ON THE PARALLELISABILITY UNDER RIEMANNIAN METRICS OF DIRECTION FIELDS OVER 3-DIMENSIONAL MANIFOLDS

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1. Introduction. In a 3-dimensional differentiable manifold over which a direction field is given, we shall consider a necessary and sufficient condition for the existence of a complete Riemannian metric which leaves the field to be a parallel field. Our purpose is to find this condition from the global structure of the manifold related with the integral curves of the field. We first treat of the structure of a 3-dimensional complete Riemannian manifold over which a parallel field of directions is given. The major part (§ 3—6) of this paper is devoted to it and the main result will be seen in Theorems 1—5. From the last section we may see that a part of our purpose is at ained. See Theorems 6—8.

We shall begin with some conventions to be used throughout this paper. By differentiability we shall always understand that of class C^∞. A *neighborhood* is an open set homeomorphic to a Euclidean space. An *isometry* is an isometric diffeomorphism. The product operation "$\times$" sometimes expresses the operation of metric product. Let E be the Euclidean 1-space with the coordinate system $\{t \mid -\infty < t < \infty\}$ and dt denotes the infinitesimal distance. Let E' be the part $\{t \mid 0 \leqq t < \infty\}$ of E. For a constant $L > 0$, let $[L]$ be the part $\{t \mid 0 \leqq t \leqq L\}$ of E. Let us suppose that indices a, α, λ, μ take the following ranges of values:

$$a = 1,\ 2\,;\ \ \alpha = 1,\ 2,\ 3\,;\ \ \lambda,\ \mu = 1,\ 2, \ldots\ldots (\text{to } \infty).$$

Take a Riemannian manifold X. For any $x, y \in X$, let $[x, y]$ denote a geodesic arc from x to y. Given a constant c and a unit tangent vector v at x, $g(x, v, c)$ is defined to be the geodesic arc issuing from x whose length is $|c|$ and whose initial vector is v or $-v$ according as $c > 0$ or < 0. Let (x, v, c) denote its terminal point. Take a point sequence $\{x_\lambda\} \subset X$ converging to a point $x \in X$, in which there exists a constant $N > 0$ such that $x \neq x_\lambda$ for all $\lambda > N$. Such a point sequence is said to be *essential*. Moreover corresponding to each $\lambda > N$, take a vector v_λ, such that $g(x, v_\lambda, c_\lambda)$ for suitable $c_\lambda > 0$ becomes a minimizing geodesic from x to x_λ. If there is the vector v at x such that $v_\lambda \to v (\lambda \to \infty)$, the unit vector v and the vector space generated from v are called the *tangential* *vector* and the *tangential straight line* of $\{x_\lambda\}$ respectively. If there are two subsequences of the sequence $\{x_\lambda\}$ with tangential vectors w, $-w$ respectively and there is no subsequence having other vector as tangential vector, the vector space generated from w is also called the *tangential straight line* of $\{x_\lambda\}$. In the case dim $X = 2$, take an isometry J of X onto itself leaving a point $x \in X$ fixed. J induces the congruent transformation in the Euclidean vector space at x tangent to X. Under a suitable frame it is represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $0 \leqq \theta \leqq \pi$. According to these matrices, J is said to be a *symmetry* or a *rotation* at x. The θ is called the *rotation angle*.

Here we shall show two lemmas, without proof. Let X, Y be 2-dimensional connected complete differentiable Riemannian manifolds. First suppose that there is given a sequence $\{J_\lambda\}$ of isometries of X onto Y. For a point $x_0 \in X$, let F_0 be a 2-frame at x_0 tangent to X. Put $y_\lambda = J_\lambda(x_0)$ and $G_\lambda = J_\lambda \cdot F_0$. If $y_\lambda \to y_0$ and $G_\lambda \to G_0 (\lambda \to \infty)$ where G_0 is a frame at $y_0 \in Y$, then we have

LEMMA 1. 1. *There exists the isometry J of X onto Y such that $J(x_0) = y_0$ and $J \cdot F_0 = G_0$* ([3], p. 404 ; [5], p. 93).

The isometry J is called the *limit* of $\{J_\lambda\}$.

Next, suppose in X that all of the rotations at $x_0 \in X$ form 1-dimensional torus group. We denote this transformation group by G. Then the following lemma seems to be already known.

LEMMA 1. 2. *X is homeomorphic onto Euclidean or elliptic or spherical 2-space, according as the cut-locus for x_0 is empty or is composed of more than one point or consists of one point alone*[1]*. Moreover by changing on X its Riemannian metric alone, it is possible to let X become Euclidean or elliptic or spherical 2-space according to the respective case above, so that G is the group of rotations there, too.*

2. S-manifold. Let V be an n-dimensional connected Hausdorff differentiable manifold over which a differentiable field of directions is given. So, to each point $x \in V$ there is assigned the direction, i.e., the oriented straight line, tangent to V at x where all of the directions form a differentiable field. This field is called the *S-field* of V and such a manifold V an n-dimensional *S-manifold*. Through each point $x \in V$ there passes a maximal integral curve of the S-field.

1) Elliptic space and spherical space mean the ones which are Riemannian spaces with constant positive curvature. For the definition of cut-loci, see [7], p. 702.

Let $S(x)$ denote it, and $S(x)$ is called the *S-orbit* passing through $x \in V$. By the *orientations* of the S-orbits we understand what are concordant with the S-field.

In V suppose that there exists a connected open submanifold V^0 which satisfies the following conditions:

1) V^0 is a union of S-orbits and dense in V;

2) V^0 is a maximal subspace which becomes a differentiable principal bundle, where each fibre is an S-orbit and the standard fibre is 1-dimensional connected Lie group.[2)]

Then V is said to be the *almost principal S-bundle* with *kernel* V^0. In this case if $V = V^0$, V is simply said to be a *principal S-bundle.*

In two S-manifolds V_1, V_2 of the same dimension, an *S-diffeomorphism* of V_1 into V_2 is a diffeomorphism of V_1 into V_2 which carries S-orbits to S-orbits.

Let D be the part in the Euclidean 3-space defined by $x^2 + y^2 < 1$ and $0 \leqq z \leqq 1$, where x, y, z denote usual orthogonal coordinates. Take a constant $\theta(0 < \theta \leqq \pi)$ such that $\pi/\theta =$ rational number. In D, identify $(x, y, 1)$ with $(x \cos\theta - y \sin\theta,\ x \sin\theta + y \cos\theta,\ 0)$ for all x, y. The manifold thus obtained is regarded as an S-manifold where each S-orbit is locally defined by $x =$ const., $y =$ const. Such a 3-dimensional S-manifold is called an *C_1-manifold.* Again in D, identify $(x, y, 1)$ with $(x, -y, 0)$ for all x, y. Then, just as defined above, we obtain a 3-dimensional S-manifold. This is called a *C_2-manifold.* In each of them, the S-orbit passing through $(0, 0, 0)$ is called its *central S-orbit.*

3. *RS*-manifold. Let M be an n-dimensional connected complete differentiable Riemannian manifold ($n > 1$) over which a parallel field of directions is given. M is also regarded as an S-manifold whose S-field is the parallel field. Accordingly, we shall call the parallel field the *S-field* of M. Thus S-orbit, $S(x)\,(x \in M)$ etc. are defined under the same sense as § 2. Every S-orbit is a geodesic of M. Take the field of $(n-1)$-dimensional tangent vector subspaces which is orthogonal to the S-field at each point of M. This field is called the *R-field* of M. The R-field becomes a parallel field over M and hence involutive (as distribution). So, through any $x \in M$ there passes its maximal integral manifold. We call this manifold with the Riemannian metric induced from M an *R-orbit* of M. Let $R(x)$ denote it. Such a manifold M is called an n-dimensional *RS-manifold.*[3)] In M, the following fact is well-known: At a point $x \in M$ there exists an admissible coordinate system $(x^\beta)\,(\beta = 1, 2, \ldots\ldots, n)$ in which the metric is expressed by the form completely decomposed as

2) By the word "maximal" it is meant that there are no subspaces, $\supset V^0$, $\neq V^0$, which have the same property. The differentiability of the principal bundle must be concordant with that of V.

3) This paper is closely related with [2], but our RS-manifolds are slightly distinct from those of [2].

$$ds^2 = g_{bc}(x^1, \ldots\ldots, x^{n-1})dx^b dx^c + (dx^n)^2$$
$$(b, c = 1, 2, \ldots\ldots, n-1)$$

and the equation $x^n =$ const. expresses a part of an R-orbit. Such a coordinate system is called a *reduced coordinate system*. Moreover let us recall that each R-orbit is totally geodesic and complete as Riemannian manifold.

From now on let $d(x)$ denote the unit tangent vector at $x \in M$ which expresses the direction through x in the S-field. For any points x, y of an R-orbit, let $d_R(x, y)$ denote the length of a minimizing geodesic in the R-orbit from x to y. Take any $x_0 \in M$. Let $I(x_0)$ denote the set $R(x_0) \cap S(x_0)$. Let $T_R(x_0)$ denote the Euclidean $(n-1)$-space tangent to $R(x_0)$ at x_0. An *R-neighborhood* of x_0 is a neighborhood in $R(x_0)$. If, for a constant $c > 0$, a part $\{x \mid x \in R(x_0), d_R(x_0, x) < c\}$ can be covered by a normal coordinate system in $R(x_0)$ with center x_0, it is called a *normal R-neighborhood* of x_0 and is denoted by $N_R(x_0\,;\,c)$. Then, the constant c is said to be a *normal R-radius* at x_0. Take an R-orbit R of M. That M is of one of the following types I—III means that for suitable L, J, there is an isometry of M onto the corresponding Riemannian manifold which maps each R-orbit onto $t =$ const. ($t \in E$ or $[L]$).

