ON A FREE RESOLUTION OF A DIHEDRAL GROUP

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A free resolution of a group G is an exact sequence

$$\cdots \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \xrightarrow{d_{i-1}} \cdots \longrightarrow X_1 \xrightarrow{d_1} X_0 \xrightarrow{\varepsilon} Z \longrightarrow 0,$$

where X_i $(i=0,1,2,\cdots)$ are free G-modules, d_i,ε are G-homomorphisms, and Z is the ring of rational integers, on which G operates trivially.

For a cyclic group, we have the well-known simple free resolution. S. Takahashi [2] constructed a free resolution of abelian groups, and applied it to local number field theory, etc.

In this note, we construct a free resolution of a dihedral group, and decide n-dimensional cohomology groups for some modules. The author is grateful to Prof. T. Tannaka and Prof. H. Kuniyoshi who gave him this theme, encouragement and many suggestions.

1. Let G be a dihedral group, i.e. a group generated by s and t with relations $s^{2^l} = 1$, $t^2 = 1$, and $tst = s^{2^l-1}$, where $l \geqq 2$.

We introduce the notations:

$$\Delta_1 = 1 - s, \qquad \Delta_2 = 1 - t, \qquad \Delta_3 = 1 - st,$$
$$N_1 = 1 + s + \cdots + s^{2^l-1},\ N_2 = 1 + t,\ N_3 = 1 + st,$$
$\Lambda_0 = Z[s]$ group ring of the subgroup generated by s over Z,
$\Lambda = Z[G]$ group ring of G over Z.

Then, it follows $\Lambda = \Lambda_0 + \Lambda_0 t$ (direct), $N_3\Delta_1 = \Delta_1\Delta_2$, $\Delta_3\Delta_1 = \Delta_1 N_2$, $N_1\Delta_3 = \Delta_2 N_1$, and $N_1 N_3 = N_2 N_1$.

LEMMA 1. *We consider the following equations in* Λ

(1) $\quad XN_i = 0 \qquad (i = 1,2,3)$

(2) $\quad Y\Delta_i = 0 \qquad (i = 1,2,3)$

(3) $\quad X\Delta_1 + Y\Delta_2 = 0$

and

$$(4) \qquad \begin{cases} XN_1 + YN_3 = 0 \\ Y(-\Delta_1) + WN_2 = 0. \end{cases}$$

Then, the solutions of these equations are as follows:

solutions of (1) *are* $X = A\Delta_i \quad (i=1,2,3)$;
solutions of (2) *are* $Y = BN_i \quad (i=1,2,3)$;
solutions of (3) *are* $\begin{cases} X = AN_1 + BN_3 \\ Y = B(-\Delta_1) + CN_2; \end{cases}$
solutions of (4) *are* $\begin{cases} X = A\Delta_1 + BN_2 \\ Y = B(-N_1) + C\Delta_3 \\ W = C\Delta_1 + D\Delta_2, \end{cases}$

where A, B, C *and* D *are arbitrary elements of* Λ.

PROOF. We prove only the third case. Let

$$X = x_0 + x_1 t$$
$$Y = y_0 + y_1 t, \qquad x_i, y_i \in \Lambda_0 \ (i = 0,1).$$

Then, from

$$(x_0 + x_1 t)\Delta_1 = -(y_0 + y_1 t)\Delta_2$$

it follows that

$$(5) \qquad x_0\Delta_1 + x_1(1 - s^{2^l-1}) = 0.$$

Hence we have

$$x_0 + x_1 t \equiv x_1 t(1 + st) \text{ mod. } \Lambda N_1.$$

Therefore, there exist elements A and B in Λ such that

$$x_0 + x_1 t = AN_1 + BN_3.$$

Conversely,

$$X = AN_1 + BN_3$$

satisfies (5) for arbitrary A and B in Λ. Then, we have

$$Y = B(-\Delta_1) + CN_2.$$

As for the last equation we can solve it similarly, and first two cases are trivial. q.e.d.

2. Let X_i be a Λ-free module with a basis $\{a_i^j\}$,
$X_i = \Lambda a_i^1 + \Lambda a_i^2 + \cdots + \Lambda a_i^{i+1}$.

