ON A DUALITY OF COHOMOLOGY GROUPS OF FROBENIUS ALGEBRAS

HISASI OGAWA

(Received June 10, 1960)

For a (quasi-) Frobenius algebra R over a commutative ring, Professor T. Nakayama [5], [6], [7] has constructed a complete resolution for R, and developed the complete cohomology theory of the algebras as a generalization of that for finite groups. Recently, K. Hirata [2] has also constructed a complete resolution for R from the viewpoint of the relative homological algebra in the sense of Hochschild [3].

In the present paper, we shall begin with definitions of "weakly projective" and "weakly injective" relative to the Frobenius algebra over a field (Def. 1 and la) and show that "weakly projective", "weakly injective" and "the identity map is a norm of some one-sided-R-epimorphism" are all equivalent to each other for any bi-R-module, (Prop. 3). Using these properties fully we shall be able to give the complete cohomology groups, defined first by Nakayama [5]. In **2** we shall construct an appropriate complete resolution for our algebras (Def. 3). In **3**, we shall develope a product theory of complete cohomology groups in the same way as for finite groups. Finally in **4** we shall show a duality as an application of this product theory in the case where our algebra is commutative (Theorem 4).

Our way to get the complete cohomology groups may be roundabout in comparision with those of [2], [5], [6], [7]. But it seems to the writer interesting that the method, used to find the periodicity of the complete cohomology groups of finite groups (XII in [1]), are applicable to our case almost in the same manner, (this must be a matter of course in some sense).

In closing the introduction the writer would like to express his gratitude to Professors T. Tannaka and T. Nakayama for their valuable advices and encouragements to him.

1. Modules. Throughout this paper we assume that all rings have a unit element which operates as the identity on all modules, (that is, unitary modules).

In the situation $({}_SA_P, {}_PB_Q, {}_TC_Q)$, there exists a right-$S$-and left-$T$-isomorphism

$$(1) \qquad s: \mathrm{Hom}_P(A, \mathrm{Hom}_Q(B, C)) \longrightarrow \mathrm{Hom}_Q(A \otimes_P B, C)$$

such that for each $\varphi: A \longrightarrow \mathrm{Hom}_Q(B, C)$ we have $(s\,\varphi)(a \otimes b) = (\varphi\, a)b$. Similarly, in the situation $({}_PA_S, {}_QB_P, {}_QC_T)$, we have a left-$S$-and right-$T$-isomorphism

$$(2) \qquad t: \mathrm{Hom}_P(A, \mathrm{Hom}_Q(B, C)) \longrightarrow \mathrm{Hom}_Q(B \otimes_P A, C)$$

such that for each $\psi : A \longrightarrow \mathrm{Hom}_Q(B, C)$ we have $(t\psi)(b \otimes a) = (\psi a)b$. Occasionally we shall use these isomorphisms without any indication and identify the isomorphic groups each other.

Now, an algebra R over a field K is called a Frobenius algebra when it is of a finite rank, say r, over K, and has a unit element 1, and further the left and the right regular representations of R in K are equivalent. The last condition is equivalent to the fact that there exist "dual" bases $(a_1, \ldots\ldots, a_r)$, $(b_1, \ldots\ldots, b_r)$, for which the relation

$$(3) \qquad xa_i = \sum_k a_k x_{ki},$$

$$b_i x = \sum_k x_{ik} b_k$$

holds for any x in R where x_{ik} in K. This is also equivalent to the fact that R^0: $\mathrm{Hom}_K(R, K)$ is left-R-isomorphic to R, where we consider R^0 as a left-R-module by the convention

$$(4) \qquad (x\alpha)y = \alpha(yx), \quad \alpha \text{ in } R^0, x, y \text{ in } R.$$

Then, there exists an automorphism $x \longrightarrow x^*$ of R, uniquely determined by R up to inner automorphisms, and we have

$$(5) \qquad x^* b_i = \sum_k b_k \bar{x}_{ki}$$

along with

$$(6) \qquad a_i x = \sum_k \bar{x}_{ik} a_k,$$

for any x in R where $\bar{x}_{ik}$ in K.

We set for any bi-R-module A,

$$(7) \qquad A^R = [a \text{ in } A \mid xa = ax \text{ for all } x \text{ in } R],$$

$$(8) \qquad \mathrm{N}a = \sum_i a_i a b_i, \quad a \text{ in } A,$$

where (a_i), (b_i) are dual bases of R as described above. Evidently N is a K-endomorphism of A and will be called the norm-homomorphism. We note that Na is independent of the choice of the dual bases. Then, [5] asserts

LEMMA 1. *For every a in A and x in R we have*

$$(9) \qquad \mathrm{N}(ax^* - xa) = 0.$$

LEMMA 2. *We have*

(10) $$\mathrm{N}A \subseteq A^R.$$

For two bi-R-module A and A', we convert the group $\mathrm{Hom}_{K-R}(A, A')$ of all right-R-homomorphisms of A into A' to a bi-R-module by setting

(11) $$(xf)(a) = xf(a),\ (fx)(a) = f(xa),\ x \text{ in } R,\ a \text{ in } A,\ f \text{ in } \mathrm{Hom}_{K-R}.$$

Similarly we define the bi-R-module structure of the group $\mathrm{Hom}_{R-K}(A, A')$ of all left-R-homomorphisms of A into A' by

(12) $$(xg)(a) = g(ax),\ (gx)(a) = g(a)x,\ g \text{ in } \mathrm{Hom}_{R-K}.$$

Then $(\mathrm{Hom}_{K-R})^R = (\mathrm{Hom}_{R-K})^R = \mathrm{Hom}_{R-R}(A, A')$: the group of all bi-$R$-homomorphisms. In particular, in view of Lemma 2, the norm of a one-sided-R-homomorphism is a bi-R-homomorphism. If further, $f\colon A \longrightarrow A'$ is a bi-R-homomorphism, we have

(13) $$\mathrm{N}(f) = \Big(\sum_i a_i b_i\Big) f = f\Big(\sum_i a_i b_i\Big).$$

Consider a sequence of bi-R-modules

$$B \xrightarrow{f} C \xrightarrow{g} D \xrightarrow{h} E,$$

where f and h are bi-R- and g is one-sided-(say, right-)R-homomorphisms, then we have

(14) $$\mathrm{N}(hgf) = h(\mathrm{N}g)f.$$

The tensor product $R \otimes R$ over K is a K-module, and the multiplication

(15) $$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_2 y_1$$

converts $R \otimes R$ into a K-algebra, which is isomorphic with the enveloping algebra R^e of R. And then $(R \otimes R)$ is regarded as a bi-R-module and as a left-$(R \otimes R)$-module by

(16) $$x_1(x_2 \otimes y_2)y_1 = (x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_2 y_1,$$

respectively. Consider a map $\varphi\colon R \longrightarrow R \otimes R$ such that $\varphi(x) = x \otimes 1$, then φ is clearly a K-algebra homomorphism. We can, therefore, regard every bi-R-module A as a left-R-module induced by φ and define the so-called φ-covariant extension ${}_{(\varphi)}A\colon (R \otimes R) \otimes_R A$, which is isomorphic with the tensor product $R \otimes A$ over K. The bi-R-structure of $R \otimes A$ is, therefore, given by

(17) $$y(x \otimes a)z = yx \otimes az,\ a \text{ in } A,\ x,\ y,\ z \text{ in } R.$$

Hence, a map $g\colon R \otimes A \longrightarrow A$ given by

(18) $$g(x \otimes a) = xa$$

is clearly a bi-R-epimorphism.

DEFINITION 1. A bi-R-module A is called to be *weakly projective* if there exists a bi-R-homomorphism ν of A into $R \otimes A$ such that $g\nu$ is the identity of A.

Obviously a bi-R-projective module is weakly projective, while, if A is weakly projective, A is so-called φ-projective (i. e. relatively projective in the sense of Hochschild [3] with respect to φ.)

PROPOSITION 1. *For any bi-R-module A, $R \otimes A$ is weakly projective.*

PROOF. Our definition of "weakly projective" is different from that of "φ-projective", but the fact that "the covariant extension $R \otimes A = {}_{(\varphi)}A$ is φ-projective" is proved by pointing out the existence of the above homomorphism ν, (see Prop. 6. 3. in p. 31 of [1]), therefore, $R \otimes A$ is weakly projective. q. e. d.