Type I: The Riemannian manifold $R \times E$.

Type II: The Riemannian manifold constructed from $R \times [L]$ by identifying (x, L) with $(x, 0)$ for all $x \in R$.

Provided that there exists a non-trivial isometry J of R onto itself, we define

Type III: The Riemannian manifold constructed from $R \times [L]$ by identifying (x, L) with $(J(x), 0)$ for all $x \in R$.

Again take any $x_0 \in M$. we shall express $S(x_0)$ by $x(s)$ $(-\infty < s < \infty)$ where $x_0 = x(0)$ and s denotes arc-length. If $S(x_0)$ is closed, it represents $S(x_0)$ many times. Let u_0 be a unit vector at x_0 tangent to $R(x_0)$ and let c be a constant. Now displace u_0 parallelly along the curve $x(s)$. Corresponding to each s, we get the vector $u(s)$ at $x(s)$ tangent to $R(x(s))$. Hence $g(x(s), u(s), c) \subset R(x(s))$. Put $z_0 = (x_0, u_0, c)$.

LEMMA 3.1. *The curve $(x(s), u(s), c)$ $(-\infty < s < \infty)$ represents $S(z_0)$ and the parameter s plays the role of the arc-length in $S(z_0)$, too* ([2], p. 333).

For an R-orbit R and an S-orbit S, we have

LEMMA 3.2. *The set $R \cap S$ is non-empty and at most countable* ([2], p. 333).

For any $x_0 \in M$ and a constant c, we have

LEMMA 3.3. *The set $\{(x, d(x), c) \mid x \in R(x_0)\}$ forms $R(y_0)$ where $y_0 = (x_0, d(x_0), c)$, and the map*

$$f\colon R(x_0) \to R(y_0) \quad \textit{defined by} \quad f(x) = (x,\ d(x),\ c)$$

is an onto-isometry ([2], p. 334).

Such a map is called the *R-map* with respet to a geodesic arc $g(x_0, d(x_0), c)$.

If the topology of an R-orbit coincides with the relative one induced from M, then this holds also good for other R-orbit and we have

LEMMA 3.4. *M is of one of types I—III* ([2], p. 335).

Next, take two RS-manifolds M_1, M_2 of the same dimension. An S-diffeomorphism of M_1 onto M_2 which carries R-orbits to R-orbits, is called an *RS-diffeomorphism*. A 2-dimensional RS-manifold is called an *RS-torus* if its underlying manifold is a torus. This is a Euclidean space form.[4] Let X be an RS-torus whose S-orbits are all non-closed, and let S_X be any one of its S-orbits. When we regard the Euclidean space form $X \times E$ as an RS-manifold where each S-orbit is defined by (S_X, t) for fixed $t \in E$, such a 3-dimensional RS-manifold is called an *A_1-manifold*. Let J be an isometry of X onto itself which is an RS-diffeomorphism preserving the orientations of the S-orbits. In $X \times [L]$, identify (x, L) with $(J(x), 0)$ for all $x \in X$. When we regard the Euclidean space form thus obtained as an RS-manifold where each S-orbit is defined by (S_X, t) for fixed $t \in [L]$, such a 3-dimensional RS-manifold is called an *A_2-manifold*. Let J_1 be an involutive isometry of X onto itself, having no fixed point, which is an RS-diffeomorphism.[5] Let X_1 be the Euclidean space form obtained by identifying $x \in X$ with $J_1(x)$ for all $x \in X$. X_1 is regarded as an RS-torus whose S-orbits are those induced from the S-orbits of X by the identification. Its S-orbits are all non-closed.[6] Let S_{X_1} be any one of them. In $X \times E'$, identify $(x, 0)$ with $(J_1(x), 0)$ for all $x \in X$. When we regard the Euclidean space form thus obtained as an RS-manifold whose S-orbits are defined by $(S_{X_1}, 0)$, (S_X, t) for fixed $t\,(\neq 0) \in E'$, such a 3-dimensional RS-manifold is called an *A_3-manifold*. Furthermore take an involutive isometry J_2 of X onto itself, having the same property as J_1.[7] By the same manner as the construction of X_1, we obtain the RS-torus X_2 if we use J_2 instead of J_1. Let S_{X_2} be any one of its S-orbits. In $X \times [L]$, identify $(x, 0)$ with $(J_1(x), 0)$ and (x, L) with $(J_2(x), L)$ for all $x \in X$. When we regard the Euclidean space form thus obtained as an RS-manifold whose S-orbits are defined by $(S_{X_1}, 0)$, (S_{X_2}, L), (S_X, t) for fixed $t\,(\neq 0, L) \in [L]$, such a 3-dimensional RS-manifold is called an *A_4-manifold*. Let Y be Euclidean, elliptic, or spherical 2-space. Take

4) Space form always means connected complete Riemannian manifold of constant curvature.

5) That the map J_1 is involutive means that $J_1(J_1(x)) = x$ for all $x \in X$. We can see that J_1 preserves the orientations of the S-orbits.

6) Note that X is a double covering manifold of X_1.

7) J_1 and J_2 may be the same one. Such a note will be omitted hereafter.

a rotation J of Y at a point $x_0 \in Y$, whose rotation angle θ satisfies $\pi/\theta =$ irrational number. In $Y \times [L]$, identify (x, L) with $(J(x), 0)$ for all $x \in Y$. When we regard the Riemannian manifold thus obtained as an RS-manifold where each R-orbit is defined by $t =$ const. ($t \in [L]$), such a 3-dimensional RS-manifold is called a B_1- or a B_2- or a B_3-*manifold* according as Y is Euclidean or elliptic or spherical. Finally, suppose that Y is spherical. Let L_0 be the half of the length of closed geodesic on Y. Let u be any tangent unit vector at x_0. In $Y \times [L]$, identify $((x_0, u, s), L)$ with $((x_0, J\cdot u, L_0 - s), 0)$ for all u and s $(0 \leqq s \leqq L_0)$. When we regard the Riemannian manifold thus obtained as an RS-manifold where each R-orbit is defined by $t =$ const. ($t \in [L]$), such a 3-dimensional RS-manifold is called a B_4-*manifold.*

4. 3-dimensional RS-manifold whose S-orbits are all non-closed. Let M be such an RS-manifold throughout this section.

HYPOTHESIS I. *There is a point $z_0 \in M$ which is not a limit point of $I(z_0)$ relative to $R(z_0)$.*

Then we have

LEMMA 4. 1. *Any point $x \in M$ is not a limit point of $I(x)$ relative to $R(x)$ and M becomes a fibre bundle where each fibre is an S-orbit* ([2], *p.* 342).

THEOREM 1. *In a 3-dimensional RS-manifold M, suppose that all the S-orbits are non-closed and that M satisfies Hypothesis I. Then, M is reduced to a principal S-bundle and the R-field defines a connection.*[8] *Furthermore M is S-diffeomorphic onto an RS-manifold of type I.*

As M becomes a fibre bundle by Lemma 4. 1, we denote its base space by B. Let $\pi: M \to B$ be the projection. Over B, a complete differentiable Riemannian metric is naturally induced from M by π. So we treat B as the Riemannian manifold. For any $b_0 \in B$ there is a neighborhood U of b_0 and a coordinate function

$$\phi: U \times E \to M \text{ where } \phi(b, E) = \pi^{-1}(b) \text{ for each } b \in U.$$

Hence, for the same U we can find an into-isometry

$$\psi: U \times E \to M \text{ where } \psi(b, E) = \pi^{-1}(b) \text{ for each } b \in U,$$

so that $\psi(U, 0)$ is an R-neighborhood and the orientation of E corresponds to that of each S-orbit by ψ. By Lemma 3. 3 we can take such ψ as coordinate function. Under such coordinate functions the former part of our theorem is easily verified. Here the principal S-bundle M has a differentiable cross-section, its fibre being solid. So the latter part is also true.

8) For the definition of connections, see [1], p. 431.

HYPOTHESIS II. *Every point $z \in M$ is a limit point of $I(z)$ relative to $R(z)$.*

This is assumed for M from here to the last of this section. For a subset W of an R-orbit, let $\overline{W}$ denote its closure relative to the R-orbit. If a sequence is composed of points of the same R-orbit (or frames tangent to the same R-orbit), its convergency is always treated relative to the R-orbit. An *R-isometry* is an isometry of an R-orbit R onto an R-orbit R'. If it is the limit of a sequence of R-maps of R onto R', it is said to be *canonical*. So all of R-maps are canonical R-isometries.[9] For two points x, y on an S-orbit of M, let $S[x,y]$ denote the subarc of the S-orbit from x to y. An R-frame F at $x \in M$ is an orthonormal 2-frame at x tangent to $R(x)$ and usually is denoted by (x, F). Take a point $x_0 \in M$. An *(x_0)-sequence* is a point sequence which is a subset of $I(x_0)$. For an (x_0)-sequence $\{x_\lambda\}$ and an R-frame F_0 at x_0, if an R-frame F_λ at x_λ is the image of F_0 by the R-map with respect to the geodesic arc $S[x_0, x_\lambda]$, then the sequence $\{(x_\lambda, F_\lambda)\}$ is called an *(x_0, F_0)-sequence* and $\{x_\lambda\}$ is called its ***base sequence***. If $\{x_\lambda\}$ is essential, the sequence $\{(x_\lambda, F_\lambda)\}$ is said to be ***base-essential***.[10] That a normal R-radius c at x_0 is ***small*** means that there is a normal R-radius at x_0 greater than c. Give a straight line l passing through x_0 and tangent to $R(x_0)$, an angle $\theta(0 < \theta \leqq \pi/8)$, and a small normal R-radius c at x_0. Let $X[x_0, l, \theta, c]$ denote the part of $R(x_0)$ which consists of all $x \in \overline{N_R(x_0\,;\,c)}$ such that an angle between l and $[x_0, x] \subset \overline{N_R(x_0\,;\,c)}$ is not greater than θ. This closed region is called an *X-region* at x_0. It consists of two sectors having x_0 alone as common, each of which is called its *Δ-region.* Let $I[x_0\,;\,c]$ denote $\overline{I(x_0)} \cap \overline{N_R(x_0\,;\,c)}$.