We now define G-homomorphisms

$$D_j(i)\ (i \geqq 2),\ \text{and}\ D'_j(i)\ (i \geqq 3)\colon X_i \to X_{i-1}, \qquad j = 1, 2, 3, 4,$$

as follows:

$D_1(i)$: $a_i^1 \to \Delta_1 a_{i-1}^1$, $a_i^2 \to \Delta_2 a_{i-1}^1 + (-N_1)a_{i-1}^2$, $a_i^3 \to N_3 a_{i-1}^2$, $a_i^k \to 0$ for $k>3$

$D_2(i)$: $a_i^1 \to N_1 a_{i-1}^1$, $a_i^2 \to N_3 a_{i-1}^1 + (-\Delta_1)a_{i-1}^2$, $a_i^3 \to N_2 a_{i-1}^2$, $a_i^k \to 0$ for $k>3$.

$D_3(i)$: $a_i^1 \to \Delta_1 a_{i-1}^1$, $a_i^2 \to N_2 a_{i-1}^1 + (-N_1)a_{i-1}^2$, $a_i^3 \to \Delta_3 a_{i-1}^2$, $a_i^k \to 0$ for $k>3$

$D_4(i)$: $a_i^1 \to N_1 a_{i-1}^1$, $a_i^2 \to \Delta_3 a_{i-1}^1 + (-\Delta_1)a_{i-1}^2$, $a_i^3 \to \Delta_2 a_{i-1}^2$, $a_i^k \to 0$ for $k>3$.

$D'_1(i)$: $a_i^3 \to \Delta_1 a_{i-1}^3$, $a^4 \to N_2 a_{i-1}^3$, other $a_i^k \to 0$

$D'_2(i)$: $a_i^3 \to N_1 a_{i-1}^3$, $a_i^4 \to \Delta_3 a_{i-1}$, other $a_i^k \to 0$

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$D'_4(i)$: $a_i^3 \to N_1 a_{i-1}^3$, $a_i^4 \to N_3 a_{i-1}^3$, other $a_i^k \to 0$

Then we have

LEMMA 2. *The kernel of the mapping $D_j(i)$ in $\Lambda a_i^1 + \Lambda a_i^2 + \Lambda a_i^3$ coincides with the image of $D_q(i+1) + D'_q(i+1)$, where $q \equiv j+1 (mod.\ 4)$.*

Moreover, if

$$Xa_i^1 + Ya_i^2 + Wa_i^3$$

belongs to the kernel of $D_j(i)$ and if

$$Wa_i^3 = D'_q(i+1)\ (Ca_{i+1}^3 + Da_{i+1}^4),$$

then we can find A and B in Λ such that

$$Xa_i^1 + Ya_i^2 + Wa_i^3 = \{D_q(i+1) + D'_q(i+1)\}(Aa_{i+1}^1 + Ba_{i+1}^2 + Ca_{i+1}^3 + Da_{i+1}^4).$$

PROOF. In fact,

$$D_1(i)(Xa_i^1 + Ya_i^2 + Wa_i^3) = 0$$

holds if and only if

$$(6) \qquad X\Delta_1 + Y\Delta_2 = 0$$

and

$$(7) \qquad Y(-N_1) + WN_3 = 0$$

hold. Applying Lemma 1 to (6), we have

$$X = AN_1 + BN_3 \text{ and } Y = B(-\Delta_1) + CN_2,$$

and from (7) we have

$$W = CN_1 + D\Delta_3.$$

Thus,

$$Xa_i^1 + Ya_i^2 + Wa_i^3 = \{D_2(i+1) + D_2'(i+1)\}(Aa_{i+1}^1 + Ba_{i+1}^2 + Ca_{i+1}^3 + Da_{i+1}^4).$$

Moreover, if X,Y and W satisfy (6) and (7) with

$$W = CN_1 + D\Delta_3,$$

then, putting W into (7), we have

$$Y = B(-\Delta_1) + CN_2,$$

and, by the same process,

$$X = AN_1 + BN_3.$$

For $j=2, 3$ and 4 we can prove the lemma similarly. q.e.d.

We remark here that $D_j'(i)$ maps Λa_i^3 and Λa_i^4 into Λa_{i-1}^3 just as $D_r(i)$ maps Λa_i^1 and Λa_i^2 into Λa_{i-1}^1, respectively, where $r \equiv j+2 \pmod{4}$.

3. Now, we construct a free resolution for a dihedral group G. Let X_i ($i = 0,1,2,\cdots$) be G-free modules described in **2.** ε is defined by

$$\varepsilon(m\sigma a_0^1) = m \text{ for } m \in Z \text{ and } \sigma \in G.$$

We define d_1 and d_2 as follows:

$$d_1(Xa_1^1) = X\Delta_1 a_0^1, \qquad d_1(X\ a_1^2) = X\Delta_2 a_0^1$$

and

$$d_2(Xa_2^1) = XN_1a_1^1, \qquad d_2(Xa_2^2) = XN_3a_1^1 + X(-\Delta_1)a_1^2$$
$$d_2(Xa_2^3) = XN_2a_1^2,$$

where $X \in \Lambda$. These mappings are clearly G-homomorphisms.

