \leqq 1)$, then for every singular q-simplex in $S(\mathfrak{S})$ $(\varphi_t T)(p)$ is continuous with respect to p and t and conditions (i)-(v) can be rewritten as follows: (i)' $\varphi_t: S(\mathfrak{S}) \to S(\mathfrak{S})$ is simplicial, (ii)' φ_0 is the identity, (iii)' $\varphi_1 T \in M(\mathfrak{S})$, (iv)' $\varphi_t T = T$ for $T \in M$ and $0 \leqq t \leqq 1$, and (v)' $\varphi_t T(\Delta_{q,i}) \subset X_i$.

This is proved similary as ([1] §5). Thus we have the following:

(1.4) *The inclusion simplicial map i: $M(\mathfrak{S}) \to S(\mathfrak{S})$ and the simplicial map φ_1: $S(\mathfrak{S}) \to M(\mathfrak{S})$ are maps such that the composition $\varphi_1 i$: $M(\mathfrak{S}) \to M(\mathfrak{S})$ is the identity, while the composition $i\varphi_1$: $S(\mathfrak{S}) \to S(\mathfrak{S})$ is chain homotopic to the identity.*

A corollary of (1.4) is

(1.5) *The inclusion map i: $M(\mathfrak{S}) \to S(\mathfrak{S})$ induces isomorphisms of the homology and cohomology groups of the $S(\mathfrak{S})$ with those of the minimal complex $M(\mathfrak{S})$.*

Let G be a local coefficient system in $S(\mathfrak{S})$ and G' be the induced local system in $M(\mathfrak{S})$.

(1.6) *The inclusion map i : $M(\mathfrak{S}) \to S(\mathfrak{S})$ induces isomorphisms*

$$i^*: H^q(S(\mathfrak{S}), G) \cong H^q(M(\mathfrak{S}), G'),$$
$$i_*: H_q(S(\mathfrak{S}), G) \cong H_q(M(\mathfrak{S}), G').$$

In particular, when $\pi_1(\mathfrak{S})$ acts as a group of operators on G and $\psi_2\pi_2(\mathfrak{S})$ acts trivially on G, the group G induces a local coefficient on $S(\mathfrak{S})$ and local coefficient system on $K(\Pi(\mathfrak{S}))$.

(1.7) *The simplical map φ_1 maps the minimal complex M_1 isomorphically onto the minimal complex M.*

Let X, A be arcwise connected topological spaces such that $X \supset A \ni x_0$. A singular q-simplex T: $\Delta_q \to X$ such that $T(\Delta_{q,q-1}) \subset A$, $T(d^j_q) = x_0$ determines an element of the homotopy group $\pi_q(X, A)$. We denote this element by $a(T)$.

Consider a map $f: \Delta_{q+1,q} \to X$ such that $f(d^0_{q+1}) = x_0$, then the map f determines an element $c(f)$ of $\pi_q(X)$.

(1.8) *Let f be a map such that $\Delta_{q+1,q} \to X$, $f(\Delta_{q+1,q-1}) \to A$ and $f(\Delta_{q+1,0}) = x_0$. Let $T^i = fe^i_{q+1}$ $(i = 0, \dots, q+1)$. If $q = 1$, T^i (i fixed) determines an element $a(T^i)$ of $\pi_1(X)$, and if $q \geqq 2$, T^i (f fixed) determines an element $a(T^i)$ of $\pi_q(X, A)$. Moreover, the following relations hold good*:

$$\text{(1.9)}\quad \begin{aligned} c(f) &= a(T^2)a(T^0)a(T^1)^{-1}, \quad q = 1 \\ j_2c(f) &= \alpha a(T^0)a(T^2)\,[a(T^1)]^{-1}\,[a(T^3)]^{-1}, \quad q = 2 \\ j_qc(f) &= \alpha a(T^0) + \sum_{j=1}^{q+1}(-1)^j a(T), \quad q \geqq 3 \end{aligned}$$

where α is the element of $\pi_1(X)$ determined by the edges d^0_{q+1}, d^1_{q+1} and the map $f, j_q: \pi_q(X) \to \pi_q(Y, A)$ is a homomorphism induced by the injection $X \to (X, A)$. (cf. [2]).

