MANIFOLDS ADMITTING CONTINUOUS FIELD OF FRAMES

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Introduction. Massey and Szczarba studied the continuous line element field on a differentiable manifold and obtained a necessary condition under which a manifold admits a continuous field of q independent line elements. ([1]) Meanwhile Adams investigated the continuous field of q frames on a sphere and clarified the relations between the number q and the dimension of the sphere. ([2]) A differentiable n-manifold admitting a continuous field of n frames is said to be parallelizable. It is well-known that a differentiable manifold admits a continuous non zero vector field if and only if its Euler characteristic is zero. The intermediate status between above two cases is in question. In this paper we shall mainly deal with the continuous field of $n-3$ frames on an n-manifold.

§1. A q frame means an ordered set of q linearly independent vectors while a q pseudo-frame means an ordered set of q vectors at least $q-1$ of which are linearly independent. The following facts are well-known:

I. *Let M_n be a compact differentiable manifold. For each q there exists a continuous field of tangent $n-q$ frames defined over the q dimensional skeleton K^q of M_n. In order that there exist such a field on any K^{q+1}, it is necessary that*

$$w_{q+1} = 0\,,$$

where $w_{q+1} \in H^{q+1}(M_n, Z_2)$ denotes the Stiefel-Whitney class. ([7] p. 199)

We have from I

COROLLARY. *If M_n admits a continuous field of $n-q$ frames* $(0 \leqq q < n)$ *it must be that*

$$w_{q+1} = w_{q+2} = \cdots = w_n = 0\,.$$

Let M_n be a compact C^∞-manifold with a Riemann metric and Ω_{ij} be its curvature form. Chern introduced in [3] a characteristic class such that

$$(1.\ 1)\qquad P'_{2q} = \alpha_q \sum_i \Omega_{i_1 i_2}\Omega_{i_2 i_3}\cdots\Omega_{i_q i_1}\,, \quad P'_{2q} \in H^{2q}(M_n, Z),$$

where α_q denotes some constant. Meanwhile the Pontryagin class is expressed as follows:

$$(1.\ 2)\qquad P_{2q} = 1_q \sum_{i,j} \delta\begin{pmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{pmatrix} \Omega_{i_1 j_1}\cdots\Omega_{i_q j_q}\,, \quad P_{2q} \in H^{2q}(M_n, Z)$$

$$1_q = ((2\pi)^q q!)^{-1}\,,$$

where $\delta\begin{pmatrix} \quad \end{pmatrix}$ denotes the generalized Kronecker symbol. It is well-known that $P_{2q} = P'_{2q} = 0$ (q=odd) and P_{2q} is a polynomial of P'_{2t} ($t \leqq q$) and conversely P'_{2q} is a polynomial of P_{2t} ($t \leqq q$). Chern proved in [3] the following theorem:

II. *Let M_n be a manifold stated above. There exists a continuous field of $(n-2m+2)$ pseudo-frames over any $4m$ dimensional skeleton if and only if $P'_{4m} = 0$ $(1 \leqq m \leqq [n/4])$. There exists a continuous field of $(n-2m+2)$ pseudo-frames over any skeleton whose dimension is less than $4m$.*

We have from I and II the

THEOREM 1. *Let M_n be a compact C^∞-Riemannian manifold. In order that M_n admit a continuous field of $n-1$ frames, it is necessary that*

$$w_2 = \cdots = w_n = 0 \quad \textit{and} \quad P_4 = P_8 = \cdots = P_{4[n/4]} = 0\,.$$

PROOF. The first part follows from I Corollary. We have from II

$$P'_{4k} = 0\,, \quad k \geqq 1$$

because we can form a continuous field of n pseudo-frames from a continuous field of $n-1$ frames.