Take an R-orbit R of M and a point $x_0 \in R$. Let F_0 be an R-frame at x_0. If $y_0 \in \overline{I(x_0)}$, then we have

LEMMA 4.2. 1) $\overline{I(x_0)} = \overline{I(y_0)}$. 2) *There exists a canonical R-isometry J of R onto itself which maps x_0 to y_0, and the inverse map J^{-1} also is a canonical R-isometry.*

If $y_0 \in I(x_0)$, it is evident.

First we prove 1). Let $\{x_\lambda\}$ be an (x_0)-sequence converging to y_0. Take any $y \in I(y_0)$. Let $'J$ be the R-map with respect to $S[y_0, y]$. Put $x'_\lambda = {}'J(x_\lambda)$. Then $\{x'_\lambda\}$ is an (x_0)-sequence converging to y. Hence $y \in \overline{I(x_0)}$. So, $\overline{I(y_0)} \subset \overline{I(x_0)}$. Next take any $x \in I(x_0)$ and let J_λ be the R-map with respect to $S[x_\lambda, x]$. Put $y_\lambda = J_\lambda(y_0)$. Then $\{y_\lambda\}$ becomes a (y_0)-sequence and $d_R(y_0, x_\lambda) = d_R(y_\lambda, x)$. So,

9) For, an R-map is considered as the limit of the sequence whose terms are all the same R-map.
10) Then this frame sequence need not converge, though $\{x_\lambda\}$ must converge.

$y_\lambda \to x$ $(\lambda \to \infty)$. Hence, $x \in \overline{I(y_0)}$ and $\overline{I(x_0)} \subset \overline{I(y_0)}$. Consequently 1) is proved.

To prove 2), take an (x_0, F_0)-sequence $\{(x_\lambda, F_\lambda)\}$ converging to an R-frame at y_0. This R-frame we denote by (y_0, F). Let J_λ be the R-map with respect to $S[x_0, x_\lambda]$. Then $J_\lambda\cdot(x_0, F_0) = (x_\lambda, F_\lambda)$. By Lemma 1.1, the limit J of $\{J_\lambda\}$ does exist and $J\cdot(x_0, F_0) = (y_0, F)$. Obviously J is a canonical R-isometry which is desired. And J^{-1} also is a canonical R-isometry, being the limit of $\{J_\lambda^{-1}\}$.

LEMMA 4.3. *There exists a base-essential (x_0, F_0)-sequence converging to (x_0, F_0).*

If we choose a suitable frame F at x_0, we can find a base-essential (x_0, F_0)-sequence $\{(x_\lambda, F_\lambda)\}$ converging to (x_0, F). Let J_λ be the R-map with respect to $S[x_\lambda, x_{\lambda+1}]$. Put $(y_\lambda, G_\lambda) = J_\lambda\cdot(x_0, F_0)$. Then, $\{(y_\lambda, G_\lambda)\}$ becomes a base-essential (x_0, F_0)-sequence converging to (x_0, F_0), since $S(x_0)$ is non-closed.

If there is an (x_0, F_0)-sequence converging to (x_0, F_0) whose base sequence has tangential vector v_0, then we have

LEMMA 4.4. *There exists an (x_0, F_0)-sequence converging to (x_0, F_0) whose base sequence has $-v_0$ as its tangential vector.*

Let $\{(x_\lambda, F_\lambda)\}$ be an (x_0, F_0)-sequence converging to (x_0, F_0) whose base sequence has tangential vector v_0. Let J_λ be the R-map with respect to $S[x_\lambda, x_0]$. Put $(y_\lambda, G_\lambda) = J_\lambda\cdot(x_0, F_0)$. Then the sequence $\{(y_\lambda, G_\lambda)\}$ becomes a base-essential (x_0, F_0)-sequence converging to (x_0, F_0). We shall prove that the base sequence $\{y_\lambda\}$ has $-v_0$ as its tangential vector. Let g_λ be a minimizing geodesic from x_0 to x_λ. Put $h_\lambda = J_\lambda\cdot g_\lambda$. Let $(x_\lambda^*, F_\lambda^*)$ and $(y_\lambda^*, G_\lambda^*)$ be the developements in $T_R(x_0)$ of (x_λ, F_λ) and (y_λ, G_λ) along g_λ and h_λ^{-1} respectively. Let J_λ^* be the congruent transformation in $T_R(x_0)$ which carries $(x_\lambda^*, F_\lambda^*)$ to (x_0, F_0). Then, $J_\lambda^*\cdot(x_0, F_0) = (y_\lambda^*, G_\lambda^*)$. Now, for sufficiently large λ, denote the points x_λ^*, y_λ^* by $x_0 + dx_0$, $x_0 + \delta x_0$ and the frames F_0, G_λ^* by (e_a), $(e_a + \delta e_a)$ respectively. Put $dx_0 = \omega^a e_a$. Then, $-\delta x_0 = \omega^a(e_a + \delta e_a)$. Neglecting its higher order, we have $\delta x_0 = -\omega^a e_a$. This implies that $\{y_\lambda\}$ has $-v_0$ as its tangential vector. So our lemma is true.

If there is an (x_0)-sequence converging to x_0 which has tangential vector v_0, then we have

LEMMA 4.5. *There exists an (x_0)-sequence converging to x_0 which has $-v_0$ as its tangential vector.*

For a suitable R-frame F at x_0, there exists an (x_0, F_0)-sequence $\{(x_\lambda, F_\lambda)\}$ converging to (x_0, F) whose base sequence has v_0 as its tangential vector. If $F_0 = F$, our lemma follows from Lemma 4.4. So suppose that $F_0 \neq F$. Let $J_{\lambda\mu}$ be the R-map with respect to $S[x_\lambda, x_\mu]$. By Lemma 1.1, for fixed λ there is the limit J_λ of $\{J_{\lambda\mu} \mid \mu = 1, 2, \ldots\ldots\}$. This is a canonical R-isometry such that

$J_\lambda\cdot(x_\lambda, F_\lambda) = (x_0, F)$. Put $(y_\lambda, G_\lambda) = J_\lambda\cdot(x_0, F)$. Then, $(y_\lambda, G_\lambda) \to (x_0, F)$ $(\lambda \to \infty)$, and by the same way as in the proof of Lemma 4.4, it is shown that the sequence $\{y_\lambda\}$ has $-v_0$ as its tangential vector. Moreover, when λ is again fixed, for any $\varepsilon > 0$ we can find $N_\lambda > 0$ such that

$$d_R(x_\mu, x_0) < \varepsilon/2, \; d_R(J_{\lambda\mu}(x_0), \; J_\lambda(x_0)) < \varepsilon/2 \quad \text{for all} \quad \mu > N_\lambda.$$

Here $d_R(J_{\lambda\mu}(x_\mu), J_{\lambda\mu}(x_0)) < \varepsilon/2$. So we have $d_R(J_{\lambda\mu}(x_\mu), y_\lambda) < \varepsilon$. As $J_{\lambda\mu}(x_\mu) \in I(x_0)$ for all μ, we can find an (x_0)-sequence converging to y_λ. From these facts our lemma is verified.

If there is an essential (x_0)-sequence converging to x_0 which has not tangential straight line, then we have

LEMMA 4.6. *$I(x_0)$ is dense in R.*

By Lemma 4.5, we can find (x_0)-sequences $Z_i (i = 1, 2, 3, 4)$ converging to x_0 and having tangential vectors v_1, v_2, $-v_1$, $-v_2$ respectively such that $0 < \widehat{v_1 v_2} < \pi$, where $\widehat{v_1 v_2}$ denotes the angle between v_1, v_2. Now suppose that $I(x_0)$ is not dense in R. Take $y_0 \in R - \overline{I(x_0)}$. There is $x' \in \overline{I(x_0)}$ such that $d_R(y_0, x') = d_R(y_0, \overline{I(x_0)})$. Put $c = d_R(y_0, x')$. Then $c > 0$ and

$$\{y \mid y \in R, \; d_R(y_0, y) < c\} \cap \overline{I(x_0)} = 0. \tag{4.1}$$

On the other hand, by Lemma 4.2 we can find a canonical R-isometry J which maps x_0 to x'. So there is a sequence $\{J_\lambda\}$ of R-maps which has J as its limit. As $J_\lambda\cdot Z_i \subset I(x_0)$, we have $J\cdot Z_i \subset \overline{I(x_0)}$. The sequences $J\cdot Z_i$ converge to x' and have tangential vectors $J\cdot v_1$, $J\cdot v_2$, $-(J\cdot v_1)$, $-(J\cdot v_2)$ respectively. Here $\widehat{(J\cdot v_1)(J\cdot v_2)} = \widehat{v_1 v_2}$. These properties of the sequences $J\cdot Z_i$ are contrary to (4.1). So, $I(x_0)$ must be dense in R.

Take x_1, $y_0 \in I(x_0)$. Let J, $'J$ be the R-maps with respect to $S[x_0, x_1]$, $S[x_0, y_0]$ respectively. Put $y_1 = {}'J(x_1)$.