Dually we shall consider the φ-contravariant extension ${}^{(\varphi)}A : \mathrm{Hom}_{R-K}(R \otimes R, A)$, which is, by t in (2), isomorphic with $\mathrm{Hom}_K(R, \mathrm{Hom}_R(R, A))$, that is, with $\mathrm{Hom}_K(R, A)$, of which bi-R-structure is given by

$$(19) \qquad (yfz)(x) = f(xy)z, \ f \text{ in } \mathrm{Hom}_K(R, \mathrm{A}), \ x, \ y, \ z \text{ in } R.$$

Then a map $h : A \longrightarrow \mathrm{Hom}_K(R, A)$ such that

$$(20) \qquad h(a)x = xa$$

is a bi-R-monomorphism, because A is unitary. It holds indeed

$$(21) \qquad h(ay)x = xay = (h(a)y)x,$$
$$h(ya)x = xya = h(a)xy = (yh(a))x, \ a \text{ in } A, \ x, \ y, \ z \text{ in } R.$$

DEFINITION 1a. A bi-R-module A is called to be *weakly injective* if there exists a bi-R-homorphism ρ of $\mathrm{Hom}_K(R, A)$ into A such that ρh is the identity of A. Evidently a bi-injective A is weakly injective. On the other hand, if A is weakly injective, then, A is so-called φ-injective (i. e. relatively injective in the sense of Hochschild [3]). Dually to the Prop. 1, we have

PROPOSITION 1a. *For any bi-R-module A,* $\mathrm{Hom}_K(R, A)$ *is weakly injective.*

Further, since the direct sum decomposition of modules yields that of functors, it is clear that

PROPOSITION 2. *A direct sum of bi-R-modules is weakly projective (weakly injective) if and only if each summand is weakly projective (weakly injective).*

From these preparations we obtain

PROPOSITION 3. *For each bi-R-module A the following properties are equivalent*:

(a) *The identity map of A is the norm of some right-R-endomorphism* ω *of A.*
(b) *A is weakly projective.*
(c) *A is weakly injective.*

PROOF. $(b) \longrightarrow (a)$. If A is weakly projective, there exists a bi-R-homomorphism $\nu : A \longrightarrow R \otimes A$ such that $g\nu(a) = a$, with $g(x \otimes a) = xa$. Since R has a finite K-basis, say (a_i), each element $\nu(a)$ can be written as a K-linear combination: $\nu(a) = \sum_i a_i \otimes g_i(a)$. Since for each x in R, $\nu(ax) = \sum_i a_i \otimes g_i(ax)$ and $\nu(a)x = \sum_i (a_i \otimes g_i(a))x = \sum_i a_i \otimes g_i(a)x$, the condition $\nu(ax) = \nu(a)x$ is equivalent to $g_i(ax) = g_i(a)x$, that is, g_i is an element of $\mathrm{Hom}_{K-R}(A, A)$ with the bi-R-module structures given in (11). On the other hand it holds that

$$\nu(xa) = \sum_i a_i \otimes g_i(xa) = \sum_i a_i \otimes (g_i x)(a), \tag{22}$$

and

$$x\nu(a) = \sum_i xa_i \otimes g_i(a) = \sum_{i,k} a_k x_{ki} \otimes g_i(a) = \sum_k a_k \otimes \Big(\sum_i x_{ki} g_i(a)\Big), \tag{23}$$

with x_{ki} in (3). The condition $\nu(xa) = x\nu(a)$ is, therefore, equivalent to

$$g_i x = \sum_k x_{ik} g_k. \tag{24}$$

From this and (3) we see that $b_i \longrightarrow g_i$ gives a right-R-homomorphism of R into $\mathrm{Hom}_{K-R}(A, A)$.

Consider ω: the image of 1 under this homomorphism. Since the image of $b_i = 1\ b_i$ is $g_i = \omega b_i$, we have

$$\begin{aligned} a = g\nu(a) &= g\Big(\sum_i a_i \otimes g_i(a)\Big) = g\Big(\sum_i a_i \otimes (\omega b_i)(a)\Big) \\ &= g\Big(\sum_i a_i \otimes \omega(b_i a)\Big) = \sum_i a_i \omega(b_i a) = (\mathrm{N}\omega)(a). \end{aligned} \tag{25}$$

$(a) \longrightarrow (b)$. If we define $g_i(a) = \omega(b_i a)$, and ν using (22), ν is a bi-R-homomorphism of A to $R \otimes A$ and $g\nu$ is the identity of A. Thus A is isomorphic with a direct summand of $R \otimes A$, which is weakly projective by Prop. 1, so that A is weakly projective by Prop. 2.

$(a) \longrightarrow (c)$. Assume $\mathrm{N}(\omega) =$ identity of A with some ω in $\mathrm{Hom}_{K-R}(A, A)$. For f in $\mathrm{Hom}_K(R, A)$ with the bi-R-structure (19), we consider a map $\mu : \mathrm{Hom}_K$

$(R, A) \longrightarrow A$ given by $\mu(f) = \sum_i a_i\omega(f(b_i))$, then μ is a bi-R-homomorphism. In fact, it is seen that

$$\mu(xf) = \sum_i a_i\omega((xf)(b_i)) = \sum_i a_i\omega(f(b_ix)) = \sum_{i,k} a_i\omega(f(x_{ik}b_k)) \tag{26}$$

$$= \sum_{i,k} a_ix_{ik}\omega(f(b_k)) = \sum_k xa_k\omega(f(b_k)) = x\mu(f),$$

$$\mu(fx) = \sum_i a_i\omega((fx)(b_i)) = \sum_i a_i\omega(f(b_i)x) \tag{27}$$

$$= \sum_i a_i\omega(f(b_i))x,$$

(because ω is a right-R-homomorphism), and $= (\mu f)x$.
If f is $h(a)$: $a \to xa$, a in A, x in R, then we have

$$\mu(f) = \sum_i a_i\omega(f(b_i)) = \sum_i a_i\omega(b_ia) = (\mathrm{N}\omega)(a) = a. \tag{28}$$

Thus $\mu(h(a)) = a$ and A is weakly injective.

$(c) \longrightarrow (a)$. Assume that A is weakly injective and μ be a bi-R-homomorphism of $\mathrm{Hom}_K(R, A)$ into A such that $\mu h(a) = a$ for $h(a)$: $a \longrightarrow xa$. Let further (ν_{ik}) be a non-singular $(r \times r)$-matrix in K such that

$$(b_1, \ldots\ldots, b_r)(\nu_{ik}) = (b_1', \ldots\ldots, b_r'),\ b_1' = 1, \tag{29}$$

with (b_i) in (3), and $(a_1', \ldots\ldots, a_r')$ be the basis of R given by

$$a_i = \sum_k \nu_{ik}a_k'. \tag{30}$$

Define f_a of $\mathrm{Hom}_K(R, A)$ for every a in A by setting

$$f_a(a_1') = a,\ f_a(a_i') = 0 \text{ for } i \neq 1. \tag{31}$$

Then $f_{ax}(a_1') = ax = (f_ax)(a_1')$, $f_{ax}(a_i') = 0 = (f_ax)(a_i')$ for $i \neq 1$, with bi-R-structure (19), i. e., $f_{ax} = f_ax$. Since μ is a bi-R-homomorphism, we have $\mu f_{ax} = \mu(f_ax) = (\mu f_a)x$. Accordingly if we put $\rho(a) = \mu(f_a)$, then $\rho(ax) = \mu(f_{ax}) = \mu(f_ax) = (\mu f_a)x = \rho(a)x$. From this we see that ρ is a right-R-endomorphism of A. With reference to the operation of R on $\mathrm{Hom}_{K-R}(A, A)$ as (11), we shall compute $\mathrm{N}(\rho)$:

$$(\mathrm{N}\rho)(a) = \sum_i (a_i\rho b_i)(a) = \sum_i a_i\rho(b_ia) = \sum_i a_i\mu(f_{b_ia}) = \mu\Big(\sum_i a_if_{b_ia}\Big). \tag{32}$$

On the other hand $\sum_i (a_if_{b_ia})(x) = xa = h(a)(x)$ for all x in R as seen below.