2. **Invariant.** Let $T \in S(\mathfrak{S})$ be a singular q-simplex. For $\alpha: [i] \to [q]$, $1 \leqq i \leqq q$. $T\alpha$ is an i-simplex of $S(\mathfrak{S})$. Let $T\alpha$ be a map such that $\Delta_i \to X_i$, $T\alpha(\Delta_{i,i-1}) \subset X_{i-1}$ and $T\alpha(d^0_i) = x_0$. Hence an element $a(T\alpha) \in \pi_i(\mathfrak{S})$ is determined.

We put $\varphi_i(\alpha) = a(T\alpha)$. φ_i (i fixed) is a function of a variable, the variable being a map from $[t]$ to $[q]$ with values in $\pi_i(\mathfrak{S})$. We assume that if α is degenerate $\varphi_i(\alpha) = 0$ for $i > 2$ and $\varphi_i(\alpha) = 1$ for $i = 1, 2$. The sequence of functions $(\varphi_1, \dots, \varphi_n, \dots)$ will be called the schema of the singular simplex T. The function φ_n is trivial for $n > q$. The function of the schema satisfies certain identities which are immediate consequences of the additivity theorem (1.8) and the definition of homomorphism Δ_i (cf. [2] §3).

Then the schema $\Phi = (\varphi_1, \varphi_2, \dots)$ of T is an n-cell of $K(\Pi(\mathfrak{S}))$.

We define $\kappa(T)$ by $\kappa(T) = \Phi$. Obviously $\kappa(T^{(i)}) = \Phi^{(i)}$, hence κ is a simplicial map. Since $M(\mathfrak{S})$ is a subcomplex of $S(\mathfrak{S})$,

$$\kappa: M(\mathfrak{S}) \to K(\Pi(\mathfrak{S}))$$

is defined. By ([5] p. 391) we have the following:

(2.1) *If T_0 and T_1 are maps such that $(\Delta_q, \Delta_{q-1}, d^0) \to (X_q, X_{q-1}, x_0)$ and $T_0^{(i)} = T_1^{(i)}$, then $j_qd(T_0, T_1) = a(T_0) - a(T_1)$, where $d(T_0, T_1) \in \pi_q(X_q)$.*

LEMMA. *If in the set system $\mathfrak{S} = \{X_i\}$ the homomorphisms $\pi_i(X_{i-1}) \to \pi_i(X_i)$, $i < p$, $q > 0$ are trivial, then there is a semi-simplicial map $\bar{\kappa}: K^q(\Pi(\mathfrak{S})) \to M(\mathfrak{S})$ such that $\kappa\bar{\kappa}$ = the identity and $\bar{\kappa}$ is determined uniquely on $K^{q-1}(\Pi(\mathfrak{S}))$.*

PROOF. $M(\mathfrak{S})$ and $K(\Pi(\mathfrak{S}))$ have exactly one 0-simplex T^0 and one 0-cell Φ^0 respectively. We define $\bar{\kappa}(\Phi^0)$ by $\bar{\kappa}(\Phi^0) = T^0$. Let Φ^1 be a 1-cell of $K(\Pi(\mathfrak{S}))$ and T' a map which represent $\gamma(\Phi^1) \in \pi_1(X)$, then there is $T^1 \in M$ homotopic with T'. We define $\bar{\kappa}(\Phi^1)$ by $\bar{\kappa}(\Phi^1) = T^1$. It satisfies $\kappa\bar{\kappa}(\Phi^1) = \Phi^1$. Suppose that $\bar{\kappa}$ is well defined for all cells of dimension $< i$ $(1 < i \leqq q)$. Let Φ be an i-cell of $K(\Pi(\mathfrak{S}))$, such that $\gamma(\Phi) \in \pi_i(X_i, X_{i-1})$, $\gamma(\Phi^{(i)}) \in \pi_{i-1}(X_{i-1}, X_{i-2})$, $\gamma(\Phi_{(0,1)}) \in \pi_1(X_1)$. By the inductive hypothesis there are $(i-1)$-simplexes

