# A CHARACTERIZATION OF THE ALGEBRA OF GENERALIZED ANALYTIC FUNCTIONS

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**1.** A complex-valued function $f$ on the unit circle of the complex plane belongs to $H^1$ if $f \in L^1$ and $\int_0^{2\pi} f(e^{i\theta})\, e^{in\theta}\, d\theta = 0$ for $n=1,2,3,\cdots$. In general, let $A$ be a logmodular algebra on a compact Hausdorff space $X$ and $m$ a representing measure for $A$ ([4]). $H^1(dm)$ is defined as the $L^1(dm)$-closure of $A$ or, equivalently, as the class of functions such that $f \in L^1(dm)$ and $\int_X fg\,dm = 0$ for every $g \in A_m$ where $A_m = \{g \mid g \in A, \int_X g\,dm = 0\}$. Clearly, the definition of $H^1(dm)$ is weaker than that of $H^1$ for the unit circle. This is because $A_{d\theta} = \{g \mid g \in A_0, \int_0^{2\pi} f(e^{i\theta})\, d\theta = 0\}$, where $A_0$ denotes the algebra of all continuous functions on the unit circle that admit analytic extensions in the disk, is spanned by $\{e^{in\theta} \mid n = 1,2,3,\cdots\}$, while existence of such a family of functions in $A_m$ is not guaranteed.

We are interested in these circumstances and intend to show that existence of a certain family of functions which spanns $A_m$ (or $A$) characterizes an algebra of all generalized analytic functions on a compact group among function algebras on a compact space $X$. We discuss this problem in §3. §4 is devoted to another considerations based on the fundamental Lemma 1.

**2.** Let $\Gamma$ be a discrete abelian group containing a subsemigroup $\Gamma_+$ such that $\Gamma_+$ contains the identity and generates $\Gamma$; let $G$ be the dual group of $\Gamma$ and $\mu$ the normalized Haar measure on $G$. (We shall throughout exclude the case where $\Gamma_+=\Gamma$). Let $A(G) = \{f \mid f \in C(G), \hat{f}(\gamma) = \int_G f(\varphi)\overline{(\varphi,\gamma)}\,d\mu(\varphi) = 0$ for all $\gamma \overline{\in} \Gamma_+\}$ where $C(G)$ is the algebra of all complex-valued continuous functions on $G$. Then by Theorem 4. 1 in [3], $A(G)$ can be regarded as the algebra of all generalized analytic functions with respect to $\Gamma_+$*). It is clear that $\Gamma_+ \subset A(G)$. If $\Gamma_+ \cap \Gamma_+^{-1} = (1)$ (1 denotes the identity), then $\Gamma$ is partially ordered; if in addition $\Gamma = \Gamma_+ \cup \Gamma_+^{-1}$ then $\Gamma$ is linearly ordered. The linearly

*) Under these assumptions, we shall call $A(G)$ the *general case.*

ordered case is treated in [5] and functions of $A(G)$ are called of analytic type. Examples are well known for both partially and linearly ordered cases. As for the (non-trivial) general case where $\Gamma_+ \cup \Gamma_+^{-1} \neq \Gamma$ and $\Gamma_+ \cap \Gamma_+^{-1} \neq (1)$, we give an example as follows: $\Gamma =$ the set of all lattice points in the plane, $\Gamma_+ = \{(m,n) \mid m = 0, 2, 3, 4, \cdots;\ n = 0, \pm 1, \pm 2, \pm 3, \cdots\}$.

By a functin algebra on a compact Hausdorff space $X$ we shall understand a uniformly closed proper subalgebra of $C(X)$ which contains the constants and separates the points of $X$. $A(G)$ is a function algebra on $G$, and even a Dirichlet algebra when $\Gamma = \Gamma_+ \cup \Gamma_+^{-1}$. We denote by $A_m$ the hyperplane of $A$ associated with a probability measure $m$ on $X$, i.e. $A_m = \{f \mid f \in A, \int_X f\,dm = 0\}$. If $m$ is a multiplicative functional on $A$, that is, $\int_X fg\,dm = \int_X f\,dm \int_X g\,dm$ for $f, g \in A$, $m$ is said to be a representing measure.

**3.** The following is proved in [5] (p. 217) with the assumption that $\Gamma$ is linearly ordered. Although the same method applies in the general case, we give the proof by direct approximation arguments.