LEMMA 4.7. *${}'J\cdot J = J\cdot{}'J$, and $d_R(x_0, x_1) = d_R(y_0, y_1)$, $d_R(x_0, y_0) = d_R(x_1, y_1)$.*

We can find constants a, b such that $S[x_0, x_1] = g(x_0, d(x_0), a)$ and $S[x_0, y_0] = g(x_0, d(x_0), b)$. Then ${}'J\cdot J$ is the R-map with respect to $g(x_0, d(x_0), a + b)$. Similarly, $J\cdot{}'J$ is the R-map with respect to $g(x_0, d(x_0), b + a)$. So ${}'J\cdot J = J\cdot{}'J$. On the other hand, we have

$${}'J(x_0) = y_0, \; {}'J(x_1) = y_1, \; J(x_0) = x_1, \; J(y_0) = {}'J\cdot J\cdot{}'J^{-1}(y_0) = y_1.$$

From this, the latter part of our lemma follows.

Under Hypothesis II, let us assume

HYPOTHESIS II_1. *For a point $z_0 \in M$, all the essential (z_0)-sequences converging to z_0 have the same tangential straight line.*

LEMMA 4.8. *For every $x \in M$, all the essential (x)-sequences converging to x have the same tangential straight line.*

To prove this, suppose that for a point $x_0 \in R(z_0)$ there exists an essential (x_0)-sequence converging to x_0 which has not tangential straight line. Then, $\overline{I(x_0)} = R(z_0)$ by Lemma 4.6 and so $\overline{I(z_0)} = R(z_0)$ by Lemma 4.2. This contradicts with Hypothesis II_1. So, for every $x \in R(z_0)$, all the essential (x)-sequences converging to x have the same tangential straight line. Therefore, by Lemma 3.3 our lemma is easily verified.

Take an R-orbit R of M and a point $x_0 \in R$. Let l_0 be the tangential straight line of all the essential (x_0)-sequences converging to x_0. Take a constant θ_0 such that $0 < \theta_0 \leqq \pi/8$. The following lemma is now evident:

LEMMA 4.9. *At x_0 there is a small normal R-radius c_0 such that*

$$X[x_0, l_0, \theta_0, c] \supset I[x_0\,;\, c] \quad \textit{for} \quad 0 < c \leqq c_0.$$

Now we take any $y \in \overline{I(x_0)}$. As there is by Lemma 4.2 a canonical R-isometry J which maps x_0 to y, c_0 is also regarded as a small normal R-radius at y. Put $l = J\cdot l_0$. Then, for the same θ_0, c, we have

LEMMA 4.10. 1) $J\cdot I[x_0\,;\, c] = I[y\,;c]$; 2) $X[y, l, \theta_0, c] \supset I[y\,;\, c]$.

Since J is the limit of a sequence of R-maps, we denote the sequence by $\{J_\lambda\}$. Put $x_\lambda = J_\lambda(x_0)$. Then, $x_\lambda \in I(x_0)$, and $x_\lambda \to y(\lambda \to \infty)$. Furthermore

$$J_\lambda\cdot I[x_0\,;\, c] = I[x_\lambda\,;\, c] = \overline{I(x_0)} \cap \overline{N_R(x_\lambda\,;\, c)} = \overline{I(y)} \cap \overline{N_R(x_\lambda\,;\, c)}.$$

Hence $$J\cdot I[x_0\,;\, c] = \overline{I(y)} \cap \overline{N_R(y\,;\, c)} = I[y\,;\, c].$$

I. e., 1) holds good. Next, by Lemma 4.9,

$$J\cdot X[x_0, l_0, \theta_0, c] \supset J\cdot I[x_0\,;\, c].$$

By 1), $$X[y, l, \theta_0, c] \supset I[y\,;\, c].$$

So, 2) also holds good.

Let Δ be a Δ-region of the X-region $X[x_0, l_0, \theta_0, c]$. Let v_0 be the unit vector at x_0 tangent to Δ and generating l_0. Put $I_\Delta[x_0\,;\, c] = \Delta \cap I[x_0\,;\, c]$.

LEMMA 4.11. *There exists a constant $L(0 < L \leqq c_0)$ such that the map*

$$f\colon I_\Delta[x_0\,;\, L] \to [L] \quad \textit{defined by } f(x) = d_R(x_0, x) \tag{4.2}$$

where $x \in I_\Delta[x_0\,;\, L]$, becomes onto and one-to-one. The inverse map f^{-1} is continuous as a map of $[L]$ into $R(= R(x_0))$.

First suppose that for any $L\,(0 < L \leqq c_0)$ the map f of (4.2) is not one-to-one (into). Then we can find two essential sequences $\{y_\lambda\}$, $\{z_\lambda\}$ converging to

x_0 such that

$$y_\lambda,\ z_\lambda \in I_\Delta[x_0\,;\ c_0/\lambda],\ d_R(x_0, y_\lambda) = d_R(x_0, z_\lambda),\ y_\lambda \neq z_\lambda.$$

By Lemma 4.9, they have the same tangential vector v_0. Hence, there is an integer $k > 0$ such that in a geodesic triangle $x_0 y_k z_k$ constructed from minimizing geodesics,

$$\text{(4.3)} \qquad \pi/4 < \text{the angle } x_0 y_k z_k < 3\pi/4,$$
$$\text{the length of the side } [y_k, z_k] < c_0.$$

On the other hand, if J is a canonical R-isometry which maps x_0 to y_k, then

$$X[y_k, J\cdot l_0, \theta_0, c_0] \supset I[y_k\,;\ c_0].$$

by Lemma 4.10. So, $x_0, z_k \in X[y_k, J\cdot l_0, \theta_0, c_0]$. However, $0 < \theta_0 \leqq \pi/8$. This is contrary to (4.3). Accordingly we can find a constant $L(0 < L \leqq c_0)$ such that the map f of (4.2) becomes one-to-one (into).

Next suppose that for our L the map f is not onto. Since $I_\Delta[x_0\,;\ L]$ is closed in R, we can find $y_0 \in I_\Delta[x_0\,;\ L]$, a, ε $(0 \leqq a < a + \varepsilon < L)$ such that

$$\text{(4.4)} \qquad f(y_0) = a,\ I_\Delta[x_0\,;\ L] \cap (N_R(x_0\,;\ a+\varepsilon) - \overline{N_R(x_0\,;\ a)}) = 0.$$

If we take a canonical R-isometry J_0 which maps x_0 to y_0, then

$$X[y_0, J_0\cdot l_0, \theta_0, c_0] \supset I[y_0, c_0].$$

So from Lemmas 4.5 and 4.8 each Δ-region of $X[y_0, J_0\cdot l_0, \theta_0, c_0]$ contains an essential (y_0)-sequence converging to y_0. Here, any (y_0)-sequence is a subset of $\overline{I(x_0)}$ and $X[y_0, J_0\cdot l_0, \theta_0, c_0]$ contains the minimizing geodesic $[x_0, y_0]$. This is contrary to (4.4), since $0 < \theta_0 \leqq \pi/8$. Accordingly the map f must be onto. So the former part of our lemma has been proved.

To prove the latter part, suppose that for the same L, the inverse map $f^{-1} : [L] \to R$ is not continuous at $t_0(0 \leqq t_0 \leqq L)$. Put $y_0 = f^{-1}(t_0)$. We can choose a sequence $\{t_\lambda\}$, $0 < t_\lambda < L$, converging to t_0 such that the sequence $\{y_\lambda\}$, $y_\lambda \equiv f^{-1}(t_\lambda)$, converges to a point $y(\neq y_0)$. So, $d_R(x_0, y) = t_0$ and $y \in I_\Delta[x_0\,;\ L]$. Hence $f(y_0) = f(y) = t_0$. As this contradicts with the fact that f is one-to-one, the map f^{-1} must be continuous. This completes the proof of our lemma.

LEMMA 4.12. *The arc* C: $g(t) \equiv f^{-1}(t)$ $(0 \leqq t \leqq L)$ *is simple and of class* C^1 *with respect to* t.

By Lemma 4.11, it is obvious that C is simple. So we prove that C is of class C^1. For each $t(0 \leqq t \leqq L)$, let J_t denote a canonical R-isometry which maps x_0 to the point $g(t)$. Put $l(t) = J_t\cdot l_0$. By Lemma 4.10, $l(t)$ is tangent to C.

Given any $\theta (0 < \theta \leqq \pi/8)$, by Lemma 4. 9 there is $c > 0$ such that $X[g(t), l(t), \theta, c] \supset I[g(t); c]$. Moreover we can find $\delta > 0$ such that $d_R(g(t), g(t+\Delta t)) < c$ for all Δt satisfying $|\Delta t| < \delta$ $(0 \leqq t + \Delta i \leqq L)$. Now take a canonical R-isometry J which maps $g(t)$ to $g(t+\Delta t)$. Then, $J \cdot l(t) = l(t + \Delta t)$, and by Lemma 4. 10,

$$X[g(t+\Delta t), l(t+\Delta t), \theta, c] \supset I[g(t+\Delta t); c].$$

Hence $X[g(t+\Delta t), l(t+\Delta t), \theta, c]$ contains the minimizing geodesic $[g(t), g(t+\Delta t)]$. So it follows that $l(t)$ can not construct an angle greater than 2θ with the developement of $l(t+\Delta t)$ in $T_R(g(t))$ along $[g(t), g(t+t)]$. This shows that $l(t) (0 \leqq t \leqq L)$ is continuous.