T_j, $j=0,\ldots,i$ in M such that $T_j=(\bar{\kappa}\Phi^{(j)})$ and we have $T^k_j=T^{(j-1)}_k$ for $k<j$. This implies the existence of a map $f\colon \Delta_{i,i-1}\to X_{i-1}$ such that $fe^j_i=T_j, j=0,\ldots, i$. A map f such that $f(\Delta_{i,i-1})\subset X_{i-1}$ determines an element $c(f)\in\pi_{i-1}(X_{i-1})$. The elements $c(f)$, $\gamma(\Phi^{(j)})$, $\gamma(\Phi_{(0,1)})$ are connected by (1.8) and the elements $\gamma(\Phi), \gamma(\Phi^{(i)}), \gamma(\Phi_{(0,1)})$ are related by

$$\Delta_2\gamma(\Phi)=\gamma(\Phi_{(0,1)})\,\gamma(\Phi^{(0)})\;[\gamma(\Phi^{(1)})]^{-1},$$
$$\Delta_3\gamma(\Phi)=[\gamma(\Phi_{(0,1)})\,\gamma(\Phi^{(0)})]\;\gamma(\Phi^{(2)})\;[\gamma(\Phi^{(1)})]^{-1}\,[\gamma(\Phi^{(3)})]^{-1},$$
$$\Delta_i\gamma(\Phi)=\gamma(\Phi_{(0,1)})\,\gamma(\Phi^{(0)})+\sum\nolimits_{j=1}^{i}(-1)^j\,\gamma(\Phi^{(j)}).$$

It follows that $\Delta_i(\gamma(\Phi))=\lambda_{i-1}(c(f))$, where λ_{i-1} are the natural homomorphisms $\lambda_{i-1}:\pi_{i-1}(X_{i-1})\to\pi_i(X_{i-1},X_{i-2})$. That is $\lambda_{i-1}\partial_i(\gamma(\Phi))=\lambda_{i-1}\,c(f)$ where ∂_i are the natural homomorphisms $\partial_i:\pi_i(X_i,X_{i-1})\to\pi_{i-1}(X_{i-1})$. But from the hypothesis the natural homomorphisms $\pi_{i-1}(X_{i-2})\to\pi_{i-1}(X_{i-1})$ are trivial, and hence from the exactness property of the homotopy sequence of the pair (X_{i-1}, X_{i-2}), λ_{i-1} are isomorophisms into and hence $\partial_i\gamma(\Phi)=c(f)$.

It follows that the mapping f has an extension $T'\colon\Delta_i\to X_i$ such that $\gamma(\Phi)=c(T')$ and there is an element $T\in M(\mathfrak{S})$ compatible and homotopic with T'. We define $\bar{\kappa}(\Phi)$ by $\bar{\kappa}(\Phi)=T$. It satisfies $\kappa\bar{\kappa}\,(\Phi)=\Phi$.

Now, we shall prove the uniqueness. If $q=1$, this is obvious. Suppose that the uniqueness has been proved for $0\leqq i<q-1$. Assume that $\bar{\kappa}(\Phi)=T$ and $\bar{\kappa}'(\Phi)=T'$, where Φ is an i-cell in $K(\Pi(\mathfrak{S}))$. Then $T^{(j)}=\bar{\kappa}\,(\Phi^{(j)})=\bar{\kappa}'(\Phi^{(j)})=T'^{(j)}$, hence T, T' is compatible. By (2.1)

$$\lambda_i d(T,T')=a(T)-a(T')=\gamma(\Phi)-\gamma(\Phi)=0.$$

Since λ_i (i fixed) is an isomorphism, $d(T,T')\in\pi_i(X_i)$ is zero. Therefore T, T' is homotopic in X_i fixing the boundary of X_i, and by virtue of the fact that $T,T'\in M(\mathfrak{S})$, it follows that $T=T'$, and hence $\kappa=\kappa'$. q. e. d.