LEMMA 1. *Let $\Gamma$, $\Gamma_+$, $G$ and $A(G)$ be as defined in* §2. *Let* $A_\mu(G) = \{f \mid f \in A(G), \int_G f(\varphi)\,d\mu(\varphi) = 0\}$, *then*

$$A(G) = \textit{closed linear span of } \Gamma_+, \textit{ and}$$

$$A_\mu(G) = \textit{closed linear span of } \Gamma_+ \backslash \{1\}.$$

PROOF. It is sufficient to show that any $f \in A(G)$, $f \neq 0$, is uniformly approximated by trigonometric polynomials consisting of elements of $\Gamma_+$. Choose a continuous function $u$ such that $\|f * u - f\|_\infty < \varepsilon/2$, and for this $u$, a trigonometic polynomial $v(\varphi) = \sum b(\gamma)(\varphi, \gamma)$ such that $\|u - v\|_\infty < \varepsilon/2\|f\|_\infty$. Then we have $\|f * v - f\|_\infty < \varepsilon$. Clearly $f * v \in C(G)$ and

$$(f * v)(\varphi) = \sum b(\gamma)\hat{f}(\gamma)(\varphi, \gamma) = \sum_{\gamma \in \Gamma_+} a(\gamma)(\varphi, \gamma).$$

The case for $A_\mu(G)$ is similar.

LEMMA 2. *Let $\Gamma_+$ be a semi-group contained in a commutative group $\Gamma$ and $1 \in \Gamma_+$. Then the following are equivalent*:

(1) $\quad \Gamma_+ \cap \Gamma_+^{-1} = (1)$

(2) $\quad \Gamma_+ \setminus \{1\}$ *is a semi-group.*

PROOF. Suppose that $\Gamma_+ \cap \Gamma_+^{-1}$ contains a $\gamma$, $\gamma \neq 1$. Then $\gamma, \gamma^{-1} \in \Gamma_+ \backslash \{1\}$, hence $\Gamma_+ \backslash \{1\}$ cannot be a semi-group. Conversely, let $\Gamma_+ \backslash \{1\}$ be not a semi-group. Then, since $\Gamma_+$ is a semi-group, there exist $\gamma_1$, $\gamma_2$ in $\Gamma_+$ such that $\gamma_1 \neq 1$, $\gamma_2 \neq 1$ and $\gamma_1 \gamma_2 = 1$. Thus, $\gamma_1^{-1} \in \Gamma_+$, so $\Gamma_+ \cap \Gamma_+^{-1} \neq (1)$.

From the above lemmas it is easily seen that if $\Gamma_+ \cap \Gamma_+^{-1} = (1)$ then the normalized Haar measure $\mu$ on $G$ is a representing measure for $A(G)$.

THEOREM 1. *Let $A$ be a function algebra on $X$. If there exist a probability measure $m$ on $X$ and a family $P$ of functions in $A_m$ each of which has the modulus* 1 *everywhere on $X$ and if they satisfy*

(a) $P \cdot P \subset P$ *(semi-group under the pointwise multiplication),*

(b) $\int_X \gamma_1(x) \overline{\gamma_2(x)}\, dm(x) = 0$ *for* $\gamma_1, \gamma_2 \in P$, $\gamma_1 \neq \gamma_2$ *(orthogonality),*

(c) $A_m =$ *closed linear span of $P$,*

*then $X$ is homeomorphic to a compact abelian group $G$ with $\widehat{G}$ partially ordered and $A$ is isometric and isomorphic to an $A(G)$ with respect to a semi-group of $\widehat{G}$. The converse is also true.*

PROOF. First, let $G$, $\Gamma$, $\Gamma_+$ and $A(G)$ with respect to $\Gamma_+$ be given and $\Gamma_+ \cap \Gamma_+^{-1} = (1)$. We take $\mu$ and $\Gamma_+ \backslash \{1\}$ for $m$ and $P$. From Lemmas 1 and 2, it is clear that they satisfy all the requirements.

Next, we shall show the sufficiency of the conditions. Let $\Gamma_+ = P \cup \{1\}$ and $\Gamma = \Gamma_+ \cdot \Gamma_+^{-1}$ (1 denotes the constant function on $X$ and "—" means the complex conjugates of functions), then $\Gamma$ is a commutative group generated by $\Gamma_+$. Since $P$ is a semi-group, Lemma 2 implies that $\Gamma_+ \cap \Gamma_+^{-1} = (1)$. We regard $\Gamma$ as equipped with the discrete topology and denote by $G$ the dual of $\Gamma$. For arbitrary $x \in X$ we define a function $\varphi_x$ on $\Gamma$ by $\varphi_x(\gamma) = \gamma(x)$, $\gamma \in \Gamma$. It is easily seen that $\varphi_x$ is a character of $\Gamma$, so that $\varphi_x \in G$. Thus, we obtain a mapping $\Phi$ of $X$ into $G$ such that $\Phi(x) = \varphi_x$, $x \in X$. Since $A$ separates the points of $X$ and $A$ is represented as the closed linear span of $\Gamma_+$ by (c), $\Gamma_+$ itself separates the points of $X$. This implies that $\Phi$ is one-to-one. Moreover $\Phi$ is continuous, because $G$ has the weak topology by members of $\Gamma$. $X$ is thus embedded in $G$.