Consequently we have $(\mathrm{N}\rho)(a) = a$. Now we must show that $\sum_i (a_i f_{b_i a})(x) = xa$.

$$(33) \qquad \sum_i (a_i f_{b_i a})(x) = \sum_i f_{b_i a}(x a_i)$$
$$= \sum_i f_{b_i a}\Big(\sum_k a_k x_{ki}\Big) = \sum_{i,k} x_{ki} f_{b_i a}\Big(\sum_t \nu_{kt} a_t'\Big)$$
$$= \sum_{i,k,t} x_{ik} f_{b_i a}(a_t')\nu_{kt} = \sum_{i,k} x_{ki} f_{b_i a}(a_1')\nu_{k1}$$
$$= \sum_{i,k} x_{ki} b_i a \nu_{k1} = \sum_k \nu_{k_1} b_k x_a = 1\, xa = xa.$$

Thus we have proved all our assertions. q. e. d.

PROPOSITION 4. *In order that a bi-R-homomorphism f of* $\mathrm{Hom}_{R-R}(A, A')$ *be the norm of an element u of* $\mathrm{Hom}_{K-R}(A, A')$ *it is necessary and sufficient that f admits a factorization*

$$(34) \qquad A \xrightarrow{h} R \otimes A' \xrightarrow{g} A'$$

where g and h are bi-R-homomorphisms, and $R \otimes A'$ is a tensor product over K.

PROOF. Suppose $f = gh$. Since $R \otimes A'$ is weakly projective by Prop. 1 there is a right-R-endomorphism ω of $R \otimes A'$ such that $\mathrm{N}\omega =$ identity of $R \otimes A'$. Then $f = gh = g(\mathrm{N}\omega)h = \mathrm{N}(g\omega h)$, by (14).

Conversely assume $f = \mathrm{N}u$ for some u in $\mathrm{Hom}_{K-R}(A, A')$. Define $g : R \otimes A' \longrightarrow A'$ such that $g(a_i \otimes a') = a_i a'$, then g is clearly bi-R-homomorphism. And define $h : A \longrightarrow R \otimes A'$ such that $h(a) = \sum_i a_i \otimes u(b_i a)$, then h is also a bi-R-homomorphism, because it holds that

$$(35) \qquad h(xa) = \sum_i a_i \otimes u(b_i xa) = \sum_i a_i \otimes u(x_{ik} b_k a) = \sum_i a_i x_{ik} \otimes u(b_k a)$$
$$= \sum_k x a_k \otimes u(b_k a) = xh(a),$$

$$(36) \qquad h(ax) = \sum_i a_i \otimes u(b_i ax) = \sum_i a_i \otimes u(b_i a)x = h(a)x.$$

We have then $gha = \sum_i a_i u(b_i a) = (\mathrm{N}u)(a) = f(a)$.

PROPOSITION 5. *If A is weakly projective, then*

$$\mathrm{Ker}\ \mathrm{N} = J^* A, \quad \mathrm{Im}\ \mathrm{N} = A^R$$

where J^*A is the bi-R-submodule generated by $(ax^* - xa)$, with arbitrary a in A, x in R. Thus A_{R^*}: (A/J^*A) is isomorphic with A^R.

PROOF. There exists a right-R-endomorphism ω of A such that $a = \sum_i a_i \omega(b_i a)$ for all a in A. By the automorphism $x \longrightarrow x^*$ of R, (6) is changed to

$$a_i^* x^* = \sum_k \bar{x}_{ik} a_k^*. \tag{37}$$

From this and (5), we see that $(b_1, \ldots\ldots, b_r)$, $(a_1^*, \ldots\ldots, a_r^*)$ are the dual bases of R as (a_i), (b_i) in (3), and then, $\mathrm{N}a = 0$ means that $\sum_i b_i a a_i^* = 0$. Then we have

$$a = \sum a_i \omega(b_i a) - \omega\Big(\sum_i b_i a a_i^*\Big) = \sum_i (a_i \omega(b_i a) - \omega(b_i a) a_i^*) \text{ in } J^*A. \tag{38}$$

Thus in view of Lemma 1 we obtain the first half part of the proposition. Next suppose a in A^R. Then

$$a = \sum_i a_i \omega(b_i a) = \sum_i a_i \omega(a b_i) = \sum_i a_i \omega(a) b_i = \mathrm{N}\omega(a). \tag{39}$$

Together with Lemma 2 we have also proved the second part.

PROPOSITION 6. *Let C be a bi-R-module, and A be a weakly projective bi-R-module. Then $A \otimes_R C$ is weakly projective and* $\mathrm{Hom}_{R-K}(A, C)$ *is weakly injective.*

PROOF. These are easily seen from the commutative diagrams:

$$\begin{tikzcd}
 & A \otimes_R C \arrow[dl] \arrow[dr, "\nu \otimes_R C"] & \\
R \otimes_K (A \otimes_R C) \arrow[dr, "g"'] & & (R \otimes_K A) \otimes_R C \arrow[ll] \arrow[dl, "g \otimes_R C"] \\
 & A \otimes_R C &
\end{tikzcd} \tag{40}$$
,

$$\begin{tikzcd}
 & \mathrm{Hom}_{R-K}(A, C) & \\
\mathrm{Hom}_{R-K}(R \otimes_K A, C) \arrow[ur, "\mathrm{Hom}_{R-K}(\nu, C)"] & & \mathrm{Hom}_K(A, \mathrm{Hom}_{R-K}(A, C)) \arrow[ul] \arrow[ll] \\
 & \mathrm{Hom}_{R-K}(A, C) \arrow[ul, "\mathrm{Hom}_{R-K}(g, C)"] \arrow[ur, "h"'] &
\end{tikzcd}$$

Now on a bi-R-module we define the new right-R-module structure induced by the isomorphism $x^* \longrightarrow x$ from the original. We shall denote the bi-R-module A thus obtained by A^*. We consider Hochschild groups for $n \geqq 0$, with some modification:

$$(41) \qquad H^*_n(R, A^*) = \mathrm{Tor}_n^{R^e}(R, A^*), \quad \mathrm{H}^n(R, A) = \mathrm{Ext}^n_{R^e}(R, A).$$

(Usually $\mathrm{H}_n(R, A) = \mathrm{Tor}_n^{R^e}(A, R)$, however, we have modified as above, in accordance with [5]).

Let N, H_0^*, H^0 be the covariant functors given by (8), the above H_0^*, H^0 respectively. Just as by the case of finite groups (p. 235 of [1]), in view of Lemmas 1 and 2, for the diagram with natural transformations of functors

$$(42) \qquad \begin{tikzcd} & \mathrm{N} \arrow[dr, "h"] & \\ \mathrm{H}_0^* \arrow[ur, "g"] \arrow[rr, "\mathrm{N}^*"'] & & \mathrm{H}^0 \end{tikzcd},$$

the derived sequences DH_0^*, DH^0, Dh, Dg and DN^* form a commutative diagram

$$(43) \qquad \begin{tikzcd} \mathrm{D}g \arrow[r] \arrow[d] & \mathrm{DN} \arrow[r] & \mathrm{D}h \arrow[d] \\ \mathrm{DH}_0^* \arrow[r] & \mathrm{DN}^* \arrow[r] & \mathrm{DH}^0 \end{tikzcd}.$$

Since all the maps in (45) are isomorphisms, we identify the above derived sequences into one sequence and denote by $\hat{\mathrm{H}}$. Thus we have Nakayama's groups [5]:

$$(44) \qquad \begin{aligned} &\hat{\mathrm{H}}^n(R, A) = \mathrm{H}^n(R, A), && n > 0, \\ &\hat{\mathrm{H}}^0(R, A) = \mathrm{Coker}(\mathrm{H}_0^* \longrightarrow \mathrm{H}^0) = \mathrm{Coker}(\mathrm{N} \longrightarrow \mathrm{H}^0) = A^R/\mathrm{N}A, && \\ &\hat{\mathrm{H}}^{-1}(R, A) = \mathrm{Ker}(\mathrm{H}_0^* \longrightarrow \mathrm{H}^0) = \mathrm{Ker}(\mathrm{H}_0^* \longrightarrow \mathrm{N}) = \mathrm{Ker}\ \mathrm{N}/J^*A, && \\ &\hat{\mathrm{H}}^n(R, A) = H^*_{-n-1}(R, A), && n < -1. \end{aligned}$$

The graded functor $\hat{\mathrm{H}}$ of bi-R-modules is an exact connected sequence of covariant functors of A, i. e., for any exact sequence $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ of bi-R-modules we have an exact sequence

$$(45) \quad \cdots \rightarrow \hat{\mathrm{H}}^{n-1}(R, A'') \rightarrow \hat{\mathrm{H}}^n(R, A') \rightarrow \hat{\mathrm{H}}^n(R, A) \rightarrow \hat{\mathrm{H}}^n(R, A'') \rightarrow \hat{\mathrm{H}}^{n+1}(R, A') \rightarrow \cdots$$

PROPOSITION 7. *If A is weakly projective (weakly injective) then* $\hat{\mathrm{H}}(R, A) = 0$.