Let Φ be a $(q+1)$-cell of $K(\Pi(\mathfrak{S}))$, then $\bar{\kappa}\Phi^{(i)}=T^{(i)}$ is a q-simplex of $M(\mathfrak{S})$. By the simpliciality of $\bar{\kappa}$, $(T^{(i)})^{(j)}=(T^{(j)})^{(i-1)}$ for $j<i$, hence a map of $(\Phi)\colon\Delta_{q+1,q}\to X_q$ is defined by $f(\Phi)e^i_{q+1}=T^{(i)}$. Then $f(\Phi)$ determines an element $c(f(\Phi))$ of $\pi_q(X_q)$. We define $k^{q+1}(\Phi)$ by $k^{q+1}(\Phi)=c(f(\Phi))\in\pi_q(X_q)$. Thus $k^{q+1}(\Phi)$ is a cochain, i.e., $k^{q+1}(\Phi)\in C^{q+1}(K,\pi_q(X_q))$. We have easily the following lemma (cf. [3] p. 503):

LEMMA. *$k^{q+1}(\Phi)$ is a cocycle.*

The cohomology class of the cocycle $k^{q+1}(\Phi)$ will be denoted by $\mathbf{k}^{q+1}(\Phi)$. It is an element of the cohomology group $H^{q+1}(K,\pi_q(X_q))$.

By the way analogous to the proof of Theorem 1 of [3] we have easily the following

THEOREM I. *If in the set system $\mathfrak{S}=\{X_i\}$ the natural homomorphisms $\pi_i(X_{i-1})\to\pi_i(X_i)$ for all $i<q, q>0$ are trivial, then the cohomology class $\mathbf{k}^{q+1}(\Phi)\in H^{q+1}(K,\pi_q(X_q))$ is a topological invariant independent of the choice of minimal complex $M(\mathfrak{S})$ and the simplicial map $\bar{\kappa}$ used in its definition. If k is any cocycle in the class $\mathbf{k}^{q+1}(\Phi)$ and $M(\mathfrak{S})$ any minimal subcomplex of $S(\mathfrak{S})$,* *then a suitable choice of* $\bar{\kappa}$ *will produce* k *as the cocycle* k^{q+1}.

3. The main theorem. Suppose that $M(\mathfrak{S})$ be a fixed minimal subcomplex of $S(\mathfrak{S})$. We assume that $\pi_i(X_{i-1}) \to \pi_i(X_i)$ are trivial for $i < q$, $q > 0$, and that a function $\bar{\kappa}$ has been selected so that to every cell Φ of $K(\Pi(\mathfrak{S}))$ of dim $\leqq q$, there corresponds a singular simplex $\bar{\kappa}(\Phi)$ of $M(\mathfrak{S})$ such that $\kappa\bar{\kappa}(\Phi) = \Phi$.

The obstruction cocycle defined by making use the function Φ is $k^{q+1} \in Z^{q+1}(K(\Pi(\mathfrak{S})), \pi_q(X_q))$. For each i-cell $i < q$, Φ of $K(\Pi(\mathfrak{S}))$ we shall denote by $[\Phi]$ the singular simplex $\bar{\kappa}(\Phi)$.

Thus $\kappa[\Phi] = \Phi$, and $[\Phi]^i = [\Phi^{(i)}]$. For each q-cell Φ of $K(\Pi(\mathfrak{S}))$ and for each $x \in \pi_q(X_q)$ we shall denote by $[\Phi, x]$ the unique q-simplex of $M(\mathfrak{S})$ compatible with $\bar{\kappa}(\Phi)$, such that $d(\bar{\kappa}(\Phi), [\Phi, x]) = x$. Thus $\kappa[\Phi, x] = \Phi$, $[\Phi, 0] = \bar{\kappa}\Phi$, $d([\Phi, x], [\Phi, y]) = y - x$, and $[\Phi, x]^{(i)} = [\Phi^{(i)}]$ for $i = 0, \dots .q$.

Every q-simplex T of $M(\mathfrak{S})$ is of the form $[\Phi, x]$; i.e. $T = [\kappa T, d(\bar{\kappa}\kappa T, T)]$. Thus a complete description of the simplexes of $M(\mathfrak{S})$ of dimension $\leqq q$ is obtained.

(3.1) *Let* $[\Phi_0, x]$, $[\Phi_1, x_1]$, $\dots$, $[\Phi_{q+1}, x_{q+1}]$ *be given. A* $(q+1)$*-simplex* T *in* $M(\mathfrak{S})$ *such that* $T^{(i)} = [\Phi_i, x_i]$ *exists, if and only if there is a* $(q+1)$*-cell of* $K(\Pi(\mathfrak{S}))$ *such that* $\Phi^{(i)} = \Phi_i$, $i_q(k^{q+1}(\Phi) + \sum_{i=0}^{q+1}(-1)^i x_i) = 0$ and if $q = 2$, $i_q(k^{q+1}(\Phi) + \alpha x_0 + \sum_{i=0}^{q+1}(-1)^i x_i) = 0$.