Accordingly, for each t we can plant the unit vector $v(t)$ generating $l(t)$ at the point $g(t)$, so that $v(0) = v_0$ and $v(t)$ is continuous over $0 \leqq t \leqq L$. Take an angle $\theta(t)$ between $v(t)$ and the geodesic circle with center x_0 passing through $g(t)$, which is continuous over $0 \leqq t \leqq L$ and satisfies $3\pi/8 \leqq \theta(t) \leqq 5\pi/8$. This is possible, since

$$X[g(t), l(t), \theta_0, L] \supset I[g(t); L]$$

for the same θ_0 as Lemma 4.9 and so $X[g(t), l(t), \theta_0, L]$ contains the minimizing geodesic $[x_0, g(t)]$.

Now, cover $\overline{N_R(x_0; L)}$ by an admissible coordinate system (x^a). Let $(x^a(t))$ denote $g(t)$. Let Δt^* denote the length of a minimizing geodesic $[g(t), g(t+\Delta t)]$ which has positive or negative sign according as $\Delta t > 0$ or < 0. Then we have

$$\begin{aligned} \frac{dx^a}{dt} = \lim \frac{x^a(t+\Delta t) - x^a(t)}{\Delta t} &= \lim \frac{x^a(t+\Delta t) - x^a(t)}{\Delta t^*} \cdot \frac{\Delta t^*}{\Delta t} \\ &= \frac{v^a(t)}{\sin \theta(t)} \end{aligned}$$

where $(v^a(t))$ denotes $v(t)$. Hence we can see that C is of class C^1. So our lemma holds good.

By Lemma 4.12, the arc C is also represented by the arc $x(s)$ $(0 \leqq s \leqq L^*; x(0) = x_0)$ of class C^1 where L^* is the length of C and the parameter s denotes arc-length. For each s let $v(s)$ denote the tangent (unit) vector of the arc $x(s)$ at the point $x(s)$. So, $v(0) = v_0$. Now, if J_s is a canonical R-isometry which maps x_0 to $x(s)$, then we have

LEMMA 4. 13. $J_s \cdot v_0 = v(s)$.

Provided that J_s is an R-map, suppose that $J_s \cdot v_0 \neq v(s)$. Then, $s \neq 0$ and $J_s \cdot v_0$ must be $-v(s)$. Hence J_s carries the subarc $\widehat{x_0 x}(s)$ of C to the subarc

$\widehat{x(s)\,x_0}$ of C. The point $x(s/2) \in C$ becomes invariant by J_s. Lemma 3.3 shows that the S-orbit $S(x(s/2))$ becomes closed. This is obviously a contradiction. So if J_s is an R-map, our lemma is true. Next, we consider the case where J_s is not R-map. We can find a sequence $\{J_\lambda\}$ of R-maps, which has J_s as its limit. For sufficiently large λ we have $J_\lambda(x_0) \in C$, and $J_\lambda(x_0) \to x(s)\,(\lambda \to \infty)$. So by using the above result, we can see that in this case, too, our lemma is true.

On the other hand, J_s^{-1} also is a canonical R-isometry which maps $x(s)$ to x_0 by Lemma 4.2 and $J_s^{-1}\cdot v(s) = v_0$. By this and Lemma 4.13, the arc C may be extended for infinitely large (absolute) values of its parameter s (arc-length). The curve $x(s)$ $(-\infty < s < \infty; x(0) = x_0)$ thus obtained is called the *cluster curve* passing through x_0. Let $Cl(x_0)$ denote the curve. Let $v(s)$ denote the tangent (unit) vector of the curve $Cl(x_0): x(s)$. For each s, we plant at the point $x(s)$ an R-frame $F(s) = (e_a(s))$ where $e_1(s) = v(s)$, so that $F(s)$ becomes continuous with respect to s.

LEMMA 4.14. 1) *For any* s', s'' $(-\infty < s', s'' < \infty)$ *there exists a canonical R-isometry J which maps* $(x(s'), F(s'))$ *to* $(x(s''), F(s''))$; 2) *J maps the curve* $Cl(x_0)$ *to itself.*

To prove 1), put $x' = x(s')$, $F' = F(s')$. By Lemma 4.3, there are base-essential (x', F')-sequences converging to (x', F'). Among them, there is a one which is represented by $\{(x(s_\lambda), F(s_\lambda))\}$ where $s_\lambda \to s'\,(\lambda \to \infty)$. For fixed λ, put $\Delta s_\lambda = s_\lambda - s'$. Let J_λ denote the R-map with respect to $S[x', x(s_\lambda)]$. So, $J_\lambda\cdot(x', F') = (x(s' + \Delta s_\lambda), F(s' + \Delta s_\lambda))$. Then, for any integer m, $(J_\lambda)^m$ is also the R-map which maps (x', F') to $(x(s' + m\,\Delta s_\lambda), F(s' + m\,\Delta s_\lambda))$. This implies the existence of an (x', F')-sequence converging to $(x(s''), F(s''))$. I.e., 1) holds good. 2) follows from Lemma 4.10.

LEMMA 4.15. *The cluster curve* $x(s)$ $(-\infty < s < \infty)$ *is a simple differentiable curve with constant curvature in* $R(= R(x_0))$.

Let $(x^*(s), F^*(s))$ be the developement of $(x(s), F(s))$ on $T_R(x_0)$ along the subarc of the curve $x(s)$ from $s = 0$ to s. For any s', s'' $(-\infty < s', s'' < \infty)$, take the congruent transformation J^* in $T_R(x_0)$ which maps $(x^*(s'), F^*(s'))$ to $(x^*(s''), F^*(s''))$. By Lemma 4.14, J^* leaves the continuous field $\{(x^*(s), F^*(s)) \mid -\infty < s < \infty\}$ of frames fixed. This shows that the curve $x^*(s)$ $(-\infty < s < \infty)$ is a circle or a straight line. Accordingly the curve $x(s)$ $(-\infty < s < \infty)$ is differentiable and has constant curvature. That it is simple is obvious from Lemmas 4.12 and 4.14.

LEMMA 4.16. 1) $Cl(x_0) = Cl(y)$ *for any* $y \in Cl(x_0)$; 2) *There exists a constant* $L > 0$ *such that the set* $\{x \mid d_R(Cl(x_0), x) < L\} \cap \overline{I(x_0)}$ *coincides with* $Cl(x_0)$ *as subset*; 3) $\overline{I(x_0)}$ *consists of some cluster curves.*

1) follows Lemma 4. 2 and the fact that the cluster curve passing through y is only one. As the constant L in 2), take L in Lemma 4. 11. Then 2) is verified by Lemma 4. 14. 3) is now evident.

Take a constant c. Let us put $y(s) = (x(s), e_2(s), c)$, using the cluster curve $x(s)\,(-\infty < s < \infty)$, and further $y_0 = y(0)$. The curve $y(s)\,(-\infty < s < \infty)$ is differentiable. We represent its arc-length from $s = 0$ to s by $f(s)$ so that $f(s) \geqq 0$ or $\leqq 0$ according as $s > 0$ or < 0, and so $f(0) = 0$.

LEMMA 4. 17. *$f(s) = ks$ where k is a positive constant.*

By Lemma 4. 14, $df/ds =$ const. $(\equiv k)$. Here $k \geqq 0$. If $k = 0$, we have $y(s) = y_0$ for all s. So, any R-map which carries $Cl(x_0)$ onto itself leaves the point y_0 fixed. This shows that the S-orbit $S(y_0)$ is closed by Lemma 3. 3. I.e., we obtain a contradiction. Accordingly, $k > 0$ and so our lemma is true.

LEMMA 4. 18. 1) *The curve $y(s)$ $(-\infty < s < \infty)$ is the cluster curve $Cl(y_0)$*; 2) *If $Cl(x_0)$ and $Cl(y_0)$ are not the same curve, they have no common point*; 3) *In $R(= R(x_0))$, all the geodesics orthogonal to $Cl(x_0)$ are also orthogonal to $Cl(y_0)$ and are simply*; 4) *Any two of them have no common point.*

For all s, $y(s) \in \overline{I(y_0)}$. So 1) follows from Lemma 4. 16. The other assertions 2)—4) are also evident.

LEMMA 4.19. *In R, any of cluster curves is an orthogonal trajectory of the system of all geodesics orthogonal to a cluster curve, say $Cl(x_0)$.*

Let α be a cluster curve in R. For a point $y_0 \in \alpha$, take $x_1 \in Cl(x_0)$ such that $d_R(y_0, x_1) = d_R(y_0, Cl(x_0))$. This is possible by Lemma 4.16, and it is permitted to assume $x_1 = x(s_1)$ for a suitable s_1. Put $d = d_R(y_0, x_1)$. Then $y_0 = (x(s_1), e_2(s_1), \varepsilon d)$ for $\varepsilon = +1$ or -1 and α is represented by $(x(s), e_2(s), \varepsilon d)$ $(-\infty < s < \infty)$. So Lemma 4. 18 proves our lemma.

Take an S-orbit S_0 of M. Let S_0^* be the closure of S_0 as a subset of M.