PROOF. Since K is a field, R^e is one-sided-R-projective, therefore, it follows from Cor. 4. 2. 1. in p. 117 of [1] that $\mathrm{Tor}_n^{R^e}(R, A) = 0$ for $n > 0$ whenever A is weakly projective. Therefore $\mathrm{Tor}_n^{R^e}(R, A^*) = 0$ for $n > 0$, because Tor (, A^*) is clearly isomorphic with Tor (, A). Now that A is also weakly injective, it follows from Cor. 4. 2. 4. in p. 118 of [1] that $\mathrm{Ext}_n^{R^e}(R, A) = 0$ for $n > 0$. Finally $\mathrm{H}^0(R, \mathrm{A}) = \mathrm{H}^{-1}(R, A) = 0$ follows from Prop. 5. q. e. d.

Given a bi-R-homomorphism $f\colon A \longrightarrow C$ we shall denote $\hat{f}$ the induced homomorphism $\hat{\mathrm{H}}(R, A) \longrightarrow \hat{\mathrm{H}}(R, \mathrm{C})$, then we have

PROPOTITION 8. *If $f\colon A \longrightarrow C$ is the norm of an element of* Hom_{K-R} (A, C) *with bi-R-module structures* (11), *then* $\hat{f} = 0$.

PROOF. It follows from Prop. 4. that f admits a factorization $A \longrightarrow R \otimes C \longrightarrow C$. Since $R \otimes C$ is weakly projective from Prop. 1, we have $\hat{\mathrm{H}}(R, R \otimes C) = 0$, and $\hat{f} = 0$.

PROPOSITION 9. *For any bi-R-module A,* $\hat{\mathrm{H}}\left(R, \sum_i a_i b_i A\right) = 0$.

PROOF. The map $\left(a \longrightarrow \sum_i a_i b_i a\right)$ is the norm of the identity map of A, which is now considered as a right-R-homomorphism. Thus the induced homomorphism $\hat{\mathrm{H}}(R, A) \longrightarrow \hat{\mathrm{H}}\left(R, \sum_i a_i b_i A\right)$ is 0 from Prop. 8.

2. Complete resolutions

$$\sigma : C \otimes_R A \longrightarrow \mathrm{Hom}_{R-K}(\mathrm{Hom}_{K-R}(C, R), A) \tag{46}$$

given by

$$\sigma(c \otimes_R a)f = f(c)a, \quad f \text{ in } \mathrm{Hom}_{K-R}(C, R), \tag{47}$$

is a bi-R-homomorphism, whenever we define bi-R-structures on Hom_{K-R}, Hom_{R-K} as (11), (12), and on $C \otimes_R A$ by

$$x(c \otimes_R a)y = xc \otimes_R ay, \quad x.y \text{ in } R, \tag{48}$$

respectively. Indeed it holds that, omitting R in $\otimes_R$,

$$\sigma[x(c\otimes a)]f = \sigma(xc\otimes a)f = f(xc)a = (fx)(c)a = \sigma(c\otimes a)(fx) = [x\sigma(ca)]f, \tag{49}$$

$$\sigma[(c\otimes a)x]f = \sigma(c\otimes ax)f = f(c)ax = [\sigma(c\otimes a)f]x = [\sigma(c\otimes a)x]f. \tag{50}$$

We note that $(C \otimes_R A)_R$ is isomorphic with $C \otimes_{R\otimes R} A$ by mapping $xc \otimes ya$ to $cy \otimes ax$.

For convenience we shall introduce

DEFINITION 2. A bi-R-module A is called to be *a bi-PF-module, r-PF, l-PF,* if A is projective, and finitely generated as a bi-R-module, a right-R-module, a left-R-module respectively.

Now if C is r-PF, then σ is a bi-R-isomorphism. In fact C is then a direct summand of a free right-R-module on a finite base, so that by the direct sum argument it suffices to show this when C is R itself, but it is trivial. In particular, taking $A = R$, we have for each r-PF-module C an isomorphism

$$(51) \qquad \sigma_0 : C \longrightarrow \mathrm{Hom}_{R-K}(\mathrm{Hom}_{K-R}(C, R), R),$$

defined by $\sigma_0(c) = f(c)$ for f in $\mathrm{Hom}_{K-R}(C, R)$, c in C.

Similarly the map

$$(52) \qquad \tau : A \otimes_R C \longrightarrow \mathrm{Hom}_{K-R}(\mathrm{Hom}_{R-K}(C, R), A)$$

given by

$$(35) \qquad \tau(a \otimes_R c)f = af(c), \ f \text{ in } \mathrm{Hom}_{R-K}(C, R),$$

is a bi-R-homomorphism whenever we define bi-R-structures on the modules as on σ. Because we have, omitting R in $\otimes_R$,

$$(54) \qquad \tau[x(a\otimes c)]f = \tau(xa\otimes c)f = xaf(c) = x(\tau(a\otimes c)f) = [\tau(a\otimes c)]f,$$

$$(55) \qquad \tau[(a\otimes c)x]f = \tau(a\otimes cx)f = af(cx) = a(xf)(c) = \tau(a\otimes c)(xf) = [\tau(a\otimes c)x]f.$$

Moreover if C is l-PF-module τ is a bi-R-isomorphism as σ was. In particular, taking $A = R$, we have, for each l-PF-module C the isomorphism

$$(56) \qquad \tau_0 : C \longrightarrow \mathrm{Hom}_{K-R}(\mathrm{Hom}_{R-K}(C, R), R)$$

defined by

$$(57) \qquad \tau_0(c)f = f(c), \ f \text{ in } \mathrm{Hom}_{R-K}(C, R), \ c \text{ in } C.$$

PROPOSITION 10. *If C is a bi-PF-module,* $\mathrm{Hom}_{R-K}(C, R)$ *is also a bi-PF-module. The same holds for* $\mathrm{Hom}_{K-R}(C, R)$.

PROOF. We may show only the first. Since K is the field, $R \otimes_K R$ is a bi-PF-module. By s in (1) $\mathrm{Hom}_{R-K}(R \otimes R, R)$ is bi-R-isomorphic with $\mathrm{Hom}_K(R, R)$. As a consequence there exists a bi-R-isomorphism

$$(58) \qquad \alpha : \mathrm{Hom}_K(R, R) \longrightarrow R \otimes_K R.$$

Thus our assertion is true for $C = R \otimes_K R$. If C is bi-PF, then C is considered as a direct summand of a free bi-R-module, i. e., $R \otimes_K R$-module on a finite base. Since the functors involved are additive, our assertion holds good also for general C.