COROLLARY. *Let M_{4k} $(k \geqq 1)$ be a compact orientable C^∞-Riemannian manifold. In order that M_{4k} admit a continuous field of $4k-1$ frames, it is necessary that $2M_{4k}$ is "bord", i.e. $2M_{4k} \sim 0$.*

PROOF. By Theorem 1 every Pontryagin numbers are zero. Hence the free part of cobordism components of M_{4k} is zero. Since every torsions of the cobordosm ring are of order 2, the statement holds. ([4])

REMARK. When $1 \leqq k \leqq 3$, M_{4k} becomes "bord", because in such a case the torsion doesn't exist.

§ 2.

THEOREM 2. *Let M_n ($n \geqq 4$) be a compact C^∞-Riemannian manifold. In order that M_n admit a continuous field of $n-3$ frames, it is necessary that*

$$\text{(i)} \qquad w_4 = w_5 = \cdots = w_n = 0\,,$$

$$\text{(ii)} \qquad P'_8 = P'_{12} = \cdots = P'_{4[n/4]} = 0 \quad \textit{and}$$

$$\text{(iii)} \qquad P_{4k} = \frac{1}{k!}(P_4)^k \qquad 1 \leqq k \leqq \left[\frac{n}{4}\right].$$

If moreover $n=4m$ and M_{4m} is orientable, then it is necessary that

$$\text{(iv)} \qquad \tau(M_{4m}) = \frac{1}{3^m m!}(P_4)^m [M_{4m}] = \frac{1}{3^m} P_{4m}[M_{4m}] \quad \textit{and}$$

$$A(M_{4m}) = \frac{(-1)^m}{3^m m!} 2^m (P_4)^m [M_{4m}] = \left(-\frac{2}{3}\right)^m P_{4m}[M_{4m}]\,,$$

where τ or A denotes the index or the A-genus respectively.

PROOF. (i) and (ii) follow from I and II as in the case of Theorem 1. We have from (1. 2)

$$\text{(2. 1)} \qquad P_4 = -\frac{1}{(2\pi)^2 2} \sum_{i,j} \Omega_{ij}\Omega_{ji}\,.$$

Meanwhile we have from (1. 2) and (ii)

$$\text{(2. 2)} \quad P_{4k} = \frac{(2k-1)(2k-3)\cdots 1}{(2\pi)^{2k}(2k)!} \sum_i \delta\begin{pmatrix} i_1 i_2 \cdots i_{2k-1} i_{2k} \\ i_2 i_1 \cdots i_{2k} i_{2k-1} \end{pmatrix} \Omega_{i_1 i_2}\Omega_{i_2 i_1} \cdots \Omega_{i_{2k-1} i_{2k}} \Omega_{i_{2k} i_{2k-1}}$$

$$= \frac{(2k-1)(2k-3)\cdots 1}{(2\pi)^{2k}(2k)!} (-1)^k \left(-\sum_{i,j} \Omega_{ij}\Omega_{ji}\right)^k$$

$$= \frac{(2k-1)(2k-3)\cdots 1}{(2\pi)^{2k}(2k)!} (2\pi)^{2k} 2^k (P_4)^k = \frac{1}{k!}(P_4)^k\,.$$

Thus (iii) holds. Let us prove (iv). We put

$$\text{(2. 3)} \qquad \sum_{k \geqq 0} P_{4k} = \prod_i (1+\gamma_i)\,.$$

Then the index is expressed as follows:

$$\tau = \left(\prod_i \frac{\sqrt{\gamma_i}}{\operatorname{tgh}\sqrt{\gamma_i}} \right) [M_{4m}] . \qquad ([5]) \tag{2. 4}$$

We have from (2. 3)

$$P_4 = \sum_i \gamma_i \quad \text{and} \quad P_8 = \sum_{i \neq j} \gamma_i \gamma_j \tag{2. 5}$$

which lead to

$$\sum_i \gamma_i^2 = \left(\sum_i \gamma_i \right)^2 - 2 \sum_{i \neq j} \gamma_i \gamma_j = P_4^2 - 2P_8 . \tag{2. 6}$$