LEMMA 4. 20. 1) *For any $x_0 \in S_0^*$ the cluster curve $Cl(x_0)$ and the S-orbit $S(x_0)$ are contained in S_0^**; 2) *S_0^* forms a 2-dimensional differentiable submanifold*; 3) *Under the Riemannian metric induced from M naturally, S_0^* is regarded as an RS-torus, whose R-and S-orbits are cluster curves and S-orbits of M respectively.*

1) follows from Lemmas 3.3 and 4.2. To prove 2) and 3), take a neighborhood U of x_0 with cubical reduced coordinate system (x^α) where $x_0 = (0,0,0)$. We denote the connected subarc of $Cl(x_0)$ passing through x_0 and contained in U by $x^\alpha(s)$ $(a < s < b; x^\alpha(0) = 0)$, where a, b are constants and the parameter s denotes arc-length. Then, $x^3(s) = 0$ for all s. Let W denote the part of U

defined by

$$(x^1(s),\ x^2(s), x^3)\ (a < s < b; -c < x^3 < c)$$

where $2c$ is the breadth of the coordinate system (x^α). Then, $W \subset S_0^*$ and an arc $(x^1(s),\ x^2(s), x^3)\,(a < s < b)$ for fixed x^3 becomes a subarc of a cluster curve and the parameter s denotes its arc-length. Now, let us treat W as a coordinate neighborhood with the coordinate system (s, x^3) and together with such coordinate neighborhoods, consider S_0^*. Then, 2) is easily shown. Moreover under the Riemannian metric induced from M naturally, we can see that S_0^* becomes a Euclidean 2-space form. Here an S-orbit of M contained in S_0^* forms a subset dense in S_0^*. Since it is a simple non-closed geodesic, the underlying manifold of S_0^* must be 2-dimensional torus. Thus we can see that 3) is also true.

By Lemma 4.20, we have an RS-torus as the closure of an S-orbit of M. Such an RS-torus is simply called an *S-closure* of M. In an S-closure S_0^*, take a *normal vector* v_0 of S_0^* at $x_0 \in S_0^*$, that is a unit vector orthogonal to S_0^*. This is tangent to $R(x_0)$ and orthogonal to $Cl(x_0)$. Let x_1 be any point of S_0^*. Let $\alpha(t)\,(0 \leqq t \leqq 1)$ be a curve in S_0^* from $x_0 = \alpha(0)$ to $x_1 = \alpha(1)$. For each t, take the normal vector $v(t)$ of S_0^* at the point $\alpha(t)$, so that $v(0) = v_0$ and it becomes continuous over $0 \leqq t \leqq 1$. If S^* is an S-closure, then we have

LEMMA 4.21. 1) *There exists a constant c such that* $(x_0, v_0, c) \in S^*$; 2) *There exists a neighborhood* U *in* S_0^* *of* x_0 *such that, if* $\{u(x) \mid x \in U\}$ *is the continuous field of normal vectors over* U *where* $u(x_0) = v_0$, *then* $(x, u(x), c) \in S^*$ *for all* $x \in U$ *and the map*

$$f : U \to S^* \text{ defined by } f(x) = (x, u(x), c)$$

becomes an into-diffeomorphism which carries the parts in U of R- and S-orbits of S_0^* *on R-and S-orbits of* S^* *respectively*; 3) $(\alpha(t), v(t), c) \in S^*$ *for* $0 \leqq t \leqq 1$.

By Lemma 3.2, we can take a point $y \in S^* \cap R(x_0)$. The cluster curve $Cl(y)$ is an orthogonal trajectory of the system of all geodesics orthogonal to $Cl(x_0)$ by Lemma 4.19. So 1) is obvious. On the other hand, 2) is verified by using Lemmas 3.1 and 4.18, and 3) follows from 2).

Under the same notations let Ψ be the set of all $s > 0$, such that at least one of two points $(x_0, \pm v_0, s)$ belongs to S_0^*. If $\Psi \neq 0$, we put $\rho(S_0^*) =$ g.l.b. Ψ. If $\Psi = 0$, we put $\rho(S_0^*) = \infty$. So, $0 \leqq \rho(S_0^*) \leqq \infty$. Next suppose that $S_0^* \neq S^*$. Let Ω be the set of all $s > 0$ such that at least one of two points $(x_0, \pm v_0, s)$ belongs to S^*. By Lemma 4.21, $\Omega \neq 0$. We put $\rho(S_0^*, S^*) =$ g.l.b. Ω. So $0 \leqq \rho(S_0^*, S^*) < \infty$. By Lemma 4.21, $\rho(S_0^*)$ and $\rho(S_0^*, S^*)$ are independent of x_0.

LEMMA 4.22. 1) $\rho(S_0^*) > 0$. 2) *If* $\rho(S_0^*) < \infty$, *at least one of two points* $(x_0, \pm v_0, \rho(S_0^*))$ *belongs to* S_0^*. 3) $\rho(S_0^*, S^*) > 0$. 4) *At least one of two points* $(x_0, \pm v_0, \rho(S_0^*, S^*))$ *belongs to* S^*.

These are easily verified by using Lemmas 4.16 and 4.21.

THEOREM 2. *In a 3-dimensional RS-manifold M, suppose that all the S-orbits are non-closed and that M satisfies Hypotheses II and* II_1. *Then M is RS-diffeomorphic onto an* A_i*-manifold* ($i = 1, 2, 3$, or 4).

To prove this, we shall classify M, with regard to the S-closures, into the following four cases:

1) The case where all the S-closures admit continuous field of normal vectors (over them). Take an S-closure S^*. Let $\{v(x) | x \in S^*\}$ be a continuous field of normal vectors. For any $x_0 \in S^*$ and $L > 0$, put $y_0 = (x_0, v(x_0), L)$. Let S_L^* be the S-closure passing through y_0. The map

$$f : S^* \to S_L^* \text{ defined by } f(x) = (x, v(x), L) \tag{4. 5}$$

is onto by Lemma 4.21. We prove that f is one-to-one. Suppose that $f(x_1) = f(x_2)$ for $x_1, x_2 \in S^*(x_1 \neq x_2)$. Then, if we consider the field $\{v(x) | x \in S^*\}$ restricted to a curve in S^* from x_1 to x_2, we can easily see that S_L^* does not admit continuous field of normal vectors. This is contrary to our case. So, f must be one-to-one. Furthermore from Lemma 4.21, it follows that f is an RS-diffeomorphism. Accordingly if $\rho(S^*) = \infty$, M is diffeomorphic onto an A_1-manifold. If $\rho(S^*) < \infty$, we have $(x_0, v(x_0), \varepsilon\rho(S^*)) \in S^*$ for $\varepsilon = +1$ or -1 by Lemma 4.22. The map f of (4.5) for $L = \varepsilon\rho(S^*)$ becomes an isometry of S^* onto itself. This is verified by Lemmas 3.1 and 4.17. So, M is then RS-diffeomorphic onto an A_2-manifold.

2) The case where an S-closure S^* only does not admit continuous field of normal vectors. Put $L = \rho(S^*)$. Of course $0 < L \leqq \infty$. Let $v(x)$ denote a normal vector of S^* at $x \in S^*$. For each $c\,(0 < c < L)$, we denote by S_c^* the S-closure passing through a point $(x_0, v(x_0), c)$ where $x_0 \in S^*$. In our case, the vector $v(x)$ is continuously displaced to $-v(x)$ along a suitable curve in S^*, preserving to be normal vector. This and Lemma 4.21 show that S_c^* consists of $(x, \pm v(x), c)$ for all $x \in S^*$. Here if $(x, v(x), c) = (x, -v(x), c)$ for $x \in S^*$, we obtain the contradiction that S_c^* does not admit continuous field of normal vectors. This is verified by using the above fact that $v(x)$ is continuously displaced to $-v(x)$, So, $(x, v(x), c) \neq (x, -v(x), c)$ for all $x \in S^*$. Next if $(x_1, v(x_1), c) = (x_2, v(x_2), c)$ for $x_1, x_2 \in S^*(x_1 \neq x_2)$, it follows that S_c^* does not admit continuous field of normal vectors, too. So, $(x_1, v(x_1), c) \neq (x_2, v(x_2), c)$ for any $x_1, x_2 \in S^*(x_1 \neq x_2)$. These show that S_c^* becomes a double

covering manifold (topologically) of S^* whose covering map p is $p(x_\varepsilon) = x$, where $x \in S^*$ and $x_\varepsilon = (x, \varepsilon v(x), c)$ for $\varepsilon = +1, -1$. By Lemma 4.21, the map p carries R- and S-orbits of S_c^* to the same ones of S^* respectively. So, the map f of S_c^* onto itself defined by

$$f(x, v(x), c) = (x, -v(x), c) \text{ for all } x \in S^*$$

becomes an involutive RS-diffeomorphism having no fixed point. From Lemmas 3.1 and 4.17, it follows that f is an isometry. In S_c^* identify $(x, v(x), c)$ with $(x, -v(x), c)$ for all $x \in S^*$. The manifold thus obtained is regarded as the RS-torus whose S-orbits are induced from those of M. This RS-torus becomes RS-diffeomorphic onto S^* by the natural correspondence. Furthermore it is evident that S_c^* is naturally RS-diffeomorphic onto $S_{c'}^*$ $(0 < c' < L)$.

Suppose that $L < \infty$. By Lemma 4.22 we have $x_0' \equiv (x_0, v(x_0), L) \in S^*$. So there exists a normal vector $v(x_0')$ of S^* such that $(x_0', v(x_0'), L) = x_0$. As $v(x_0')$ is continuously displaced to $v(x_0)$ along a suitable curve in S^* preserving to be normal vector, the S-closure passing through the point $(x_0, v(x_0), L/2)$ does not admit continuous field of normal vectors. This contradicts with our case. Therefore, $L = \infty$. Hence it is proved that M is RS-diffeomorphic onto an A_3-manifold.

3) The case where two S-closures S_0^*, S_1^* only do not admit continuous field of normal vectors. Put $L = \rho(S_0^*, S_1^*)$ and take $x_0 \in S_0^*$. Let v_0 be a normal vector of S_0^* at x_0. By Lemmas 4.21 and 4.22, two points $(x_0, \pm v_0, L)$ belong to S_1^*. For each c $(0 < c < L)$, let S_c^* be the S-closure passing through a point $(x_0, v(x_0), c)$. Just as 2), S_c^* becomes a double covering manifold (topologically) of S_0^* and further of S_1^*. Thus it is proved that M is RS-diffeomorphic onto an A_4-manifold.

