NOTES ON THE DUALITY THEOREM OF NON-COMMUTATIVE NON-COMPACT TOPOLOGICAL GROUPS

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In a recent note [3], Kazuo Suzuki has given an interesting application of the Tannaka duality theory for compact groups (Cf. [4]). If G_1 and G_2 are compact groups, let $\mathrm{Hom}(G_1, G_2)$ denote the set of homomorphisms of G_1 into G_2. Let G_1^* (respectively G_2^*) denote the set of finite dimensional representations of G_1 (respectively G_2). Let $\mathrm{Hom}(G_2^*, G_1^*)$ denote the set of maps of G_2^* into G_1^* which preserve the representation theoretic operations such as the direct sum and the tensor product. Then Kazuo Suzuki shows that there is a canonical one-to-one correspondence between $\mathrm{Hom}(G_1, G_2)$ and $\mathrm{Hom}(G_2^*, G_1^*)$. Natural topologies may be specified in $\mathrm{Hom}(G_1, G_2)$ and $\mathrm{Hom}(G_2^*, G_1^*)$ such that this correspondence is a homeomorphism.

Recently the author has developed a theory [2] which may be looked upon as a partial generalization of the Tannaka duality theory, to the case of infinite dimensional representations of separable locally compact groups. The purpose of this note is to illustrate this theory by obtaining an analogue of K. Suzuki's theorem for non-compact groups.

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1. Following the procedure of Kazuo Suzuki, we begin by outlining the duality theory on which our result will be based.

Let G denote a separable locally compact group. Let $\mathfrak{H}$ denote a fixed infinite dimensional separable Hilbert space. Let G^c denote the set of all strongly continuous unitary representations of G, with representation space $\mathfrak{H}$. In this theory G^c will play the role of our "dual object" and will be called the *concrete dual of* G.

(Remark: The consideration of G^c also plays a fundamental role in decomposition theory [1]. The use of a fixed Hilbert space is needed to make the collection of representations a well defined set. An analogous procedure should be used even in the Tannaka theory in order that G^* be a well defined set.)

By a *representation J of G^c*, we shall mean a mapping J of G^c into the invertible operators on $\mathfrak{H}$, satisfying the following properties:

(i) $\sup\{\|J(L)\| : L \in G^c\} < +\infty$

(ii) *If M and N are elements of G^c and if U is an isometric linear mapping of the representation space of $M \oplus N$ onto $\mathfrak{H}$, then*

$$J(U(M \oplus N)U^{-1}) = U(J(M) \oplus J(N))U^{-1}.$$

(iii) *If M and N are elements of G^c and if U is an isometric linear mapping of the representation space of $M \otimes N$ onto $\mathfrak{H}$, then*

$$J(U(M \otimes N)U^{-1}) = U(J(M) \otimes J(N))U^{-1}.$$

In short, a representation of G^c is a bounded mapping of G^c into the collection of invertibie operators on $\mathfrak{H}$, which preserves direct sums and tensor products. Let $\mathfrak{U} = \mathfrak{U}(G)$ denote the set of all such representations of G^c. We define multiplication in $\mathfrak{U}$ point-wise. Thus if J and K are two elements of $\mathfrak{U}$, then JK is defined to be that representation for which $JK(L) = J(L)K(L)$ for every L in G^c. $\mathfrak{U}$ is given a topology defined as the smallest topology such that the maps $J \to J(L)$, where L is an element of G^c, are continuous in the strong (equivalently weak, σ-weak or σ-strong) operator topology. The following facts about $\mathfrak{U}$ are proven in [2]. $\mathfrak{U}$ is a topological group. In fact $\mathfrak{U}$ is isomorphic and homeomorphic to the group of all unitary operators of some von Neumann algebra, where the topology of this unitary group is taken to be any of the operator topologies, strong, σ-strong, weak or σ-weak. Further G may be embedded in $\mathfrak{U}$ in a canonical way by associating with each x in G, the representation $\hat{x}$ of G^c defined by $\hat{x}(L) = L_x$ for all L in G^c. In this way G is embedded isomorphically and homeomorphically in $\mathfrak{U}$. Further every separable strongly continuous unitary representation of G has a unique extension to $\mathfrak{U}$. This correspondence between the representation theory of G and that of $\mathfrak{U}$ preserves all the usual representation theoretic operations, such as direct sum and tensor product. Thus from an abstract point of view, G and $\mathfrak{U}$ have the same concrete dual, i.e., $G^c = \mathfrak{U}^c$. We call the group $\mathfrak{U}$ the *fulfillment* of G. Thus the duality procedure described here leads back to an enlargement $\mathfrak{U}$ of the original group. However a strict duality does hold for any group $\mathfrak{U}$ which is the fulfillment of a separable locally compact group G. Indeed the group of representations of $\mathfrak{U}^c$ is the same as the group of representations of G^c, which is just $\mathfrak{U}$. This raises the interesting question as to when two separable locally compact groups will have the same fulfillment, or, what is equivalent, when two separable locally compact groups will have isomorphic concrete duals.