Now we must show that (58) is isomorphism. In the situation $({}_K R, {}_K R_K, {}_K K)$ there exists by (2) an isomorphism

$$(59)\qquad t: \mathrm{Hom}_K(R, \mathrm{Hom}_K(R, K)) \longrightarrow \mathrm{Hom}_K(R \otimes_K R, K)$$

such that for each $\psi: R \longrightarrow \mathrm{Hom}_K(R, K)$ we have $(t\psi)(b \otimes a) = (\psi a)b$. Since R and $R \otimes_K R$ are Frobeniusean, $\mathrm{Hom}_K(R, K)$ is left-R-isomorphic with R with operators (4), and $\mathrm{Hom}_K(R \otimes_K R, K)$ is left-$R \otimes_K R$-isomorphic with $R \otimes_K R$ with bi-R-operators (4), (16), that is, $R \otimes_K R$ is bi-R-isomorphic with $\mathrm{Hom}_K(R \otimes_K R, K)$, whose bi-$R$-structures are given by

$$(60)\qquad (xf)(b \otimes a) = f(bx \otimes a),\ (fx)(b \otimes a) = f(b \otimes xa),$$

where f in $\mathrm{Hom}(\dots)$, x, a, b in R. Finally, $\mathrm{Hom}_K(R, \mathrm{Hom}_K(R, K))$ is considered as a bi-R-module with operators given by

$$(61)\qquad (x\psi)a = x(\psi a),\quad (\psi x)a = \psi(xa).$$

From these, t is a bi-R-isomorphism. Indeed, it results that

$$(62)\quad (t(x\psi))(b\otimes a) = (x\psi)(a)b = (x(\psi a))b = (\psi a)bx = (t\psi)(bx\otimes a) = (xt\psi)(b\otimes a),$$

$$(63)\quad (t(\psi x))(b\otimes a) = (\psi x)(a)b = \psi(xa)b = (t\psi)(b\otimes xa) = (t\psi x)(b\otimes a).$$

Consequently, α is also a bi-R-isomorphism, and the proof is complete. q. e. d.

Since R is always K-projective, $R \otimes R$ is a PF-module as a one-sided-R-module. Again by the direct sum argument, if C is bi-PF, then C and $\mathrm{Hom}_{R-K}(C, R)$ are PF-modules as a one-sided-R-modules. Accordingly even if we replace C of (46) by $\mathrm{Hom}_{R-K}(C, R)$, the bi-R-isomorphisms σ and τ_0 hold. Thus there exists a bi-R-isomorphism

$$(64)\qquad \mathrm{Hom}_{R-K}(C, R) \otimes_R A \longrightarrow \mathrm{Hom}_{R-K}(C, A).$$

On the other hand, for each bi-R-projective C, Prop. 6 shows that $\mathrm{Hom}_{R-K}(C, R)$ is weakly injective (weakly projective), and that $\mathrm{Hom}_{R-K}(C, R) \otimes_R A$ is also weakly projective. It follows from Prop. 5 that there exists an isomorphism

$$(65)\qquad (\mathrm{Hom}_{R-K}(C, R) \otimes_R A)_{R^*} \longrightarrow (\mathrm{Hom}_{R-K}(C, A))^R,$$

where the first group is isomorphic with $\mathrm{Hom}_{R-K}(C, R) \otimes_{R\otimes R} A^*$, and the second group is isomorphic with $\mathrm{Hom}_{R-R}(C, A)$. In this manner we have finally obtained an isomorphism

$$(66)\qquad \rho: \mathrm{Hom}_{R-K}(C, R) \otimes_{R\otimes R} A^* \longrightarrow \mathrm{Hom}_{R-R}(C, A)$$

such that

$$(67)\qquad \rho(f \otimes a)c = \sum_i f(ca_i)ab_i,$$

for every f in $\mathrm{Hom}_{R-K}(C, R)$, where (a_i), (b_i) are dual bases.

Now that K is a field, a left acyclic bi-R-complex over R with an augmentation ε

$$(X_L) \qquad \cdots\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \longrightarrow \cdots\cdots \longrightarrow X_0 \xrightarrow{\varepsilon} R$$

is exact and may be regarded as a K-projective resolution of the zero module. It follows that the identity map of (X_L) is homotopic with the zero map. This yields for (X_L) a contract homotopy, that is, a sequence of K-homomorphisms $s_n: X_n \longrightarrow X_{n+1}$ and $\sigma: R \longrightarrow Y_0$ such that

$$(68) \qquad d_{n+1}s_nx + s_{n-1}d_nx = x - \sigma\varepsilon x, \text{ for } x, \text{ for } x \text{ in } X_n.$$
$$\varepsilon\sigma x = x.$$

Then the sequence

$$(X_L^0) \qquad R \xrightarrow{\varepsilon^0} X_0^0 \longrightarrow \cdots\cdots \longrightarrow X_n^0 \xrightarrow{d_n^0} X_{n+1}^0 \longrightarrow \cdots\cdots$$

where $X_n^0 = \mathrm{Hom}_{R-K}(X_n, R)$, $d_n^0 = \mathrm{Hom}_{R-K}(d_{n+1}, R)$, is a right acyclic bi-$R$-complex over R with the augmentation $\varepsilon^0 = \mathrm{Hom}_{R-K}(\varepsilon, R)$, because $[\mathrm{Hom}_{R-K}(\sigma, R), \mathrm{Hom}_{R-K}(s_n, R)]$ yields a contract homotopy for (X_L^0). Similarly from a right acyclic bi-R-complex over R with augmentation μ

$$(X_R) \qquad R \xrightarrow{\mu} X_{-1} \longrightarrow \cdots\cdots \longrightarrow X_{-n+1} \longrightarrow X_{-n} \longrightarrow \cdots\cdots$$

we have a left acyclic bi-R-complex over R

$$(X_R^0) \qquad \cdots\cdots \longrightarrow X_{-n}^0 \longrightarrow X_{-n+1}^0 \longrightarrow \cdots\cdots \longrightarrow X_{-1}^0 \longrightarrow R \longrightarrow 0,$$

where $X_n^0 = \mathrm{Hom}_{R-K}(X_{-n}, R)$, and if X_{-n} is a bi-PF-module, so is X_{-n}^0.

DEFINITION 3. A complete resolution for a Frobenius algebra R is an exact sequence

$$(X) \qquad \cdots \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots\cdots \longrightarrow X_0 \xrightarrow{d_0} X_{-1} \longrightarrow \cdots\cdots \longrightarrow X_{-n} \longrightarrow \cdots$$

of bi-PF-modules together with an element e in $(X_{-1})^R$ such that the image of d_0 is generated by e. Further since X_{-1} is bi-PF-module, much more l-PF, X_{-1} is a direct summand of a left-R-free module on a finite base. We see therefore that $xe \neq 0$ for x in R, $x \neq 0$. The map d_0 then admits a factorization

$$(69) \qquad X_0 \xrightarrow{\varepsilon} R \xrightarrow{\mu} X_{-1},$$

where ε is a bi-R-epimorphism while μ is a bi-R-monomorphism given by $\mu 1 = e$.

We consider the exact sequences

$$(X_L) \qquad \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots\cdots \longrightarrow X_0 \xrightarrow{\varepsilon} R \longrightarrow 0,$$

$$(X_R) \qquad 0 \longrightarrow R \xrightarrow{\mu} X_{-1} \longrightarrow \cdots\cdots \longrightarrow X_{-n+1} \longrightarrow X_{-n} \longrightarrow \cdots\cdots$$

The sequence (X_L) provides a projective resolution of R by means of bi-PF-module. From Prop. 10 and the facts before Def. 3 we see that the sequence

$$(X_R^0) \qquad \cdots\cdots \longrightarrow X_{-n}^0 \longrightarrow X_{-n+1}^0 \longrightarrow \cdots\cdots \longrightarrow X_{-1}^0 \longrightarrow R \longrightarrow 0$$

provides also a projective resolution of R by means of bi-PF-modules.

Conversely, given two resolution (X_L) and (X_L') of R by bi-PF-modules, we can construct a complete resolution combining (X_L) with the $(X_L'^0)$ suitable renumbered.

Given a complex X and a bi-R-module A, consider the complex

$$(70) \qquad \mathrm{Hom}_{R-R}(X, A).$$

For $n \geqq 0$ we leave the group $\mathrm{H}_{R-R}(X_n, A)$ as it is, and $n < 0$ we replace $\mathrm{Hom}_{R-R}(X_n, A)$ by the isomorphic group $X_n^0 \otimes_{R\otimes R} A^*$ using the isomorphism ρ in (66).

