FURTHER REMARKS ON THE PAPER "UNIFORM CONVERGENCE OF SOME TRIGONOMETRICAL SERIES", TÔHOKU MATH. JOURN., 6(1954), 162-173.

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The series $\sum_{n=1}^{\infty} a_n$ is said to be evaluable (R, p) to zero if the series

$$\sum_{n=1}^{\infty} a_n \left(\frac{\sin nt}{nt}\right)^p$$

converges in some interval $0 < t < t_0$ and its sum tends to zero as $t \to 0$.

In the preceding paper [1], the author proved the following

THEOREM 1. *Let $\beta > 0$ and let $0 < \alpha < 1$. If*

$$\tau_n^\beta = \sum_{\nu=1}^{n} A_{n-\nu}^{\beta-1} \nu a_\nu = o(n^{\beta\alpha}),$$

where A_n^γ is Andersen's notation, and

$$\sum_{\nu=n}^{\infty} |a_\nu - a_{\nu+1}| = O(n^{-\alpha}),$$

then the series $\sum_{n=1}^{\infty} a_n$ is evaluable $(R, 1)$.

Later, the author [2] stated the following

THEOREM 2. *Let $0 < \beta \leqq 1$ and let*

$$\tau_n^\beta = O(n^\beta \lambda_n),$$

where $\lambda_n > 0$ and $\sum \lambda_n/n$ converges. Then the series $\sum_{n=1}^{\infty} a_n$ is evaluable $(R, 1)$.

And I proposed a problem: *whether we may replace $\beta \leqq 1$ by $\beta > 1$ in* Theorem 2. The solution is given by

THEOREM 3. *Let $\beta \geqq 1$. Then, there exists a series $\sum_{n=1}^{\infty} a_n$ such that*

$$\tau_n^\beta = o(n/\log n) \tag{1}$$

and the series $\sum_{n=1}^{\infty} a_n$ is not evaluable $(R, 1)$.

Further we may prove, by the method analogous to one by which we obtain Theorem 3, that, if $\beta \geqq p$ and $p = 2, 3, 4,$ then, there exists a series $\sum_{n=1}^{\infty} a_n$ such that

(2) $$\tau_n^\beta = o(n^p/\log n)$$

and the series $\sum_{n=1}^{\infty} a_n$ is not evaluable (R, p). These results are intresting in some meaning. If (2) holds when $\beta > p \geqq 1$, then we have evidently

$$\tau_n^\beta = O(n^\beta/(\log n)^{1+\varepsilon}), \qquad \varepsilon > 0.$$

This implies the summability $|C, \beta|$ of the series $\sum_{n=1}^{\infty} a_n$. Thus, Theorem 3 and its supplementary remarks show that, when $\beta > p$, the summability $|C, \beta|$ does not always imply the summability (R, p). Therefore we see that the following Obreschkoff's theorem [3] is best possible in their kinds.

THEOREM 4. *The summability $|C, p|$ implies the summability (R, p).*

We shall now prove Theorem 3. For the proof, we may suppose, without loss of generality, that β is an integer. Let $\varphi(n, t) = (\sin nt)/n^2 t$. Then Abel's transformation shows that

$$\begin{aligned}\sum_{\nu=1}^{n} a_\nu \frac{\sin \nu t}{\nu t} &= \sum_{\nu=1}^{n} \nu a_\nu \varphi(\nu, t) \\ &= \sum_{\nu=1}^{n-\beta} \tau_\nu^\beta \Delta^\beta \varphi(\nu, t) + \tau_{n-\beta+1}^\beta \Delta^{\beta-1}\varphi(n-\beta+1, t) + \dots + \tau_n^1 \varphi(n, t)\end{aligned}$$

where $\Delta^\gamma \varphi(n, t)$ denotes the γ-th difference of $\varphi(n, t)$ with respect to n. By (1), when t is fixed,

$$\tau_{n-\gamma+1}^\gamma \Delta^{\gamma-1}\varphi(n-\gamma+1, t) = o\left(\frac{n}{\log n} \cdot \frac{1}{n^2}\right) = o(1),$$

when $\gamma = 1, 2, 3, \dots, \beta$. Therefore the series

(3) $$\sum_{n=1}^{\infty} a_n \frac{\sin nt}{nt} \quad \text{and} \quad \sum_{n=1}^{\infty} \tau_n^\beta \Delta^\beta \varphi(n, t)$$

are equiconvergent for a fixed t. Thus, for the proof, it is sufficient to prove that (1) does not always imply the convergence of the second series in (3) in an arbitrary neighbourhood of the origin. Let us write

(4) $$\sum_{n=1}^{\infty} \tau_n^\beta \Delta^\beta \varphi(n, t) = \sum_{n=1}^{\infty} \varepsilon_n c_n(t),$$

where $\varepsilon_n = n^{-1}\tau_n^\beta \log(n+1)$ and $c_n(t) = n\Delta^\beta \varphi(n, t)/\log(n+1)$. Then we have

$\varepsilon_n = o(1)$ as $n \to \infty$, by (1). In order that the sequence-to-function transform (4) is convergence-preserving, by Kojima-Schur's theorem, $\sum_{n=1}^{\infty} |c_n(t)|$ must uniformly bounded in $0 < t < t_0$. But this series is divergent at some point in an arbitrary neighbourhood of the origin. This is shown by the following. Let $t = 2\pi/k$, where k is an arbitrary positive integer. Since

$$\Delta^\beta \varphi(kn, t) = \frac{k}{2\pi} \sum_{\nu=0}^{\beta} (-1)^\nu \binom{\beta}{\nu} (kn+\nu)^{-2} \sin(2\nu\pi/k),$$

we have

$$\lim_{n\to\infty} \frac{2\pi}{k} (kn)^2 \Delta^\beta \varphi(kn, t) = \sum_{\nu=0}^{\beta} (-1)^\nu \binom{\beta}{\nu} \sin(2\nu\pi/k)$$
$$= \mathfrak{I}\{(1 - e^{2\pi i/k})^\beta\} \neq 0.$$

Hence, there exist an integer n_0 and some constant, independent on $n, C > 0$ such that

$$|\Delta^\beta \varphi(kn, t)| \geqq CK^{-1} n^{-2}$$

when $n \geqq n_0$. Thus we have

$$\sum_{n=1}^{\infty} |c_n(t)| = \sum_{n=1}^{\infty} \frac{n}{\log(n+1)} |\Delta^\beta \varphi(n, t)|$$
$$\geqq \sum_{n=1}^{\infty} \frac{kn}{\log(kn+1)} |\Delta^\beta \varphi(kn, t)|$$
$$\geqq C \sum_{n=n_0}^{\infty} (n \log(kn+1))^{-1} = +\infty.$$

Thus, Theorem 3 is completely proved.

REFERENCES

[1] H. HIROKAWA, Uniform convergence of some trigonometrical series, Tôhoku Math. Journ. 6(1954), 162–173.
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[3] N. OBRESCHKOFF, Über das Riemannsche Summierungsverfahren, Math. Zeits. 48(1942), 441–454.

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