Meanwhile we have from (iii)

$$P_8 = \frac{1}{2} P_4^2 . \tag{2. 7}$$

From (2. 6) and (2.7) we have

$$\sum_i \gamma_i^2 = 0 . \tag{2. 8}$$

In such a way we can prove from (iii) and (2. 3) that every symmetric functions of γ_i's are zero except for the elementary ones. Therefore we can regard γ_i^t ($t \geqq 2$) as zero in the following computations. Since

$$\frac{\sqrt{\gamma_i}}{\operatorname{tgh}\sqrt{\gamma_i}} = 1 + \frac{1}{3} \gamma_i + \cdots \tag{2. 9}$$

we have from (2. 3) and (2. 4)

$$\tau = \left[\prod_i \left(1 + \frac{1}{3} \gamma_i + \cdots \right) \right] [M_{4m}] = \frac{1}{3^m} P_{4m} [M_{4m}] . \tag{2. 10}$$

In such a way we have

$$\begin{aligned} A(M_{4m}) &= \left[\prod_i \frac{2\sqrt{\gamma_i}}{\sinh 2\sqrt{\gamma_i}} \right] [M_{4m}] = \left[\prod_i \left(1 - \frac{2}{3} \gamma_i + \cdots \right) \right] [M_{4m}] \\ &= \left(- \frac{2}{3} \right)^m P_{4m} [M_{4m}] . \qquad \text{Q. E. D.} \end{aligned} \tag{2. 11}$$

REMARK. We have from (iv)

$$A/\tau = (-1)^m 2^m . \tag{2. 12}$$

Thus in such a case the A-genus is divisible by 2^m.

COROLLARY 1. *Let M_n be a compact orientable C^∞-Riemannian manifold. If M_n admits a continuous field of $n-3$ frames and $H^{4k}(M_n, Z)=0$ for some k $(1 \leqq k \leqq [n/4])$, then*

$$P_{4l} = 0 \ (l \geqq k) \quad \textit{and} \quad 2M_n \sim 0 .$$

PROOF. We have from Theorem 2 (iii)

$$P_{4l} = \frac{1}{l!} P_4{}^l = 0 \ \ (l \geqq k) \tag{2. 13}$$

and hence if $n = 4m$, every Pontryagin numbers become zero. Hence we have $2M_n \sim 0$ as in the case of Theorem I Corollary.

COROLLARY 2. *Let M_{4m} $(m > 1)$ be a compact orientable C^∞-Riemannian manifold admitting a continuous field of $4m-3$ frames. If moreover either $\tau(M_{4m})$ or $A(M_{4m})$ is zero, then $2M_{4m} \sim 0$.*

PROOF. From Theorem 2 (iii) and (iv) we see that every Pontryagin numbers of M_{4m} are zero.

COROLLARY 3. *Let M_{4n} $(n \geqq 1)$ be a compact orientable C^∞-Riemannian manifold admitting a continuous field of $4n-3$ frames. If moreover M_{4n} is differentiably imbedded in an Euclidian space E_{6n}, then $2M_{4n} \sim 0$.*

PROOF. The dual Pontryagin classes are defined by

$$\sum_{k \geqq 0} \overline{P}_{4k} \sum_{l \geqq 0} (-1)^l P_{4l} = 1 , \quad \overline{P}_{4k} \in H^{4k}(M_n, Z) . \tag{2. 14}$$

We have from Theorem 2 (iii)

$$\sum_{l \geqq 0} (-1)^l P_{4l} = e^{-P_4} . \tag{2. 15}$$

Hence we have from (2. 14)

$$\sum_{k \geqq 0} \overline{P}_{4k} = e^{P_4} , \text{ i.e.} \tag{2. 16}$$

$$\overline{P}_{4k} = \frac{1}{k!} P_4{}^k . \tag{2. 17}$$

If $M_{4n} \subset E_{6n}$ differentiably, then we have

(2. 18) $$\bar{P}_{4n} = 0\,, \qquad ([6])$$

i.e.

(2. 19) $$P_4^{\,n} = 0\,.$$

Hence every Pontryagin numbers become zero. Q. E. D.

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