For $i \geqq 3$, we put

$$d_i = D_j(i) + d'_{i-2}\,,$$

where $j \equiv i$ (mod.4), and d'_{i-2} is the G-homomorphism which maps

$$\Lambda a_i^3 + \Lambda a_i^4 + \cdots \quad \text{to} \quad \Lambda a_{i-1}^3 + \Lambda a_{i-1}^4 + \cdots$$

just as d_{i-2} maps

$$\Lambda a_{i-2}^1 + \Lambda a_{i-2}^2 + \cdots \quad \text{to} \quad \Lambda a_{i-3}^1 + \Lambda a_{i-3}^2 + \cdots$$

PROPOSITION 1. *The above defined* $\{X_i, d_i, \varepsilon\}$ *gives a free resolution of a dihedral group* G.

PROOF. The kernel of ε is clearly the image of d_1. For $i=1$ and $i=2$ the exactness follows from (3) and (4) of Lemma 1, respectively. When $i=3$,

$$d_3(Xa_3^1 + Ya_3^2 + Wa_3^3 + Va_3^4) = 0$$

holds if and only if

$$D_3(3)(Xa_3^1 + Ya_3^2 + Wa_3^3) = 0 \tag{8}$$

and

$$d'_1(3)\,(Wa_3^3 + Va_3^4) = 0 \tag{9}$$

hold.

As we have already showed the exactness for $i=1$, there exist from (9) C, D and E in Λ such that

$$Wa_3^3 + Va_3^4 = d'_2\,(Ca_4^3 + Da_4^4 + Ea_4^5),$$

and then

$$Wa_3^3 = D'_4(4)(Ca_4^3 + Da_4^4) \tag{10}$$

as we remarked in **2**. Applying Lemma 2 to (8) and (10), we can find A,B in Λ such that

$$Xa_3^1 + Ya_3^2 + Wa_3^3 = \{D_4(4) + D'_4(4)\}(Aa_4^1 + Ba_4^2 + Ca_4^3 + Da_4^4).$$

Thus,

$$Xa_3^1 + Ya_3^2 + Wa_3^3 + Va_3^4 = \{D_4(4) + d_2'\}(Aa_4^1 + Ba_4^2 + Ca_4^3 + Da_4^4 + Ea_4^5).$$

We now proceed by induction on i. Assume that the exactness has been proved for $n \leqq i-1$.

Let

$$\alpha = Xa_i^1 + Ya_i^2 + Wa_i^3 + \cdots + Va_i^{i+1}$$

be an element of Ker $d_i \subset X_i$. Then from $d_i(\alpha)=0$ we have

(11) $$D_j(i)(Xa_i^1 + Ya_i^2 + Wa_i^3) = 0, \quad j \equiv i \ (\text{mod. } 4)$$

and

(12) $$d_{i-2}' \ (Wa_i^3 + \cdots + Va_i^{i+1}) = 0.$$

From (12), by assumption, we can find $C, D, \cdots, E$ in Λ such that

(13) $$Wa_i^3 + \cdots + Va_i^{i+1} = d_{i-1}' \ (Ca_{i+1}^3 + Da_{i+1}^4 + \cdots + Ea_{i+1}^{i+2}).$$

Since

$$d_{i-1}' = D_q(i+1) \quad \text{on } \Lambda a_{3+1}^i + \Lambda a_{i+1}^4, \ q \equiv i+1 \ (\text{mod.} 4),$$

we have from (13)

$$Wa_i^3 = D_q'(i+1)(Ca_{i+1}^3 + Da_{i+1}^4).$$

From this and (11), by Lemma 2, there exist A and B in Λ such that

$$\begin{aligned} &Xa_i^1 + Ya_i^2 + Wa_i^3 \\ &\quad = \{D_q(i+1) + D_q'(i+1)\}(Aa_{i+1}^1 + Ba_{i+1}^2 + Ca_{i+1}^3 + Da_{i+1}^4). \end{aligned}$$

Thus we have

$$\alpha = \{D_q(i+1) + d_{i-1}'\}(Aa_{i+1}^1 + Ba_{i+1}^2 + Ca_{i+1}^3 + Da_{i+1}^4 + \cdots + Ea_{i+1}^{i+2}),$$

where $D_q(i+1) + d_{i-1}' = d_{i+1}$, $q \equiv i+1$ (mod. 4). Finally,

$$d_{i-1}d_i(\alpha) = 0 \text{ for arbitrary } \alpha \in X_i$$

follows immediately from Lemma 2 and the assumption of induction. q.e.d.