In view of the factorization (69) of d_0, we have a commutative diagram

$$(71) \qquad \begin{array}{ccccc} X_{-1}^0 \otimes_{R\otimes R} A^* & \xrightarrow{\sigma} & \mathrm{Hom}_{R-R}(X_{+1}, A) & \longrightarrow & \mathrm{Hom}_{R-R}(X_0, A) \\ \downarrow & & \downarrow & \nearrow & \\ R \otimes_{R\otimes R} A^* & \xrightarrow{\sigma} & \mathrm{Hom}_{R-R}(R, A). & & \end{array}$$

Thus (70) admits the factorization

$$(73) \qquad \cdots \rightarrow X_{-1}^0 \otimes_{R\otimes R} A^* \rightarrow \mathrm{H}_0(R^e, A^*) \xrightarrow{\mathrm{N}^*} \mathrm{H}_0(R^e, A) \rightarrow \mathrm{Hom}_{R-R}(X_0, A) \rightarrow \cdots.$$

Denoting this complex again by $\mathrm{Hom}_{R-R}(X, A)$, we obtain

THEOREM 1. *For any bi-R-module A, the group $\hat{\mathrm{H}}^n(R, A)$ may be computed as $\mathrm{H}^n(\mathrm{Hom}_{R-R}(X, A))$, where X is any complete resolution of R. If $f\colon A \longrightarrow A'$ is a bi-R-homomorphism then $\hat{f}$ may be computed from $\mathrm{Hom}_{R-R}(X, A) \longrightarrow \mathrm{Hom}_{R-R}(X, A')$. If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence of bi-R-modules, then the connecting homomorphisms $\hat{\mathrm{H}}(R, A'') \longrightarrow \hat{\mathrm{H}}(R, A')$ may be computed from the exact sequence*

$$(74) \qquad 0 \longrightarrow \mathrm{Hom}_{R-R}(X, A') \longrightarrow \mathrm{Hom}_{R-R}(X, A) \longrightarrow \mathrm{Hom}_{R-R}(X, A'') \longrightarrow 0.$$

For example, the complete resolution obtained by taking the standard complex $S(R)$ as (X_L) and its dual $(S(R))^0$ as (X_R), yields the complex $\mathrm{Hom}_{R-R}(X, A) = [(S(R))^0 \otimes_{R\otimes R} A^*, \mathrm{N}^*, \mathrm{Hom}_{R-R}(S(R), A)]$ as denoted in (73), which may be regarded as $[S(R) \otimes_K A^*, \mathrm{N}^*, \mathrm{Hom}_K(S(R), A)]$, where $S(R) = R \otimes_K S(R) \otimes_K R$, by appropriate isomorphisms. (In this form $\hat{\mathrm{H}}$ was also defined in [5].)

The proof is obtained by applying Prop. 10. 4. in p. 103 of [1].

3. Products for Frobenius Algebras. Now that complete resolutions for our algebras have been obtained, we can develop such a product theory of complete cohomology groups for our algebras as for finite groups in the almost same way. Namely for two bi-R-modules A and A' we consider the tensor product over R with usual R-operators, i. e., $x(a \otimes a')y = xa \otimes a'y$, x, y in R.

If a in A^R, a' in A'^R then, with omitting R in $\otimes_R$, $x(a \otimes a') = xa \otimes a' = a \otimes xa' = (a \otimes a')x$ and therefore $a \otimes a'$ in $(A \otimes_R A')^R$. If a in A and $a' = \mathrm{N}b'$ with b' in A', then $a \otimes a' = a \otimes \sum_i a_i b' b_i = \sum_i aa_i \otimes b' b_i = \sum_i a_i (a \otimes b') b_i = \mathrm{N}(a \otimes b')$ so that $a \otimes a'$ belongs to $\mathrm{N}(A \otimes_R A')$. Similarly if a in $\mathrm{N}A$, a' in A', $a \otimes a'$ in $\mathrm{N}(A \otimes_R A')$. Thus there exists a homomorphism

$$\xi : (A^R/\mathrm{N}A) \otimes_K (A'^R/\mathrm{N}A') \longrightarrow (A \otimes_R A')^R/\mathrm{N}(A \otimes_R A'), \tag{75}$$

or

$$\xi : \mathrm{H}^0(R, A) \otimes_K H^0(R, A') \longrightarrow \mathrm{H}^0(R, A \otimes_R A'). \tag{76}$$

THEOREM 2. *There is a unique family of K-homomorphisms*

$$\xi^{p,q} : \mathrm{H}^p(R, A) \otimes_K \mathrm{H}^q(R, A') \longrightarrow \mathrm{H}^{p+q}(R, A \otimes_B A') \tag{77}$$

defined for each pair of bi-R-modules A and A' and all integers p, q such that $\xi^{0,0}$ coincides with ξ and $\xi^{p,q}$ commutes with the connecting homomorphisms with respective to the variables A and A'. For instance with respect to A, let

$$0 \longrightarrow A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow 0 \tag{78}$$

be an exact sequence of bi-R-modules such that the sequence

$$0 \longrightarrow A_1 \otimes_R A' \longrightarrow A \otimes_R A' \longrightarrow A_2 \otimes_R A' \longrightarrow 0 \tag{79}$$

is exact. Then the diagram

$$\begin{array}{ccc} \hat{\mathrm{H}}^p(R, A_2) \otimes_K \hat{\mathrm{H}}^q(R, A') & \xrightarrow{\xi^{p,q}} & \hat{\mathrm{H}}^{p+q}(R, A_2 \otimes_R A') \\ \downarrow{\scriptstyle \delta \otimes i} & & \downarrow{\scriptstyle \Delta} \\ \hat{\mathrm{H}}^{p+1}(R, A_1) \otimes_K \hat{\mathrm{H}}^q(R, A') & \xrightarrow{\xi^{p,q}} & \hat{\mathrm{H}}^{p+q+1}(R, A_1 \otimes_R A') \end{array} \tag{80}$$

is commutative. Here δ is the connecting homomorphism relative to (78), *Δ is that of* (79) *and i is the appropriate identity map.*

To prove this it is sufficient to replace Z by R, $\otimes_Z$ by $\otimes_R$(but in $H \otimes H$,

$\otimes_Z$ by $\otimes_K$), $Z(\Pi)$ by $R\otimes R$ in the proof of Theorem 4. 1 in p. 242 of [1], except for the uniqueness property. Let A, $A_1,\ldots\ldots, A_n$, B, C etc. be bi-R-modules. Let $\mathrm{U}_1,\ldots\ldots, \mathrm{U}_k$, V each represent an exact connected sequence of covariant functors of A. A map

$$F: \mathrm{U}_1 \otimes_K \ldots\ldots \otimes_K \mathrm{U}_k \longrightarrow \mathrm{V} \tag{81}$$

is a family of homomorphisms

$$F: \mathrm{U}_1^{i_1}(A_1) \otimes_K \ldots\ldots \otimes_K \mathrm{U}_k^{i_k}(A_k) \longrightarrow \mathrm{U}^{i_1+\cdots+i_k}(A_1 \otimes_R \ldots\ldots \otimes_R A_k) \tag{82}$$

which is natural relative to bi-R-homomorphisms of the variables $A_1,\ldots\ldots, A_k$ and which commutes with the connecting homomorphisms in the following sense:

If $0 \longrightarrow A_j' \longrightarrow A_j \longrightarrow A_j'' \longrightarrow 0$ is an exact sequence of bi-R-modules which splits as a one-sided-(say left) R-sequence, then the diagram

$$\begin{array}{ccc} \mathrm{U}_1(A_1)\otimes_K\cdots\otimes_K\mathrm{U}_j(A_j'')\otimes_K\cdots\otimes_K\mathrm{U}_k(A_k) & \longrightarrow & \mathrm{V}(A_1\otimes_R\cdots\otimes_R A_j''\otimes_R\cdots\otimes_R A_k) \\ \downarrow & & \downarrow \\ \mathrm{U}_1(A_1)\otimes_K\cdots\otimes_K\mathrm{U}_j(A_j')\otimes_K\cdots\otimes_K\mathrm{U}_k(A_k) & \longrightarrow & \mathrm{V}(A_1\otimes_R\cdots\otimes_R A_j'\otimes_R\cdots\otimes_R A_k) \end{array} \tag{83}$$

is commutative.