Now, let $\nu = m\Phi^{-1}$, then $\nu$ is a positive measure on $G$ the support of which is contained in $\Phi(X)$. For $\gamma \in \Gamma$ we have $\gamma = \gamma_1 \overline{\gamma_2}$, $\gamma_1, \gamma_2 \in \Gamma_+$, so by (b)

$$\int_G (\varphi_x, \gamma)\, d\nu(\varphi) = \int_X \gamma_1(x) \overline{\gamma_2(x)}\, dm(x) = 0 \qquad \text{for} \qquad \gamma \neq 1$$

and

$$\int_G d\nu(\varphi) = \int_X dm(x) = 1\,.$$

It follows that $\nu = \mu$, the nomalized Haar measure on $G$, and this implies that $\Phi(X) = G$.

Let $A(G)$ be the algebra of all generalized analytic functions with respect to $\Gamma_+$. If we define the mapping $\Psi$ by $\Psi(f) = f \circ \Phi^{-1}$, $f \in C(X)$, it is easily seen from Lemma 1 and (c) that $\Psi(A) = A(G)$. That $\Psi(A_m) = A_\mu(G)$ is also clear. Thus, the proof is completed (we have also proved that $m$ becomes a representing measure for $A$).

If we require $\Gamma$ to be linearly ordered in Theorem 1 it must be satisfied that $P \cdot P^{-1} \setminus \{1\} \subset P \cup P^{-1}$, since this is equivalent to $\Gamma = \Gamma_+ \cup \Gamma_+^{-1}$. Furthermore, it is well known that $\Gamma$ is linearly ordered if and only if $G = \widehat{\Gamma}$ is connected ([5]). The linearly ordered analogue of Theorem 1 is the following

THEOREM 2. *Let $A$ be a function algebra, and let $m$ and $P$ satisfy the following conditions*:

(a) $P \cdot P \subset P$.

(b′) *For* $\gamma_1, \gamma_2 \in P$ $(\gamma_1 \neq \gamma_2)$, $\gamma_1 \gamma_2 \in P$ *or* $\gamma_1 \bar{\gamma}_2 \in P$.

(c) $A_m =$ *closed linear span of* $P$.

*Then $X$ is homeomorphic to a compact connected group $G$ and $A$ is represented as $A(G)$ with respect to a semi-group $\Gamma_+$ which defines a linear order on $\Gamma$.*

*Furtheremore,* (c) *can be equivalently replaced by the condition*:

(c′) *$m$ is a representing measure for $A$ and $P$ separates the points of $X$.*

*The converse is also true.*

PROOF. We have only to prove that (a), (b′) and (c′) imply the conclusion above, since other implications are easily verified. We conclude from (a), (b′) and (c′) that $\Phi(X) = G$ as in the proof of Theorem 1, and $\Gamma$ is linearly ordered by (b′), so $G$ is connected. Lemma 1 implies that $\Psi(A) \supset A(G)$. For the converse, let $f \in A$. If $\gamma \notin \Gamma_+$ $(\Gamma_+ = P \cup \{1\})$ then $\bar{\gamma} \in P$ by (b′), so we have

$$\int_G (f \circ \Phi^{-1})(\varphi)\overline{(\varphi, \gamma)}\, d\mu(\varphi) = \int_X f(x)\, \overline{\gamma(x)}\, dm(x) = 0\,,$$

since $m$ is a representing measure and $f\bar{\gamma} \in A_m$. Thus, $\Psi(f) \in A(G)$, which completes the proof.

We have treated above, merely for the sake of convenience, the case in which $\Gamma_+ \cap \Gamma_+^{-1} = (1)$. As for the general case, the analogous result is stated as follows: *Let $m$ be a probability measure on $X$ and $P$ a family of functions in $A$ each of which is of modulus 1 on $X$ such that they satisfy* (a′) $1 \in P$, $P \cdot P \subset P$, (b) *for* $\gamma_1, \gamma_2 \in P$, $(\gamma_1 \neq \gamma_2)$, $\int_X \gamma_1 \bar{\gamma}_2 \, dm = 0$ *and* (c″) $A =$ *closed linear span of $P$. Then $X$ is considered as a compact group and $A$ is represented as an $A(G)$, and vice versa.* Theorem 1 is nothing but the case where $P \setminus \{1\}$ itself is a semi-group.