4) The case where three (or more) S-closures do not admit continuous field of normal vectors. Let S_0^*, S_1^*, S_2^* be such ones. We have $\rho(S_0^*) = 2\rho(S_0^*, S_1^*)$. Similarly, $\rho(S_0^*) = 2\rho(S_0^*, S_2^*)$. So, $\rho(S_0^*, S_1^*) = \rho(S_0^*, S_2^*)$. Hence, $S_1^* = S_2^*$. As this is a contradiction, our case does not occur. This completes the proof of our theorem.

In the next place, under Hypothesis II we assume

HYPOTHESIS II_2. *For every point z of M there is an essential (z)-sequence converging to z which has not tangential straight line.*

Then, by Lemma 4.6 any S-orbit is dense in M as its subset.

THEOREM 3. *In a 3-dimensional RS-manifold M, suppose that all the S-orbits are non-closed and that M satisfies Hypotheses II and II_2. Then all the R-orbits, and so M too, are Euclidean space form.*

Let R be any R-orbit of M. Take $x_0 \in R$. We have $\overline{I(x_0)} = R$. For any $x \in I(x_0)$, since the R-map with respect to $S[x_0, x]$ is an isometry, the curvature of R at x_0 is equal to that of R at x. So the curvature of R is constant from its continuity.

Let F_0 be an R-frame at x_0. By Lemma 4.3 there is a base-essential (x_0, F_0)-sequence converging to (x_0, F_0). Now take an element (x_1, F_1) of the sequence which is sufficiently near to (x_0, F_0). Let J be the R-map such that $J\cdot(x_0, F_0) = (x_1, F_1)$. Put $x_2 = J(x_1)$, $x_3 = J(x_2)$. Denote the minimizing geodesic joining x_0 to x_1 by $[x_0, x_1]$. Moreover put $[x_1, x_2] = J\cdot[x_0, x_1]$ and $[x_2, x_3] = J\cdot[x_1, x_2]$. In the product curve $[x_0, x_1]\cdot[x_1, x_2]\cdot[x_2, x_3]$, the angle $x_0x_1x_2$ is equal to the angle $x_1x_2x_3$ together with their orientations, (x_1, F_1) being sufficiently near to (x_0, F_0). Let g_1, g_2 denote the geodesics passing through x_1, x_2 and bisecting the angles $x_0x_1x_2$, $x_1x_2x_3$ respectively. Then $J\cdot g_1 = g_2$.

First, consider the case where R is an elliptic space form. Then g_1, g_2 intersect each other. We have $J(z) = z$ for the intersecting point z. So, $S(z)$ is closed. This is obviously a contradiction. Secondly consider the case where R is a hyperbolic space form. We can find $z_0 \in g_1$ which is sufficiently near to x_1 such that

$$d_R(z_0, J(z_0)) = d_R(x_1, x_2). \tag{4.6}$$

Of course $J(z_0) \in g_2$. Let us take an (x_0)-sequence $\{z_\lambda\}$ converging to z_0. By Lemma 4.7, $d_R(z_\lambda, J(z_\lambda)) = d_R(x_1, x_2)$. So, $d_R(z_0, J(z_0)) = d_R(x_1, x_2)$. This is contrary to (4.6). Accordingly R must be a Euclidean space form. Hence our theorem holds good.

REMARK. Let us give a model of the RS-manifold in Theorem 3. In a 2-dimensional torus group G there exists a element $g \in G$ such that the subgroup generated from g forms a subset dense in G[11]. We consider G as Euclidean space form, naturally. Let J be the isometry of G onto itself which is the parallel translation carrying the zero element of G to g. In $G \times [L]$, identify (x, L) with $(J(x), 0)$ for all $x \in G$. The Euclidean space form thus obtained is regarded as the RS-manifold where each R-orbit is defined by $t =$ const. ($t \in [L]$). Then the S-orbits are all non-closed and this RS-manifold satisfies Hypotheses II and II_2.

5. 3-dimensional RS-manifold among whose S-orbits there are both of closed one and non-closed one. In such an manifold M, let M^0 be the subspace which is the union of all of non-closed S-orbits.

LEMMA 5.1. *M^0 is a connected open submanifold of M whose closure is*

11) We owe this fact to K. Masuda. It is easily proved by using Theorem 33 in [4], p. 136.

M, and the maximal subset of M in which each point x is a limit point of I(x) relative to R(x) ([2], p. 344).

Take $x_0 \in M - M^0$, and put $R = R(x_0)$ and $R^0 = R \cap M^0$. R^0 is open and dense in R. Let L be the length of the closed geodesic $S(x_0)$. Let J be the R-map with respect to $g(x_0, d(x_0), L)$. Then, J leaves x_0 fixed and we have

LEMMA 5.2. *J is a rotation of R at x_0 and its rotation angle θ satisfies $\pi/\theta =$ irrational number. Hence all of the rotations of R at x_0 form 1-dimensional torus group.*

For a normal R-radius c at x_0, we have $N_R(x_0; c) \cap R^0 \neq 0$. Take a point $x \in N_R(x_0; c) \cap R^0$. Now, if J is a symmetry, then $S(x)$ becomes closed. However, $S(x)$ is not closed. So, J must be a rotation of R at x_0, and its rotation angle θ must satisfy $\pi/\theta =$ irrational number. Hence, the latter part of our lemma follows from Lemma 1.1.

If $R - R^0$ consists of x_0 alone, then we have

LEMMA 5.3. *R is homeomorphic onto Euclidean or elliptic 2-space.*

By Lemmas 1.2 and 5.2, R becomes homeomorphic onto Euclidean, elliptic or spherical 2-space. However, R can not become homeomorphic onto spherical 2-space. For, if R is homeomorphic onto it, the cut-locus for x_0 consists of a point alone. Denote the point by x_1. Then, $x_0 \neq x_1$ and $J(x_1) = x_1$. Hence $S(x_1)$ becomes closed and so $x_1 \in R - R^0$. This contradicts with the assumption. Accordingly our lemma is true.

If $R - R^0$ contains another point x_1, then we have

LEMMA 5.4. *$R - R^0$ consists of the two points x_0, x_1 only and R is homeomorphic onto spherical 2-space.*

First note that, at x_1, too, we may take up such a rotation of R as J at x_0 and the same property as Lemma 5.2 holds good.

1) The case where $x_1 \in I(x_0)$. Let g_0 be a minimizing geodesic from x_0 to x_1. Denote the unit vector at x_0 tangent to g_0 by u_0 and the length of g_0 by $L_0(>0)$. So, $x_1 = (x_0, u_0, L_0)$. We displace u_0 parallelly along $g(x_0, d(x_0), L)$ $(= S(x_0))$. For each $s\,(0 \leqq s \leqq L)$ let $u(s)$ be the vector at the point $(x_0, d(x_0), s)$ obtained by this displacement. By Lemma 3.1, we have

$$((x_0, d(x_0), s),\ u(s), L_0) \in S(x_0) \text{ for all } s\,(0 \leqq s \leqq L)$$

and $J(x_1) = x_1$. Hence for any integer m, $J^m(x_1) = x_1$ and $J^m \cdot g_0$ is a minimizing geodesic from x_0 to x_1. Any two of the geodesics $J^m \cdot g_0$ have x_0, x_1 only as common points. By Lemma 5.2, R must be homeomorphic onto spherical 2-space. And then it is easy to see that $I(x_0)$ consists of x_0, x_1 only.

2) The case where $x_1 \notin I(x_0)$. Now suppose that $J(x_1) \neq x_1$. By Lemma 3.1, $J(x_1) \in I(x_1)$, and $J(x_1) \in R - R^0$. Just as in 1), $I(x_1)$ consists of two points x_1 and $J(x_1)$ only. Hence $J^2(x_1) = x_1$ and $J^{2m}(x_1) = x_1$ for any integer m. On the other hand, the rotation angle of J^2 is 2θ or $2\pi - 2\theta$ according as $\theta \leqq \pi/2$ or $> \pi/2$, and $\pi/2\theta =$ irrational number by Lemma 5.2. Hence, x_1 is invariant by all the rotations at x_0. This contradicts with $J(x_1) \neq x_1$. So, $J(x_1) = x_1$ and $J^m(x_1) = x_1$. Accordingly, as in 1), R must be homeomorphic onto spherical 2-space.

Moreover from 1) and 2), it follows directly that $R - R^0$ consists of x_0, x_1 only. Therefore our lemma holds good.

THEOREM 4. *In a 3-dimensional RS-manifold M, suppose that among the S-orbits there are both of closed one and non-closed one. Then M is RS-diffeomcrphic onto a B_i-manifold* ($i = 1, 2, 3,$ or 4).

To prove this, we shall use the previous notations. By Lemma 5.4, $S(x_0) \cap R$ consists of at most two points. So the topology of R coincides with the relative one induced from M. This is seen by using Lemma 3.3. Accordingly by Lemma 3.4, M is of type III. If $R - R^0$ consists of a point alone, M is RS-diffeomorphic onto a B_1- or a B_2-manifold by Lemmas 1.2 and 5.3. If $R - R^0$ consists of two points, M is RS-diffeomorphic onto a B_3- or B_4-manifold by Lemmas 1.2 and 5.4. This completes the proof of our theorem.

6. 3-dimensional *RS*-manifold whose *S*-orbits are all closed. In such an RS-manifold M, take any $x_0 \in M$. Let L be the length of the closed S-orbit $S(x_0)$.