2. We next apply the duality theory described in the previous section, to obtain an analogue of the theorem of K. Suzuki.

DEFINITION OF TOPOLOGY IN THE CONCRETE DUAL. Let G be a separable locally compact group and let G^c denote its concrete dual. We define the weak topology of G^c to be the smallest topology such that the maps $L \to L_x$ are continuous relative to the strong operator topology, for all x in G. (Since the elements of G^c are all unitary representations, one obtains an equivalent definition if one uses any one of the four operator topologies, weak, strong, σ-weak, or σ-strong.) Thus a basic neighborhood of a representation M in G^c is of the form

$$U(M\colon x_1, \cdots, x_n;\ \psi_1, \cdots, \psi_n) = \{L : \|(L_{x_i} - M_{x_i})\psi_i\| \leqq 1, \quad 1 \leqq i \leqq n\}$$

where $x_1, \cdots, x_n$ are elements of G and $\psi_1, \cdots, \psi_n$ are elements of $\mathfrak{H}$.

DEFINITION OF HOMOMORPHISM BETWEEN CONCRETE DUALS. Let G_1 and G_2 denote two separable locally compact groups and let $G_1{}^c$ and $G_2{}^c$ denote their concrete duals. A *homomorphism* of $G_1{}^c$ into $G_2{}^c$ is a mapping Φ of $G_1{}^c$ into $G_2{}^c$ which satisfies the following three axioms.

1. *If M and N are elements of $G_1{}^c$ and U is a linear isometry mapping the representation space $\mathfrak{H}(M \otimes N)$ of $M \otimes N$, onto $\mathfrak{H}$, then*

$$\Phi(U(M \otimes N)U^{-1}) = U(\Phi(M) \otimes \Phi(N))U^{-1}.$$

2. *If M and N are elements of $G_1{}^c$ and U is a linear isometry mapping the representation space $\mathfrak{H}(M \oplus N)$ of $M \oplus N$, onto $\mathfrak{H}$, then*

$$\Phi(U(M \oplus N)U^{-1}) = U(\Phi(M) \oplus \Phi(N))U^{-1}.$$

3. *Φ is continuous.*

REMARK. The reader should compare the above definition with the definition given by Kazuo Suzuki [3] in the case of finite dimensional representations. Our axioms 1 and 2 are exact analogues of Suzuki's axioms 1 and 2. We have not assumed the analogue of Suzuki's axiom 3, as that will appear later as a corollary. The analogue of Suzuki's axiom 4 is unnecessary here, due to our restriction to unitary representations. We do not need the analogue of Suzuki's axiom 5, as all our representations are acting on the same fixed Hilbert space. On the other hand, our consideration of continuous infinite dimensional representations has required the addition of a continuity assumption, which is our third axiom.

