MOD 3 PONTRYAGIN CLASS AND INDECOMPOSABILITY OF DIFFERENTIABLE MANIFOLDS.

YASURÔ TOMONAGA

(Received March 10, 1964)

Introduction. We have already dealt with the indecomposability of differentiable manifolds twice ([1], [2]). In this paper we shall show an application of the mod q Pontryagin class, where q denotes a prime number bigger than 2, on this problem. The mod q Pontryagin classes were systematically investigated by Hirzebruch ([3], [4]). In particular the vanishment of mod 3 dual-Pontryagin class of the highest dimension is fundamental for our purpose.

1. Let q be a prime number bigger than 2 and let X_n be a compact orientable differentiable n-manifold. For any cohomology class $v \in H^{n-2r(q-1)}(X_n, Z_q)$ it holds that

$$(1.\ 1) \qquad \mathfrak{P}_q^r v = s_q^r v \ ([3], [4]),$$

where $\mathfrak{P}_q^r$ denotes the Steenrod power ([7])

$$(1.\ 2) \qquad \mathfrak{P}_q^r : H^t(X_n, Z_q) \longrightarrow H^{t+2r(q-1)}(X_n, Z_q)$$

and s_q^r denotes a mod q polynomial of Pontryagin classes such that

$$(1.\ 3) \qquad s_q^r = q^r L_{\frac{1}{2}r(q-1)}(p_1, \cdots, p_t) \quad \text{mod } q\,, \qquad t = \frac{1}{2}r(q-1)$$

where

$$(1.\ 4) \qquad \prod_i \left(\sqrt{\gamma_i}/tgh\sqrt{\gamma_i}\right) = \sum_{j\geqq 0} L_j(p_1, \cdots, p_j),$$

$$(1.\ 5) \qquad p = \sum_{i\geqq 0} p_i = \prod_i (1+\gamma_i) \qquad \text{and}$$

$$(1.\ 6) \qquad p_i \in H^{4i}(X_n, Z).$$

The dimension of s_q^r is equal to $2r(q-1)$. We put

$$(1.\ 7) \qquad \sum_{i\geqq 0} b_{q,i} = \prod_i (1+\gamma_i^l), \qquad b_{q,j}^* \in H^{4j}(X_n, Z_q)$$

where $*$ denotes the reduction modulo q and

$$l = \frac{1}{2}(q-1). \tag{1. 8}$$

It is known that

$$b^*_{q,j} = \sum \mathfrak{P}^i_q s^r_q \tag{1. 9}$$

where the summation is extended over all i, r with

$$2j = (i + r)(q - 1). \text{ ([4])} \tag{1. 10}$$

In the case $q = 3$, (1. 7) takes the form

$$\sum_{j\geqq 0} b_{3,j} = \prod_i (1+\gamma_i) = \sum_{i\geqq 0} p_i \tag{1. 11}$$

and we have from (1. 9)

$$p^*_j = \sum_{j=i+r} \mathfrak{P}^i_3 s^r_3 . \tag{1. 12}$$

We define $\bar{b}_{q,j}$ and $\bar{s}^r_q$ by

$$\Big(\sum_{r\geqq 0} s^r_q\Big)\Big(\sum_{r\geqq 0} \bar{s}^r_q\Big) = 1 , \qquad \bar{s}^r_q \in H^{2r(q-1)}(X_n, Z_q) \tag{1. 13}$$

and

$$\Big(\sum_{j\geqq 0} b_{q,j}\Big)\Big(\sum_{j\geqq 0} \bar{b}_{q,j}\Big) = 1 \tag{1. 14}$$

which leads to

$$\Big(\sum_{j\geqq 0} b^*_{q,j}\Big)\Big(\sum_{j\geqq 0} \bar{b}^*_{q,j}\Big) = 1 . \tag{1. 14)'$$