LEMMA. *Let* f_0 *and* f_1 *be two maps such that* $\Delta_{q+1,q} \to X$, $f_0(\Delta_{q+1,0}) = f_1(\Delta_{q+1,0}) = x_0$ *and* $f_0 = f_1$ *on* $\Delta_{q+1,q-1}$, $q > 1$. *Let* $T^i_j = f_j e^i_{q+1}$ *be maps such that* $\Delta_q \to X$ *for* $i = 0, \dots, q+1, j = 0, 1$. *Since* T^i_0 *and* T^i_1 *are compatible,* $d(T^i_0, T^i_1)$ *is defined. Let* α *be the element of* $\pi_1(X_q)$ *determined by the edge* d^0_{q+1}, d^1_{q+1} *and either of the maps* f_0 *or* f_1 *(which agree on this edge). Then*

$$c(f_1) - c(f_0) = \alpha\, d(T^0_0, T^0_1) + \sum_{i=1}^{q+1}(-1)^i\, d(T^i_0, T^i_1). \quad (cf. [3], \text{ p. } 515)$$

PROOF OF (3.1). The necessity can be proved in the same way as the proof of ((4.1)[3]). We shall prove now the sufficiency. Let Φ and $\bar{\kappa}(\Phi^{(i)})$ are a $(q+1)$-cell of $K(\Pi(\mathfrak{S}))$ and a q-simplex of $M(\mathfrak{S})$ respectively. Then we have $d(\bar{\kappa}(\Phi^{(i)}), f_i) = x_i$ for $x_i \in \pi_q(X_q)$, where f_i is a simplex of $M(\mathfrak{S})$ such that $f_i = [\Phi^{(i)}, x_i]$. Map f: $\Delta_{q+1,q} \to X_q$ is defined such that $fe^i_{q+1} = f_i$. The map f will be extended to a map $\bar{f}$: $\Delta_{q+1} \to X_{q+1}$. To prove it we consider the map $\bar{\kappa}(\Phi^{(i)})$: $\Delta_q \to X_q$, then the map defines a map g: $\Delta_{q+1,q} \to X_q$ such that $g^i = ge^i_{q+1} = \bar{\kappa}(\Phi^{(i)}) = [\Phi^{(i)}, 0]$. Since $fe^i_{q+1} = f_i = [\Phi^{(i)}, x_i]$, we have $f = g$ on $\Delta_{q+1,q}$. By the above Lemma

$$c(f) - c(g) = \sum_{i=0}^{q+1}(-1)^i\, d(g_i, f_i) = \sum_{i=0}^{q+1}(-1)^i x_i.$$

Therefore $c(f) = k(\Phi) + \sum_{i=0}^{q+1}(-1)^i x_i$,

$$i_q c(f) = i_q\ (k(\Phi) + \sum_{i=0}^{q+1}(-1)^i x_i) = 0,$$

hence f can be extended to map $\bar{f}$: $\Delta_{q+1} \to X_{q+1}$. If we take a simplex of $M(\mathfrak{S})$ compatible and homotopic to f_i, then we have

$$T^{(i)} = (\bar{f})^i = f_i = [\Phi^{(i)}, x_i] = [\Phi_i, x_i].$$

The case $q = 2$ is proved similarly.

These considerations lead to a description of the cochains of dimension $\leqq q$ on the $M(\mathfrak{S})$ with coefficients in any group G. Indeed the cochains of dimension $< q$ may be identified with the corresponding cochains of $K(\Pi(\mathfrak{S}))$. The cochains of dimension q are G-valued functions $f(\Phi, x)$ of two variables of which the first is a q-simplex of $K(\Pi(\mathfrak{S}))$ while the second is an element of $\pi_q(X_q)$.