THEOREM. *Let G_1 and G_2 denote separable locally compact groups, and let $\mathfrak{U}_2$ denote the fulfillment of G_2. Then there is a canonical one-to-one correspondence between the set* $\mathrm{Hom}(G_1, \mathfrak{U}_2)$ *of all continuous homomorphisms of G_1 into $\mathfrak{U}_2$, and the set* $\mathrm{Hom}(G_2^c, G_1^c)$ *of all homomorphisms of G_2^c into G_1^c.*

PROOF. Suppose φ is an element of $\mathrm{Hom}(G_1, \mathfrak{U}_2)$. Each representation L in G_2^c had a unique extension to $\mathfrak{U}_2$, by theorem 8. 3 of [2]. Thus $L_{\varphi(x)}$ is a well defined operator on $\mathfrak{H}$, for each x in G_1. Define φ', a map of G_2^c into G_1^c by $\varphi'(L)_x = L_{\varphi(x)}$ for all L in G_2^c and x in G_1. A simple verification, left to the reader, shows that $\varphi'(L)$ is an element of G_1^c, for every L in G_2^c.

Further φ' is a homomorphism of G_2^c into G_1^c. We leave the simple verification of axioms 1 and 2 to the reader. Let M be an element of G_2^c. We will show that φ' is continuous at M. Let U_1 denote a basic neighborhood of $\varphi'(M)$ in G_1^c. Thus U_1 is of the form:

$$U_1(\varphi'(M): x_1, \cdots, x_n; \psi_1, \cdots, \psi_n)$$
$$= \{L: L \in G_1^c, \|(L_{x_i} - \varphi'(M)_{x_i})\psi_i\| \leqq 1, \quad 1 \leqq i \leqq n\}$$

where $\psi_i \in \mathfrak{H}$, $x_i \in G_1$, $1 \leqq i \leqq n$. Then let U_2 denote the neighborhood of M in G_2^c of the form:

$$U_2(M: \varphi(x_1), \cdots, \varphi(x_n); \psi_1, \cdots, \psi_n)$$
$$= \{L: L \in G_2^c, \|(L_{\varphi(x_i)} - M_{\varphi(x_i)})\psi_i\| \leqq 1, \quad 1 \leqq i \leqq n\}.$$

Thus clearly $\varphi'(U_2) \subset U_1$, φ' is continuous and hence φ' is contained in $\mathrm{Hom}(G_2^c, G_1^c)$.

Suppose next that Φ is an element of $\mathrm{Hom}(G_2^c, G_1^c)$. We define Φ', a map of G_1 into $\mathfrak{U}_2$, by defining, for each x in G_1, $\Phi'(x)$ to be the representation of G_2^c (and thus an element of $\mathfrak{U}_2$) defined by $\Phi'(x)(L) = [\Phi(L)]_x$, for all L in G_2^c. We leave to the reader the simple verification that $\Phi'(x)$ is an element of $\mathfrak{U}_2$, for each x in G_1. We also leave to the reader the simple verification that Φ' is a homomorphism. We next verify that Φ' is continuous.

Recall the definition of topology in $\mathfrak{U}_2$. It is the smallest topology such that the maps $J \to J(L)$ on $\mathfrak{U}_2$ are continuous for all L in G_2^c, relative to the strong (equivalently the weak) operator topology. Suppose $\{x_i\}$ is a sequence in G_1 which converges to x in G_1. Then for all L in $G_2{}^c$, $\Phi'(x_i)(L)=[\Phi(L)]_{x_i}$ converges strongly to $[\Phi(L)]_x = \Phi'(x)(L)$. Thus $\Phi'(x_i)$ converges to $\Phi'(x)$ and hence Φ' is continuous.

We next note that $(\Phi')' = \Phi$. Indeed, suppose L is an element of $G_2{}^c$. Then for all x in G_1, we have

$$[(\Phi')'(L)]_x = L_{\Phi'(x)} = \Phi'(x)(L) = [\Phi(L)]_x\,.$$

Hence $(\Phi')'(L) = \Phi(L)$ for all L, or $(\Phi')' = \Phi$. Thus every element of $\mathrm{Hom}(G_2{}^c, G_1{}^c)$ is of the form φ' for some φ in $\mathrm{Hom}(G_1, \mathfrak{U}_2)$. Thus the mapping $\varphi \to \varphi'$ maps $\mathrm{Hom}(G_1, \mathfrak{U}_2)$ *onto* $\mathrm{Hom}(G_2{}^c, G_1{}^c)$.