THEOREM 3. (*Uniqueness theorem*) *Assume that the functors* $\mathrm{U}_1,\ldots\ldots, \mathrm{U}_k$, V *satisfy*

$$\mathrm{U}_i(\mathrm{Hom}_K(R, A)) = 0, \quad \mathrm{V}(R \otimes_K A) = 0 \tag{84}$$

for any bi-R-module A. If

$$F, G: \mathrm{U}_1 \otimes_K \ldots\ldots \otimes_K \mathrm{U}_k \longrightarrow \mathrm{V} \tag{85}$$

are two maps that F and G coincide on $\mathrm{U}_1^0 \otimes_K \ldots\ldots \otimes_K \mathrm{U}_K^0$, *then* $F = G$. *Here* $R\otimes_K A$ *and* $\mathrm{Hom}_K(R, A)$ *are treated as bi-R-modules with operators*

$$y(x \otimes a)z = yx \otimes az, \ (yfz)x = f(xy)z. \tag{86}$$

This may be proved again by the replacement as Theorem 2 in the proof of Theorem 5. 1 in p. 245 of [1].

The uniqueness of the products asserted in Theorem 2 follows readily by taking $\mathrm{U}_1 = \mathrm{U}_2 = \mathrm{V} = \mathrm{H}$. Actually, $\mathrm{Hom}_K(Q, A)$, $R\otimes_K A$ are always weakly projective and weakly injective by Prop. 1 and la, and it follows from Prop. 7 that the assumption of above theorem is satisfied.

For a in $\mathrm{H}^p(R, A)$, b in $\mathrm{H}^q(R, A')$ we shall denote the product $\xi^{p,q}(a\otimes_K b)$ of $H^{p+q}(R, A\otimes A')$ by the symbol ab.

PROPOSITION 11. *If R is a commutative Frobenius algebra, the elements ab of* $\mathrm{H}^{p+q}(R, A\otimes_R A')$ *and* $(-1)^{pq}$ *ba of* $\mathrm{H}^{p+q}(R, A'\otimes_R A)$ *correspond to each other under the natural isomorphim* $A\otimes_R A' \longrightarrow A'\otimes_R A$.

PPOPOSITION 12. *For a in* $\mathrm{H}^p(R, A)$, *b in* $\mathrm{H}^q(R, A')$, *c in* $\mathrm{H}^r(R, A'')$ *we have* $a(bc) = (ab)c$, *if we identify* $a \otimes (b \otimes c) = (a \otimes b) \otimes c$.

In the group $\mathrm{H}^0(R, R) = C(R)/\sum_i a_i R b_i$, where $C(R)$ is the center of R, the coset containing unit element 1 denote again by 1.

PROPOSITION 13. *If a in* $\mathrm{H}^p(R, A)$ *then* $a = 1a = a1$ *if we identify* R, $R \otimes_R A$, $A \otimes_R R$ *each other.*

These can be obtained from the uniqueness theorem. For example, in the proof of Prop. 12, put $\mathrm{U}_1 = \mathrm{U}_2 = \mathrm{U}_3 = \mathrm{V} = \mathrm{H}$ and $F(a \otimes b \otimes c) = a(bc)$, $G(a \otimes b \otimes c) = (ab)c$.

4. Duality. Now we can use the bi-R-homomorphism

(87) $$\varphi : \mathrm{Hom}_{K-R}(A, C) \otimes_R A \longrightarrow C, \quad ({}_R A_R, {}_R C_R)$$

given by $f \otimes_R a$ to obtain the modified product

(88) $$\hat{\mathrm{H}}(R, \mathrm{Hom}_{K-R}(A, C)) \otimes_K \hat{\mathrm{H}}(R, A) \longrightarrow \hat{\mathrm{H}}(R, C),$$

where R operates on $\mathrm{Hom}_{K-R}(A, C)$ as $(xfy)(a) = xf(yx)$ and on $\mathrm{Hom}_{K-R}(A, C) \otimes_R A$ as $x(f \otimes a)y = (xf) \otimes ay$. (As for the modified products, see (6), (17) in [8].) We shall still use the symbol fa to denote the image of $f \otimes a$ under (88).

PROPOSITION 14. *Let* $0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{j} A'' \longrightarrow 0$ *be an exact sequence of bi-R-modules such that the sequence*

(89) $$0 \longrightarrow \mathrm{Hom}_{K-R}(A'', C) \xrightarrow{j'} \mathrm{Hom}_{K-R}(A, C) \xrightarrow{i'} \mathrm{Hom}_{K-R}(A', C) \longrightarrow 0$$

is bi-R-exact. Let a be in $\hat{\mathrm{H}}^p(R, \mathrm{Hom}_{K-R}(A', C))$, *$b$ in* $\hat{\mathrm{H}}^q(R, A'')$. *Then*

(90) $$(\delta a)b + (-1)^p a(\delta b) = 0,$$

where δ is the appropriate connecting homomorphism.

This can be proved by replacing Z by R, Hom by Hom_{K-R} in the proof of Prop. 6.1 in p. 247 of [1].

We shall also define the mappings

(91) $\gamma_{p,q} : \hat{\mathrm{H}}^p(R, \mathrm{Hom}_{K-R}(A, C)) \longrightarrow \mathrm{Hom}_K(\hat{\mathrm{H}}^q(R, A), \hat{\mathrm{H}}^{p+q}(R\ C))$, by setting

(92) $$(\gamma_{p,q} a)(b) = ab.$$

PROPOSITION 15. *If for fixed bi-R-module C and a pair of integers p, q, the mapping $\gamma_{p,q}$ is an isomorphism for all bi-R-modules A, then the same holds* *for all* p', q' *with* $p' + q' = p + q$.

PROOF. The exact bi-R-sequence $0 \longrightarrow B \longrightarrow R \otimes_K A \xrightarrow{g} C \longrightarrow 0$, where $g(x \otimes a) = xa$ and $B = \operatorname{Ker} g$, splits as a one-sided-(say right) R-sequence, therefore, the sequence

(93) $$0 \longrightarrow \operatorname{Hom}_{K-R}(A, C) \longrightarrow \operatorname{Hom}_{K-R}(R \otimes_K A, C) \longrightarrow \operatorname{Hom}_{K-R}(B, C) \longrightarrow 0$$

is exact. Thus by Prop. 13 we have the commutative diagram (with R omitted)

(94) $$\begin{array}{ccc} \mathrm{H}^p(\operatorname{Hom}_{K-R}(B, C)) & \xrightarrow{\gamma_{p,q}} & \operatorname{Hom}_K(\mathrm{H}^q(B), \mathrm{H}^{p+q}(C)) \\ \downarrow (-1)^{p+1}\delta & & \downarrow \operatorname{Hom}_K(\delta, \hat{\mathrm{H}}^{q+p}(C)) \\ \mathrm{H}^{p+1}(\operatorname{Hom}_{K-R}(A, C)) & \xrightarrow{\gamma_{p+1,q-1}} & \operatorname{Hom}_K(\mathrm{H}^{q-1}(A), \mathrm{H}^{p+q}(C)). \end{array}$$

Since $R \otimes_K A$ is weakly projective, it follows from Prop. 6 that $\operatorname{Hom}_{K-R}(R \otimes_K A, C)$ is weakly injective. Consequently both connecting homomorphisms involved are isomorphisms. Since $\gamma_{p,q}$ is an isomorphism by assumption, it follows that $\gamma_{p+1,q-1}$ is an isomorphism. The proof that $\gamma_{p-1,q+1}$ is an isomorphism is similarly obtained by using the exact sequence $0 \xrightarrow{h} A \longrightarrow \operatorname{Hom}_K(R, A) \longrightarrow B' \longrightarrow 0$, where $(hax) = xa$, $B' = \operatorname{Coker} h$ as in Def. 2 *a*.