Such a function f is a cocycle on $M(\mathfrak{S})$ under following assumptions. For every $(q+1)$-cell Φ of $K(\Pi(\mathfrak{S}))$ and for every system of elements x_0, $\dots$, $x_{q+1} \in \pi_q(X_q)$ such that

$$(*) \qquad i_q(k^{q+1}(\Phi) + \sum_{i=0}^{q+1} (-1)^i x_i) = 0$$

the equality

$$(**) \qquad \sum_{i=0}^{q+1} (-1)^i f(\Phi^{(i)}, x_i) = 0$$

holds for $q > 2$. If $q = 2$ then in (*), (**) we have to replace the terms x_0 and $f(\Phi^{(0)}, x)$ by αx_0 and $\alpha f(\Phi^{(0)}, x)$, where $\alpha \in \pi_1(X_q)$ is the element representing the 1-cell $\Phi_{(0,1)}$.

A function $f(\Phi, x)$ yields a coboundary in $M(\mathfrak{S})$ proveded there is a cochain $g \in C^{q-1}(K(\Pi(\mathfrak{S})), G)$ such that $(Sg)\ (\Phi) = f(\Phi, x)$ for all Φ and x.

Therefore, we define the new complex K^* as follows: each $(q-1)$-cell of K^* corresponds 1 to 1 to each q-cell $\Phi^q \in K(\Pi(\mathfrak{S}))$, a q-cell of K^* is the symbol $\Phi = [\Phi, x]$, $(q+1)$-cell of K^* is $\Phi = [\Phi, x]$ such that its faces $[\Phi^{(0)}, x_0], \dots, [\Phi^{(q+1)}, x_{q+1}]$ satisfy the condition $i_q(k^{q+1}(\Phi)) + \sum_{i=0}^{q+1} (-1)^i x_i) = 0$.

Resuming the above results we have the following theorem:

THEOREM II. *Let $\mathfrak{S} = \{x_i\}$ be a set system, and let natural homomorphisms $\pi_i(X_{i-1}) \to \pi_i(X_i)$ for $i < q, q > 0$ be trivial. Then for any coefficient group G, the cohomology group $H^i(S(\mathfrak{S}), G)$ is isomorphic to $H(K^*, G)$ for $i \leqq q$, i.e.*

$$H^i(S(\mathfrak{S}), G) \cong H(K^*, G) \text{ for } i \leqq q.$$

4. Algebraic considerations. We consider the following algebraic situation. Let $\mathfrak{G} = (\pi_i, G)$ be a group system and G be an abelian group and suppose that a cocycle $Z^{q+1}\ (K(\mathfrak{G}), \pi_q(X_q))$ is given for $1 < q$. We consider a function $f(\Phi, x)$ with values in G, of two variables, the first of which is a q-cell of $K(\mathfrak{G})$, while the second is an element of $\pi_q(X_q)$. These functions $f(\Phi, x)$ are subject to the following condition:

(4.1) *For every $(q+1)$-cell Φ of $K(\mathfrak{G})$ and for every system of elements x_0,*

$\dots, x_{q+1} \in \pi_q(X_q)$ *the equality*

(i) $$i_q(k(\Phi) + \sum_{i=0}^{q+1}(-1)^i x_i) = 0$$

implies

(ii) $$\sum_{i=0}^{q+1}(-1)^i f(\Phi^{(i)}, x_i) = 0,$$

where i_q is the injection homomorphism $i_q : \pi_q(X_q) \to \pi_q(X_{q+1})$. If $q = 2$, in (i) *and* (ii) *we have to replace the terms x_0 and $f(\Phi^{(0)}, x_0)$ by αx_0, and $\alpha f(\Phi^{(0)}, x_0)$ respectively, where $\alpha \in \pi_1(X_q)$ is the element represented by $\Phi_{(0,1)}$.*

The following lemma shows that these functions $f(\Phi, x)$ break up into the sum of functions of one variable each.

(4.2) *Every function $f(\Phi, x) \in G$ satisfying* (4.1) *may be represented as*

(iii) $$f(\Phi, x) = \rho(x) + r(\Phi)$$

where

(iv) $$\rho \in \mathrm{Hom}(\pi_q(X_q), G), \quad \rho(i_q^{-1}(0)) = 0, \quad r \in C^q(K(\mathfrak{G}), G),$$

(v) $$\delta r = \rho k.$$

Conversely every pair (ρ, r) satisfying (iv) *and* (v) *yields by* (iii) *a function $f(\Phi, x)$ satisfying* (4.1). *The representation* (iii) *is unique and is given by*

(vi) $$\rho(x) = f(\Phi, x) - f(\Phi, 0), \quad r(\Phi) = f(\Phi, 0).$$

We obtain the proof of (4.1) by modifying the proof of (5.2) of [3].