4. In this section we shall discuss an application of Lemma 1. Let $A$ be given as a logmodular algebra in Theorem 1. Let $H^2(dm) = L^2(dm)$-closure of $A$, and $H_m^2(dm) = L^2(dm)$-closure of $A_m$, where $m$ is a given measure in Theorem 1 (and this is in fact a representing measure). Then $L^2(dm)$ is decomposed as $L^2(dm) = H^2(dm) \oplus \overline{H}_m^2(dm)$ ([4]). The conclusion of Theorem 1 and Lemma 1 imply that $L^2(G) = L^2(dm) = L^2$-span of $\Gamma_+ \oplus L^2$-span of $P^{-1}$. Since the character group constitutes a complete orthonormal family of $L^2(G)$, it follows that $\Gamma$ coincides with $\Gamma_+ \cup \Gamma_+^{-1}$. Thus, for a logmodular algebra $A$, Theorem 1 is reduced to Theorem 2. If in particular the algebra $A(G)$ of generalized analytic functions on $G$ is logmodular, it is necessarily a Dirichlet algebra.

The decomposition of $L^2(dm)$ is possible for any function algebra which has the property that every complex homomorphism has a unique representing measure ([6]). It follows as above that if $A(G)$ has this property $\Gamma = \Gamma_+ \cup \Gamma_+^{-1}$ holds. This also holds in the general case:

THEOREM 3. *For an algebra $A(G)$ with respect to a semi-group $\Gamma_+$, the following are equivalent:*

(1) $\Gamma_+ \cap \Gamma_+^{-1} = \Gamma$.

(2) *$A(G)$ is a Dirichlet algebra.*

(3) *$A(G)$ is a logmodular algebra.*

(4) *Every homomorphism of $A(G)$ admits a unique representing measure.*

PROOF. It is sufficient to prove that (4) implies (1) (a Dirichlet algebra is always logmodular). For the case $\Gamma_+ \cap \Gamma_+^{-1} = (1)$, this implication is already proved, so we shall reduce the general case to it*). Let $\Gamma_0 = \Gamma_+ \cap \Gamma_+^{-1}$, $\Lambda = \Gamma/\Gamma_0$

*) The following argument was suggested by that of [3].

and $\Lambda_+ = \Gamma_+/\Gamma_0$. $\Lambda_+$ is a semi-group generating $\Lambda$ and $\Lambda_+ \cap \Lambda_+^{-1} = (\dot{1})$ ($\dot{1}$ means the identity of $\Lambda$). Let $\mathfrak{G} = \widehat{\Lambda}$ and let $A(\mathfrak{G})$ be the algebra of all generalized analytic functions on $\mathfrak{G}$ with respect to $\Lambda_+$. We denote by $\mathfrak{A}(G)$ the collection of all functions in $C(G)$ whose restrictions to $\mathfrak{G}$ belong to $A(\mathfrak{G})$. This is cleary a uniformly closed subalgebra of $C(G)$. We show that $A(G) \subset \mathfrak{A}(G)$. To see this, it is sufficient to verify that $\Gamma_+ \subset \mathfrak{A}(G)$. Let $\mu_g$ denote the normalized Haar measure on $\mathfrak{G}$. Let $\gamma_0 \in \Gamma_+$ and $\dot{\gamma} \bar{\in} \Lambda_+$.

$$(\gamma_0 | \mathfrak{G})^\wedge(\dot{\gamma}) = \int_{\mathfrak{G}} (\varphi, \gamma_0)\overline{(\varphi, \dot{\gamma})}\, d\mu_{\mathfrak{G}}(\varphi)$$

$$= \int_{\mathfrak{G}} (\varphi, \dot{\gamma}_0\dot{\gamma}^{-1})\, d\mu_{\mathfrak{G}}(\varphi) = 0,$$

since $\dot{\gamma}_0 \neq \dot{\gamma}$. Thus, $\gamma_0 \in \mathfrak{A}(G)$.

Now, let $A(G)$ satisfy the condition (4). Let $h$ be any homomorphism of $A(\mathfrak{G})$ and let $m_1, m_2$ be its representing measures. Then for any $f \in A(G)$, we have

$$h(f | \mathfrak{G}) = \int_{\mathfrak{G}} f(\varphi)\, dm_1(\varphi) = \int_{\mathfrak{G}} f(\varphi)\, dm_2(\varphi)\,.$$

Let $h'(f) = h(f | \mathfrak{G})$, $f \in A(G)$, then $h'$ is a homomorphism of $A(G)$ and $m_1, m_2$ are its representing measures, hence it follows that $m_1 = m_2$. Thus we have $\Lambda = \Lambda_+ \cup \Lambda_+^{-1}$ so $\Gamma = \Gamma_+ \cup \Gamma_+^{-1}$ which completes the proof.

REMARK. An example is constructed in [1] which shows that a homomorphism of a function algebra (and even an $A(G)$) admits non-unique representing measures. From above considerations we can easily derive a set of simple examples given by $A_k =$ closed linear span of $\{1, e^{i(k+1)\theta}, e^{i(k+2)\theta}, \cdots\}$, $k = 1, 2, 3, \cdots$.

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