It is well known that

$$\text{(1. 15)} \quad \begin{cases} \text{(i)} \ \ \mathfrak{P}^0_q = \text{identity} ; \\ \text{(ii)} \ \ \mathfrak{P}^i_q u_k = 0 , \ \ 2i > k , \ \ u_k \in H^k(X_n, Z_q), \\ \text{(iii)} \ \ \mathfrak{P}^r_q(uv) = \sum_{s=0}^{r} \mathfrak{P}^s_q u \cdot \mathfrak{P}^{r-s}_q v . \ \text{([7])} \end{cases}$$

We have from (1. 9), (1. 13), (1. 14) and (1. 15)

$$\bar{b}^*_{q,j} = \sum_{2j=(i+r)(q-1)} \mathfrak{P}^i_q \bar{s}^r_q \tag{1. 16}$$

because

$$1 = \Big(\sum_{i\geqq 0} \mathfrak{P}^i_q\Big)\Big(\sum_{j+k\geqq 0} s^j_q \bar{s}^k_q\Big) = \sum_{j+k\geqq 0}\sum_{i\geqq 0}\sum_{r+s=i} \mathfrak{P}^r_q s^j_q \mathfrak{P}^s_q s^k_q \tag{1. 17}$$

$$= \Big(\sum_{r,j} \mathfrak{P}^r_q s^j_q\Big)\Big(\sum_{s,k} \mathfrak{P}^s_q \bar{s}^k_q\Big) = \Big(\sum_{l\geqq 0} b^*_{q,l}\Big)\Big(\sum \mathfrak{P}^s_q \bar{s}^k_q\Big).$$

In particular $\bar{b}^*_{3,j}$ equals the mod 3 dual-Pontryagin class $\bar{p}^*_j(\in H^{4j}(X_n, Z_3))$, where

$$\sum_{j\geqq 0} \bar{p}_j \sum_{i\geqq 0} p_i = 1.$$

We have from (1. 16)

$$\bar{b}^*_{q,k} = \sum_{2k=(i+j)(q-1)} \mathfrak{P}^i_q s^j_q = \bar{s}_q^{\frac{2k}{q-1}} + \sum_{i=1}^{2k/(q-1)} \mathfrak{P}^i_q \bar{s}_q^{\frac{2k}{q-1}-i}. \tag{1. 18}$$

On the other hand (1. 13) leads to

$$0 = \bar{s}_q^{\frac{2k}{q-1}} + \sum_{i=1}^{2k/(q-1)} s^i_q \bar{s}_q^{\frac{2k}{q-1}-i}. \tag{1. 19}$$

Hence we have

$$\bar{b}^*_{q,k} = \sum_{i=1}^{2k/(q-1)} \Big(\mathfrak{P}^i_q \bar{s}_q^{\frac{2k}{q-1}-i} - s^i_q \bar{s}_q^{\frac{2k}{q-1}-i}\Big). \tag{1. 20}$$

When $4k=n$, we have from (1. 1) and (1. 20)

$$\bar{b}^*_{q,k} = 0, \tag{1. 21}$$

i.e. $\bar{b}_{q,k}[X_{4k}]$ is divisible by q. In particular

$$\bar{p}_k[X_{4k}] = 0 \quad \bmod 3. \tag{1. 22}$$

2. If a differentiable manifold X_n is a product of two differentiable manifolds X_r and X_s, we say that the X_n is decomposable and if not we say that X_n is indecomposable ([1]).

We deal with a compact orientable differentiable X_{4k}. Suppose that such an X_{4k} be decomposable, i.e.

$$X_{4k} = X_r \times X_s\,. \tag{2. 1}$$

Then we have for any multiplicative series $\sum_{i\geqq 0} K_i(p_1, \cdots, p_i)$ ([6])

$$K_k(p_1, \cdots, p_k)[X_{4k}] = K_r(p_1, \cdots, p_r)[X_r]\, K_s(p_1, \cdots, p_s)[X_s] \tag{2. 2}$$

provided that $r\equiv 0 \bmod 4$. If $r\not\equiv 0 \bmod 4$, then the cobordism components of X_r consist only of torsions and hence the same thing holds for X_{4k}. Therefore all Pontryagin numbers of X_{4k} are zero, in particular $K_k[X_{4k}]$ equals zero. In the case of $\sum_{j\geqq 0} \bar{b}_{q,j}$ we have from (1. 21) and (2. 2) the