We next verify that this mapping is one-to-one. Suppose φ_1 and φ_2 are two elements of $\mathrm{Hom}(G_1, \mathfrak{U}_2)$ such that $\varphi_1' = \varphi_2'$. Then for all L in $G_2{}^c$ and x in G_1, we have $\varphi_1'(L)_x = \varphi_2'(L)_x$ or $L_{\varphi_1(x)} = L_{\varphi_2(x)}$. Considering $\varphi_1(x)$ and $\varphi_2(x)$ as elements of $\mathfrak{U}_2$ we have $\varphi_1(x)(L) = \varphi_2(x)(L)$ for all x in G_1 and all L in $G_2{}^c$. Thus $\varphi_1(x) = \varphi_2(x)$ for all x in G_1.

COROLLARY 1. *Every homomorphism Φ of $G_1{}^c$ into $G_2{}^c$ satisfies the following property. If U is a unitary operator on $\mathfrak{H}$, then $\Phi(U^* LU) =U^* \Phi(L)U$ for all L in $G_1{}^c$.*

PROOF. Every Φ in $\mathrm{Hom}(G_1{}^c, G_2{}^c)$ is of the form $\Phi=\varphi'$ where φ is an element of $\mathrm{Hom}(G_2, \mathfrak{U}_1)$. But it is trivial to verify that φ' has the stated property.

REMARK. There are natural topologies in $\mathrm{Hom}(G_1, \mathfrak{U}_2)$ and $\mathrm{Hom}(G_2{}^c, G_1{}^c)$, namely the topology of point-wise convergence. Relative to these topologies, the correspondence of the theorem is a homeomorphism. Indeed, suppose η is an element of $\mathrm{Hom}(G_1, \mathfrak{U}_2)$. Let U' denote a basic neighborhood of η' in $\mathrm{Hom}(G_2{}^c, G_1{}^c)$. Then U' is of the form $U'(\eta') = \{\varphi' : \varphi' \in \mathrm{Hom}(G_2{}^c, G_1{}^c)$ and $\|(\varphi'(L^i)_{x_i} - \eta'(L^i)_{x_i})\psi_i\| \leqq 1,\ 1 \leqq i \leqq n\}$ where $x_i \in G_1$, $\psi_i \in \mathfrak{H}$, $L^i \in G_2{}^c$, $1 \leqq i \leqq n$. But under the correspondence of the theorem, this basic neighborhood of η' corresponds precisely to the basic neighborhood U of η defined by

$$U(\eta) = \{\varphi : \varphi \in \mathrm{Hom}(G_1, \mathfrak{U}_2) \text{ and } \|(L^i{}_{\varphi(x_i)} - L^i{}_{\eta(x_i)})\psi_i\| \leqq 1\,, \quad 1 \leqq i \leqq n\}\,.$$

REMARK. Since, from an abstract point of view, a separable locally compact group G and its fulfillment $\mathfrak{U}$ have identical concrete duals, i.e., $G^c = \mathfrak{U}^c$, we have the following identifications:

$$\mathrm{Hom}(G_2{}^c, G_1{}^c) = \mathrm{Hom}(\mathfrak{U}_2{}^c, G_1{}^c) = \mathrm{Hom}(G_2{}^c, \mathfrak{U}_1{}^c) = \mathrm{Hom}(\mathfrak{U}_2{}^c, \mathfrak{U}_1{}^c)\,.$$

Using this identification, and proving the theorem in the same manner as before but with $\mathfrak{U}_1$ in place of G_1, we have that Hom $(\mathfrak{U}_1, \mathfrak{U}_2)$ is in one-to-one correspondence with Hom $(\mathfrak{U}_2^c, \mathfrak{U}_1^c) =$ Hom (G_2^c, G_1^c). This result, along with our original theorem, gives us the following corollary.

COROLLARY 2. Hom $(G_1, \mathfrak{U}_2)$ *is in one-to-one correspondence with* Hom $(\mathfrak{U}_1, \mathfrak{U}_2)$. *Every continuous homomorphism of* G_1 *into* $\mathfrak{U}_2$ *has a unique extension to a continuous homomorphism of* $\mathfrak{U}_1$ *into* $\mathfrak{U}_2$.

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