4. By our exact sequence, we define cohomology groups as usual. Let A be a G-module, and $A_r = \text{Hom}^G(X_r, A)$ be the group of all G-homomorphisms from X_r to A.

Then, we can consider $f \in A_r$ to be a vector $(\alpha_1, \alpha_2, \cdots, \alpha_{r+1})$ with elements in A, where $f(a_r^1) = \alpha_1$, $f(a_r^2) = \alpha_2$, $\cdots, f(a_r^{r+1}) = \alpha_{r+1}$. And coboundary operator $\delta_r : A_r \to A_{r+1}$ is translated into a mapping between additive groups of vectors.

From this we have

PROPOSITION 2. *If A is a Z-torsion free module, on which G operates trivially,*

$$H^0(G,A) = A/2^{l+1}A$$
$$H^{4n+1}(G,A) = 2n\cdot(A/2A)$$
$$H^{4n+2}(G,A) = 2(n+1)\cdot(A/2A)$$
$$H^{4n+3}(G,A) = (2n+1)\cdot(A/2A)$$
$$H^{4n+4}(G,A) = A/2^lA + 2(n+1)\cdot(A/2A),$$

where $+$ means direct sum, and $m\cdot B$ is direct sum of m copies of module B.

PROOF. First of all, we remark the following fact: consider a subgroup

$$C = \{(\alpha_1, \alpha_2);\ \alpha_1, \alpha_2 \in A \text{ and } 2\alpha_1 - 2^l\alpha_2 = 0\}$$

of additive group of vectors and a subgroup

$$D = \{(2^l\alpha, 2\alpha);\ \alpha \in A\}$$

of C, then

$$C/D \cong A/2A.$$

Using these facts, the results follow from direct computations, for $n=0,1,2$.

Then we have from the proof of Theorem 1,

$$H^n(G,A) = A/2^lA + H^{n-2}(G,A) \quad \text{(direct) for } n \equiv 0 \text{ (mod.4)}$$
$$H^n(G,A) = A/2A + H^{n-2}(G,A) \quad \text{(direct) for } n \equiv 1 \text{ (mod.4)}$$
$$H^n(G,A) = A/2A + \overline{H}^{n-2}(G,A) \quad \text{(direct) for } n \equiv 2 \text{ (mod.4)}$$
$$H^n(G,A) = A/2A + H^{n-2}(G,A) \quad \text{(direct) for } n \equiv 3 \text{ (mod.4)},$$

where $\overline{H}^{n-2}(G,A)$ is the group obtained by replacing only the first component of $H^{n-2}(G,A)$ by $A/2A$. From these equations we have the results. q.e.d.

COROLLARY 1. *Let Z be the ring of rational integers, then*

$$H^0(G,Z) = Z/2^{l+1}Z,$$
$$H^{4n}(G,Z) = Z/2^lZ + 2n\cdot(Z/2Z),$$
$$H^{2n+1}(G,Z) = n\cdot(Z/2Z),$$

and

$$H^{4n+2}(G,Z) = 2(n+1)\cdot(Z/2Z).$$

Let k be a p-adic number field and K its Galois extension. T. Tannaka

(3) considered the structure of the group $G(k/K) = \{\alpha; \alpha \in K, N_{K/k}(\alpha) = 1\}$ and decided it using a suitable factor set and $K.^{1-\sigma}$ H. Kuniyoshi (1) decided the structure of $G(k/K)$ in another form when the Galois group G is abelian. When G is a dihedral group, we can decide the group $G(k/K)$ as analogous form in [1].

COROLLARY 2. *Let k be a p-adic number field, and K be a normal extension, of which Galois group is a dihedral group. Then*

$$G(k/K)/K^{1-\sigma} = Z/2Z,$$

where

$$K^{1-\sigma} = \{\Pi\alpha^{1-\sigma};\ \alpha \in K,\ \sigma \in G\}.$$

PROOF. The left-hand side is $H^{-1}(G,K^*)$, where K^* is the multiplicative group of K. And this group is isomorphic with $H^{-3}(G,Z)$ by Tate's theorem, while

$$H^{-3}(G,Z) = H^{3}(G,Z)$$

and the last group is isomorphic to $Z/2Z$ by Corollary 1. q.e.d.

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