THEOREM. *Let X_{4k} be a compact orientable differentiable manifold. If $\bar{b}_{q,k}[X_{4k}]$ is not divisible by q^2, then such an X_{4k} is indecomposable.*

COROLLARY. *If $\bar{p}_k[X_{4k}]$ is not divisible by 9, then such an X_{4k} is indecomposable.*

We shall show some applications of above Corollary.

EXAMPLE 1. In the case $W=F_4/\mathrm{Spin}\,(9)$ ([5], p. 534) the Pontryagin classes are given by

$$p_1 = p_3 = 0, \quad p_2 = 6u, \quad p_4 = 39u^2, \quad u^2[W] = 1, \quad u \in H^8(W, Z). \tag{2. 3}$$

Hence we have

$$\bar{p}_4 = -p_4 + 2p_1p_3 - 3p_1^2p_2 + p_2^2 + p_1^4 = -3u^2\,, \text{ i.e.} \tag{2. 4}$$

$$\bar{p}_4[W] = -3\,. \tag{2. 5}$$

Thus W is indecomposable.

EXAMPLE 2. Let $P_{2m+1}(c)$ be the complex projective space of complex dimension $2m+1$. The total Pontryagin class of $P_{2m+1}(c)$ is given by

$$p = (1+g^2)^{2m+2}\,, \qquad g \in H^2(P_{2m+1}(c), Z)\,. \tag{2. 6}$$

Let X_{4m} be a compact orientable differentiable submanifold of $P_{2m+1}(c)$ and let λg be the cohomology class corresponding to the homology class represented by X_{4m}. Then the dual-Pontryagin class of X_{4m} is given by ([6])

$$\begin{aligned} \bar{p}_m[X_{4m}] &= [\lambda g(1+\lambda^2 g^2)(1+g^2)^{-(2m+2)}][P_{2m+1}(c)] \\ &= \left[\lambda g(1+\lambda^2 g^2) \sum_{r\geqq 0} (-1)^r \frac{(2m+2)\cdots(2m+r+1)}{r!}\, g^{2r}\right][P_{2m+1}(c)] \end{aligned} \tag{2. 7}$$

$$= (-1)^m \lambda \left\{ \binom{3m+1}{m} - \binom{3m}{m-1} \lambda^2 \right\}$$

by virtue of

$$g^{2m+1}[P_{2m+1}(c)] = 1 . \tag{2. 8}$$

Hence, if $\lambda \left\{ \binom{3m+1}{m} - \binom{3m}{m-1} \lambda^2 \right\} \not\equiv 0 \mod 9$, *then such an* X_{4m} *is indecomposable.*

REFERENCES

[1] Y. TOMONAGA, Indecomposability of differentiable manifolds, Tôhoku Math. Journ., 14(1962), 328-331.

[2] Y. TOMONAGA, A-genus and indecomposability of differentiable manifolds, to appear in the Proc. Amer. Math. Soc..

[3] F. HIRZEBRUCH, On Steenrod's reduced powers, the index of inertia and Todd genus, Proc. Nat. Acad. Sci. U.S.A., 39(1953), 951-956.

[4] F. HIRZEBRUCH, Some problems on differentiable and complex manifolds, Ann. of Math., 69(1954), 213-236.

[5] A. BOREL & F. HIRZEBRUCH, Characteristic classes and homogeneous spaces I, Amer. Journ. of Math., 80(1958), 458-538.

[6] F. HIRZEBRUCH, Neue topologische Methoden in der algebraischen Geometrie, 1956, Springer.

[7] J. ADEM, The relations on Steenrod powers of cohomology classes, Algebraic geometry and topology, Princeton Univ. Press, 1957, 191-238.

UTSUNOMIYA UNIVERSITY.