We now form the group $Z_q(k, G)$ as the group of all those pair (p, r) in the direct sum Hom $(\pi_q(X_q), G) + C^q(K(\Pi(\mathfrak{S}), G)$ such that $\delta r = \rho k$, $\rho(i_q^{-1}(0)) = 0$. Any pair $(0, r)$ with $\delta r = 0$ satisfies the last conditions, hence each cocycle $r \in Z^q(K(\Pi(\mathfrak{S}), G)$ may be identified with the element $(0, r)$, accordingly $Z^q(K(\Pi(\mathfrak{S})), G)$ is a subgroup of $Z^q(k, G)$. Since $B^q(K(\Pi(\mathfrak{S}), G)$ is a subgroup of $Z^q(k, G)$ we may form the factor group

$$E^q(k, G) = Z^q(k, G)/B^q(K(\Pi(\mathfrak{S}), G).$$

Then $H^q(K(\Pi(\mathfrak{S}), G)$ is a subgroup of E^q. The following theorem is proved:

THEOREM III. *The group system $\Pi(\mathfrak{S})$, groups $\pi_q(X_q)$, G and the cocycle $k \in Z^{q+1}(K(\Pi(\mathfrak{S}), \pi_q(X_q))$ determine an abelian group $E^q(k, G)$ and a homomorphism χ of this group into* Hom $(\pi_q(X_q), G)$. *The kernel of this homomorphism is the group $H^q(K(\Pi(\mathfrak{S}))$ regarded as a subgroup of E^q.*

The image of χ is the subgroup $A(k)$ of $\mathrm{Hom}(\pi_q(X_q), G)$ which consists of every homomorphism $\rho : \pi_q(X_q) \to G$ such that ρk is a coboundary: $\rho k \in B^{q+1}(K(\Pi(\mathfrak{S})), G)$. Thus (E^q, χ) is an abelian extension of $H^q(K(\Pi(\mathfrak{S}), G)$ by $A(k)$. The subgroup $A(k)$ of $\mathrm{Hom}(\pi_q(X_q), G)$ and the extension in question are independent of the choice of the cocycle k within its cohomology class in $H^{q+1}(K(\Pi(\mathfrak{S})), G)$.

THEOREM IV. *Let* $\mathfrak{S} = \{X_i\}$ *be a set system and let natural homomorphisms* $\pi_i(X_{i-1}) \to \pi_i(X_i)$ *for* $i < q, q > 0$ *are trivial. Then for any coefficient group* G, *the cohomology group* $H^i(S(\mathfrak{S}), G)$ $(i < q)$ *is determined by* $\mathfrak{S}, G$ *as*

$$H^i(S(\mathfrak{S}),\ G) \approx H^i(K(\Pi(\mathfrak{S}), G) \quad i < q,$$

while $H^q(S(\mathfrak{S}), G)$ *is determined by the characteristic cohomology class* $\mathbf{k}^{q+1} \in H^{q+1}(K(\Pi(\mathfrak{S})), G)$ *as*

$$H^q(S(\mathfrak{S}), G) \cong E^q(k, G),$$

where k *is any cocycle in the cohomology class* $\mathbf{k}^{q+1}$.

Proofs of Theorem III, IV are analogous to that of Theorem II, Theorem IV in [3].

BIBLIOGRAPHY

[1] S. EILENBERG AND J. A. ZILLER, Semi-simplicial complexes and singular homology, Ann. of Math., 51(1950), 499-513.

[2] A. L. BLAKERS, Some relations between homotopy groups and homology groups, Ann. of Math., 49(1948), 428-461.

[3] S. EILENBERG AND S. MACLANE, Relations between homology and homotopy group of spaces I, Ann. of Math., 46(1945), 480-509; II, Ann. of Math, 51(1950), 514-533.

[4] S. EILENBERG, Singular homology theory, Ann. of Math.,45(1944),407-477.

[5] W. S. MASSEY, Exact couples in algebraic topology, Ann. of Math., 56(1952),363-396.

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