PROPOSITION 16. *The mapping*

(95) $$\gamma_{0,q} : \hat{\mathrm{H}}^0(R, \operatorname{Hom}_{K-R}(A, C)) \longrightarrow \operatorname{Hom}_K(\hat{\mathrm{H}}^q(R, A), \hat{\mathrm{H}}^q(R, C))$$

composed with the natural epimorphism

(96) $$\operatorname{Hom}_{R-R}(A, C) \longrightarrow \mathrm{H}^0(R, \operatorname{Hom}_{K-R}(A, C))$$

which to each f *in* $\operatorname{Hom}_{R-R}(A, C)$ *assigns the induced homomorphism*

(97) $$f\colon \hat{\mathrm{H}}^q(R, A) \longrightarrow \hat{\mathrm{H}}^q(R, C).$$

PROOF. Let the map $g\colon R \longrightarrow \operatorname{Hom}_{K-R}(A, C)$ be given by $g\,1 = f$, therefore, $g(x)a = xf(a)$, x in R, a in A, and let the map $h\colon A = R \otimes_R A \longrightarrow \operatorname{Hom}_{K-R}(A, C) \otimes_R A$ be induced by g. We obtain a commutative diagram

(99)

$$\begin{array}{ccccc} \hat{\mathrm{H}}^0(R, R) \otimes_M \hat{\mathrm{H}}^q(R, A) & \longrightarrow & \hat{\mathrm{H}}^q(R, R \otimes_R A) & \longrightarrow & \hat{\mathrm{H}}^q(R, A) \\ \downarrow \hat{g} \otimes i & & \downarrow \hat{h} & & \downarrow \hat{f} \\ \hat{\mathrm{H}}^0(R, \operatorname{Hom}_{K-R}(A, C)) \otimes_K \hat{\mathrm{H}}^q(R, A) & \longrightarrow & \hat{\mathrm{H}}^q(R, \operatorname{Hom}_{K-R}(A, C) \otimes_R A) & \longrightarrow & \hat{\mathrm{H}}^q(A, C). \end{array}$$

If a in $\hat{\mathrm{H}}^0(R, \operatorname{Hom}_{K-R}(A, C))$ is the element determined by f and 1 in $\hat{\mathrm{H}}(R, R)$ is the unit element, then $g\,1 = a$. Thus for each b in $\hat{\mathrm{H}}^q(R, A)$ we have

$$(100) \qquad (\gamma_{0,q}a)b = \hat{\varphi}(ab) = \hat{\varphi}(\hat{g}lb) = \hat{\varphi}\hat{h}(1\,b) = \hat{f}(1\,b) = \hat{f}b,$$

because $1\,b = b$ from Prop. 12.

Now we assume that R is commutative, thus the center $C(R)$ is R itself, therefore, Prop. 11, 12, 13 yield that each $\hat{\mathrm{H}}^n$ may be considered as a bi-R-module induced by $R = C(R) \longrightarrow C(R)/\mathrm{N}R = \mathrm{H}^0(R, R)$. Then we see also from Prop. 11, 12, 13, for x in R.

$$(101) \qquad \begin{aligned}(\gamma a)(bx) &= a(bx) = (ab)x = ((\gamma a)b)x,\\ (\gamma a)(xb) &= a(xb) = (ax)b = (xa)b = x(ab) = x((\gamma a)b),\end{aligned}$$

and these show that Im γ belongs to $\mathrm{Hom}_{R-R}(\ldots)$.

THEOREM 4 (*Duality theorem*). *Assume that R is commutative. Let C be a bi-R-module such that $J^*C = 0$,* i. e. *$xc = cx^*$ for all x in R with the isomorphism $x \longrightarrow x^*$, and which is one-sided-(say right) R-injective ($=$ projective, see* [4]). *Then for all bi-R-module A the homomorphism*

$$(102) \qquad \gamma_{p-1,-p} : \hat{\mathrm{H}}^{p-1}(R, \mathrm{Hom}_{K-R}(A, C)) \longrightarrow \mathrm{Hom}_{R-R}(\hat{\mathrm{H}}^{-p}(R, A), \hat{\mathrm{H}}^{-1}(R, C))$$

given by $(\gamma a)b = ab$ is an isomorphism.

PROOF. From the assumption $J^*C = 0$, we see that $\hat{\mathrm{H}}^{-1}(R, C)$ is the kernel of N. Since $\sum_i a_ib_ia = 0$ for all a in $\hat{\mathrm{H}}$ by Prop. 9, it follows that for any bi-R-homomorphism f: $\hat{\mathrm{H}}(R, A) \longrightarrow C$, and for a in $\hat{\mathrm{H}}(R, A)$,

$$\mathrm{N}(f(a)) = \sum_i b_if(a)a_i^* = \sum_i a_ib_if(a) = f\Big(\sum_i a_ib_ia\Big) = 0.$$

Therefore, every homomorphism $\mathrm{H}(R, A) \longrightarrow C$ is a homomorphism $\mathrm{H}(R, A) \longrightarrow_N C$. Thus (102) may be considered as follows

$$(103) \qquad \gamma_{p-1,-1} : \hat{\mathrm{H}}^{p-1}(R, \mathrm{Hom}_{K-R}(A, C)) \longrightarrow \mathrm{Hom}_{R-R}(\hat{\mathrm{H}}^{-p}(R, A), C).$$

In view of Prop. 15, it suffices to show that $\gamma_{0,-1}$ is an isomorphism. Since $[\mathrm{Hom}_{K-R}(A, C)]^R = \mathrm{Hom}_{R-R}(A, C)$ we have

$$(104) \qquad \gamma_{0,-1} : \mathrm{Hom}_{R-R}(A, C)/\mathrm{N}\mathrm{Hom}_{K-R}(A, C) \longrightarrow \mathrm{Hom}_{R-R}({}_{\mathrm{N}}A/J^*A, C).$$

It follows from Prop. 16 that $\gamma_{,1}$ is obtained by restricting bi-R-homomorphisms : $A \longrightarrow C$ to the subgroup ${}_{\mathrm{N}}A$. Now consider any bi-, and then of course right-, R-homomorphism f: ${}_{\mathrm{N}}A \longrightarrow C$ with $f(J^*A) = 0$. Since C is right-injective, f admits a right-R-homomorphism $g : A \longrightarrow C$. Then $g(J^*A) = 0$, therefore, we see that

$$g(xa) - xg(a) = g(xa) - g(a)x^* = g(xa) - g(ax^*) = g(xa - ax^*) = 0,$$

this shows that g is in $\mathrm{Hom}_{R-R}(A, A)$, and $\gamma_{0,-1}$ is an epimorphism. Next consider g of $\mathrm{Hom}_{R-R}(A, C)$ with $g(_N A) = 0$. Since the sequence $0 \longrightarrow {}_K A \longrightarrow A \xrightarrow{\mathrm{N}} A$ is an exact sequence of K-modules and since C is right-R-injective, it follows that

$$\mathrm{Hom}_{K-R}(A, C) \xrightarrow{\mathrm{N}} \mathrm{Hom}_{K-R}(A, C) \longrightarrow \mathrm{Hom}_{K-R}(A, C) \longrightarrow 0$$

is exact. Thus there exists an element h in $\mathrm{Hom}_{K-R}(A, C)$ such that the composition $A \xrightarrow{\mathrm{N}} A \xrightarrow{h} C$ is g. Then

$$(\mathrm{N}h)a = \sum_i a_i h(b_i a) = \sum_i h(b_i a)a_1^* = h\Big(\sum_i a_i a a_i^*\Big) = h(\mathrm{N}a) = g, \tag{110}$$

and g is in $\mathrm{NHom}_{K-R}(A, C)$. Thus $\gamma_{0,-1}$ is a monomorphism. This shows that all our assertions hold good.

REFERENCES

[1] H. CARTAN-S. EILENBERG, Homological Algebra, Princeton 1956.

[2] K. HIRATA, On relative homological algebra of Frobenius extensions, Nagoya Math. J., 15(1959), 17-28.

[3] G. HOCHSCHILD, Relative homological algebra, Trans. of Amer. Math. Soc., 82(1956), 58-67.

[4] F. KASCH, Grundlagen einer Theorie der Frobeniuserweiterungen, Math. Ann., 127 (1954)

[5] T. NAKAYAMA, On the complete cohomology theory of Frobenius algebras, Osaka Math. J., 9(1957), 165-187.

[6] ―――――, On algebras with complete homology. Abh. Math. Sem. Hamburg, 22 (1958), 300-307.

[7] ―――――, Note on complete cohomology of a quasi-frobenius algebra, Nagoya Math. J., 13(1958), 115-121.

[8] H. YAMASAKI, On products of Hochschild groups. Tôhoku Math. J., 11(1959), 147-161.

IWATE UNIVERSITY.