LEMMA 6.1. 1) *There is an R-neighborhood U_R of x_0 such that the map*

$$f : U_R \times [L] \rightarrow M \text{ defined by } f(x, t) = (x, d(x), t)$$

where $x \in U_R$ and $t \in [L]$, is an into-isometry provided that U_R is doubly treated in M as the images by f at $t = 0$, L ([2], p. 343). 2) *Among the S-orbits of M, there are S-orbits with the longest length* ([2], p, 346).

Here, the map which assigns to each $x \in U_R$ the point $f(x, L)$ becomes an isometry of U_R onto itself. So, the map f induces the congruent transformation f^* on $T_R(x_0)$. Relative to a suitable frame at x_0, f^* is a symmetry or a rotation whose rotation angle θ satisfies $\pi/\theta =$ rational number. We describe as the *main part* of M the subspace of M which consists of all the S-orbits with the longest length.

LEMMA 6.2. *The main part of M is a connected open submanifold dense in M and a maximal subspace which becomes a fibre bundle where each fibre is an S-orbit* ([2], p. 346).

THEOREM 5. *In a 3-dimensional RS-manifold M, suppose that all the S-orbits are closed. Then the main part M^0 of M is reduced to the principal S-bundle whose standard fibre is the additive group of mod L_0 (L_0 denotes the length of an S-orbit of M^0) and the R-field defines a connection in M^0. If $M - M^0 \neq 0$, then for an S-orbit $S \subset M - M^0$ there exists an S-diffeomorphism of a C_i-manifold ($i = 1$, or 2) into M which carries its central S-orbit to S.*

The former part is verified from Lemmas 3.3 and 6.2, by regarding the orientations of the S-orbits. The latter part follows from Lemmas 6.1 and 6.2.

7. Necessary and sufficient condition. In an S-manifold V, a *n.a.s.c.* means a necessary and sufficient condition that V admits a complete differentiable Riemannian metric leaving its S-field to be a parallel field.

THEOREM 6. *In a 3-dimensional S-manifold V, suppose that all the S-orbits are non-closed and that a certain S-orbit is not dense in M as subset. Then a n.a.s.c. is that V be S-diffeomorphic onto an RS-manifold of type I or an A_i-manifold ($i = 1, 2, 3$, or 4).*

THEOREM 7. *In a 3-dimensional S-manifold V, suppose that among the S-orbits there are both of closed one and non-closed one. Then a n, a.s.c. is that V be S-diffeomorphic onto a B_i-manifold ($i = 1, 2, 3$, or 4).*

Theorems 6 and 7 follow from Theorems 1, 2 and 4.

THEOREM 8. *In a 3-dimensional S-manifold V, suppose that all the S-orbits are closed and that V is compact. Then a n.a.s.c. is that*

1) *V be an almost principal S-bundle,*

2) *V admit an involutive differentiable field of tangent vector 2-subspaces transversal to the S-orbits which defines in the kernel V^0 a connection,*

3) *if $V - V^0 \neq 0$, for an S-orbit $S \subset V - V^0$ there exist an S-diffeomorphism of a C_i-manifold ($i = 1$ or 2) into V which carries its central S-orbit to S.*

The necessity of Theorem 8 is evident by Theorem 5. So we shall here prove the sufficiency. To do this, we call the field in 2) as the Q-field. Through each $x \in V$, there passes a maximal integral manifold of the Q-field. Let $Q(x)$ denote it. $Q(x)$ is called a Q-orbit of V. The quotient space of V, which is considered as the set of all the S-orbits, is denoted by B. Let π be the natural map of V onto B. Since V is compact and connected, so is B. At each $x \in V$, there is an admissible coordinate system (x^α) such that the system of equations $x^a =$ const. defines a subarc of an S-orbit and the equation $x^3 =$ const. defines a neighborhood of a Q-orbit. We can prove $\pi \cdot Q = B$ for any Q-orbit Q.

Since the standard fibre G of V^0 is a 1-dimensional torus group, G is regarded as the additive group of mod L for a suitable $L > 0$. So, each element of G will be represented by $a(0 \leqq a < L)$. Using this representation, we give G the metric under which the distance from 0 to a is a or $L - a$ according as $a \leqq L/2$ or $\geqq L/2$. For any $g \in G$, let R_g denote the right translation of V^0 by g.

A) The case where $V = V^0$. Then B becomes a compact differentiable manifold. Give B a differentiable Riemannian metric. Now, at each $b_0 \in B$ we can find an admissible coordinate neighborhood U_1 of b_0 and a coordinate function

$$\phi_1 : U_1 \times G \to \pi^{-1}(U_1)$$

$$(\phi_{1,b}(g) \equiv \phi_1(b, g) \in \pi^{-1}(b) \textit{ for all } b \in U_1,\ g \in G)$$

where ϕ_1 is a diffeomorphism. Take other pair (U_2, ϕ_2), $U_2 \ni b_0$, which has the same property as (U_1, ϕ_1). For $x, y \in \pi^{-1}(b_0)$ we put

$$g_1 = \phi_{1,b_0}^{-1}(x),\ h_1 = \phi_{1,b_0}^{-1}(y),\ g_2 = \phi_{2,b_0}^{-1}(x),\ h_2 = \phi_{2,b_0}^{-1}(y).$$

Then, $g_2 = g_{21}(b_0) + g_1$ and $h_2 = g_{21}(b_0) + h_1$ where $g_{21}(b_0) \equiv \phi_{2,b_0}^{-1} \cdot \phi_{1,b_0} \in G$. Hence, $g_2 - h_2 = g_1 - h_1$. This implies that if on the S-orbit $\pi^{-1}(b_0)$ we induce the metric from G by ϕ_{1,b_0}, it is independent of coordinate functions. Let us give such a metric (arc-length) to every S-orbit. On the other hand, each Q-orbit Q becomes a covering manifold of B whose covering map is π/Q. The map is locally a diffeomorphism, since π is differentiable. We shall induce on Q the Riemannian metric from B by π/Q. Thus all the Q-orbits become differetiable Riemannian manifolds. Take two Q-orbits Q_1, Q_2, then there is $g \in G$ such that $R_g \cdot Q_1 = Q_2$. This right translation R_g is regarded as an isometry and the arc-lengths from Q_1 to Q_2 along the S-orbits (under their orientations) are equal to one another. Accordingly, we can give V the differentiable Riemannian metric

$$ds^2 = ds_1^2 + ds_2^2 \tag{6. 1}$$

where ds_1, ds_2 denote the metrics of Q- and S-orbits respectively. This metric is complete, V being compact. It is now obvious that the metric is a required one.

B) The case where $V \neq V^0$. Put $B_0 = \pi \cdot V^0$. From 3) and the compactness of V, it follows that the subset $B - B^0$ consists of a finite number of points and a finite number of simple closed curves. Denote them by

$$b_j\ (j = 1,2,\ldots\ldots, j_0) \qquad \text{and} \qquad \beta_k\ (k = 1, 2,\ldots\ldots, k_0)$$

respectively. Indeed, all the curves β_k form the boundary of B. For each j, there exists an S-diffeomorphism f_j of a C_1-manifold V_j into V which carries its central S-orbit to $\pi^{-1}(b_j)$. Here, V_j will be considered as a manifold with the Euclidean metric which is naturally induced by its construction in § 2. Put $W_j = \pi \cdot f_j \cdot V_j$. We can see that a Euclidean metric is induced from V_j on $W_j - b_j$ by the map $\pi \cdot f_j$. In Euclidean 3-space, take the cylinder

$$D(\delta_0) \equiv \{(x, y, z) \mid x^2 + y^2 = 1,\ 0 \leqq z \leqq \delta\} \text{ for } \delta_0 > 0$$

where x, y, z are usual orthogonal coordinates. Let C^* be the boundary curve in $D(\delta_0)$ defined by $z = 0$. Then there are neighborhoods U_j of b_j and homeomorphisms h_k of $D(\delta_0)$ into B, which satisfy the following conditions:

a) $\overline{U}_j \subset W_j$ where $\overline{U}_j$ is the closure in B of U_j;

b) $h_k \cdot C^* = \beta_k$;

c) The compact subsets $\overline{U}_1, \ldots, \overline{U}_{j_0}, H_1, \ldots,\ H_{k_0} (H_k \equiv h_k \cdot D(\delta_0))$ do not intersect with one another.

Here if we choose a suitable $\delta_1 (0 < \delta_1 < \delta_0)$, we can find an open set[12] of B containing

$$W \equiv (\overline{U}_1 - b_1) \cup \cdots \cup (\overline{U}_{j_0} - b_{j_0}) \cup H_1' \cup \cdots \cup H_{k_0}'$$
$$(H_k' \equiv h_k \cdot D(\delta_1), \text{ using } h_k \text{ above})$$

and having a Euclidean metric, which leaves all β_k to be closed geodesics and which on each $\overline{U}_j - b_j$ coincides with that of $W_j - b_j$ induced from V_j. Let us give B, except the subset $\{b_j \mid j = 1, 2, \cdots, j_0\}$, a differentiable Riemannian metric which on W coincides with the Euclidean metric above. This is possible by theorems (pp. 25, 55) in [6]. Hence B^0 becomes a differentiable Riemannian manifold. So by the same manner as A) we can introduce onto V^0 the differentiable Riemannian metric which takes the same form as (6. 1). By regarding 3) of our theorem, this metric on V^0 will be concordantly extended over V. The metric thus extended becomes a complete differentiable Riemannian metric, since V is compact, and a metric which is required over V.

12) This need not be connected.

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