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ASYMPTOTIC BEHAVIORS IN FUNCTIONAL DIFFERENTIAL EQUATIONS.*)

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Introduction. The asymptotic equivalence between a linear system and its perturbed system has been discussed by many authors (for references, see [10]). Two systems

$$\dot{x} = X(t, x) \tag{1}$$

and

$$\dot{x} = Y(t, x) \tag{2}$$

are said to be asymptotically equivalent, if the following condition is satisfied: For any solution $x(t; x_0, t_0)$ of one of the systems (1) and (2), we can find a solution of the other system, which tends to $x(t; x_0, t_0)$ as $t \to \infty$. However, for example, the systems

$$\dot{x} = 0$$

and

$$\dot{x} = x^2 e^{-t}$$

are not asymptotically equivalent in the sense above. Clearly, in the condition above, if we take t_0 suitably large according to the norm of x_0, then we can have the same conclusion. In the case, we shall say that they are eventually asymptotically equivalent.

In the previous papers, we have discussed the eventually asymptotic equivalence between more general systems and their perturbed systems, under the assumption that perturbation terms satisfy a special type of Lipschitz conditions [8,9] or some type of integrabilities [10].

In this article, under much weaker condition, we shall discuss the eventually asymptotic equivalence between systems of functional differential equations and

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their perturbed systems, by applying a kind of boundary value problem, which Hukuhara considered in studying the behaviors of solutions of ordinary differential equations [7]. In Section 2, we shall discuss this boundary value problem for a system of functional differential equations.

As an application, we shall obtain a result for the case where the unperturbed system is linear (in Section 4).

In [3], Hale has discussed the stability with an asymptotic amplitude and an asymptotic phase near an integral manifold of periodic solutions of an autonomous system. As another application of our results, we shall consider a converse of Hale's result in some sense, under the same assumptions as given by Hale (in Section 6).

1. Notations and definitions. The following notations will be used throughout this paper: E^p is the Euclidean p-space, and for $x \in E^p$ $|x|$ is the Euclidean norm. For a given constant $h \geqq 0$, $C(E^p)$ denotes the space of continuous functions mapping the interval $[-h, 0]$ into E^p, and for $\varphi \in C(E^p)$

$$\|\varphi\| = \sup\{|\varphi(\theta)| \mid -h \leqq \theta \leqq 0\}.$$

In the case where $h = 0$, $C(E^p)$ is identical with E^p and $\|\varphi\| = |\varphi(0)|$. $C_\alpha(E^p)$ and $\bar{C}_\alpha(E^p)$ will denote the sets of $\varphi \in C(E^p)$ for which we have $\|\varphi\| < \alpha$ and $\|\varphi\| \leqq \alpha$, respectively, while if α is infinite, both $C_\alpha(E^p)$ and $\bar{C}_\alpha(E^p)$ are identical with $C(E^p)$. For any E^p-valued continuous function $x(s)$ defined on $a \leqq s \leqq b$, $b - a \geqq h$, and for any t, $a + h \leqq t \leqq b$, the symbol x_t will denote the function such that

$$x_t(\theta) = x(t+\theta) \quad \text{for all} \quad \theta \in [-h, 0],$$

and hence $x_t \in C(E^p)$. Here, we shall call x_t the segment of $x(s)$ at $s = t$. Similarly, the segment of $x(s; \xi)$, with a parameter ξ, at $s = t$ will be represented by $x_t(\xi)$.

For the convenience, we shall use the following notations: For a subset $S \subset C(E^p)$ and a continuous function $T'(\varphi)$ mapping S into $[0, \infty)$, let $G(T'; S)$ be defined by

$$G(T'; S) = \{(t, \varphi) \mid \varphi \in S,\ t > T'(\varphi)\} \subset [0, \infty) \times C(E^p).$$

Specially, if S in $G(T'; S)$ is $C_\alpha(E^n)$ (or $\bar{C}_\alpha(E^n)$) and $T'(\varphi) = T(\|\varphi\|)$, we shall denote it by $\Delta_\alpha(T)$ (or $\bar{\Delta}_\alpha(T)$). Moreover, let

$$
(1.1) \qquad \begin{cases} D_\alpha(T) = \{(t, \varphi_1, \varphi_2) | (t, \varphi_1), (t, \varphi_2) \in \Delta_\alpha(T)\}, \\ \Omega_\alpha(H) = \{(\varphi, \psi) | \varphi \in C_\alpha(E^n), \ \psi \in C_{H(\|\varphi\|)}(E^m)\}, \\ \Omega_\alpha(T, H) = \{(t, \varphi, \psi) | (t, \varphi) \in \Delta_\alpha(T), (\varphi, \psi) \in \Omega_\alpha(H)\} \end{cases}
$$

for a continuous function $H(r) > 0$. Here, it is note that $D_\alpha(T) = G(T'; S')$ with $S' = C_\alpha(E^n) \times C_\alpha(E^n) \subset C(E^{2n})$, $T'(\varphi_1, \varphi_2) = \max\{T(\|\varphi_1\|), T(\|\varphi_2\|)\}$ and $\Omega(T, H) = G(T''; S'')$ with $S'' = \Omega_\alpha(H) \subset C(E^{n+m})$, $T''(\varphi, \psi) = T(\|\varphi\|)$. In the case where α is infinite, we shall omit the suffix α in the above. If $\Delta_\alpha(T)$, $C_\alpha(E^n)$ etc. in the right-hand sides of (1.1) are replaced by $\overline{\Delta}_\alpha(T)$, $\overline{C}_\alpha(E^n)$ etc., then we shall denote them by $\overline{D}_\alpha(T)$, $\overline{\Omega}_\alpha(H)$ and $\overline{\Omega}_\alpha(T, H)$. For a $\varphi \in C(E^n)$, φ is said to be constant, if $\varphi(\theta)$ is constant on $[-h, 0]$, and for any $x \in E^p$ we shall denote the constant function with the value x on $[-h, 0]$ by $<x>$, that is, $<x> \in C(E^p)$ is the function such that

$$
<x>(\theta) = x \quad \text{for all} \quad \theta \in [-h, 0].
$$

Let $f(t, \varphi)$ be a function mapping $G(T; S) \subset [0, \infty) \times C(E^p)$ into E^p, and let $\dot{x}(t)$ denote the right-hand derivative of the function $x(s)$ at $s = t$. Consider a system of functional differential equations

$$
(1.2) \qquad \dot{x}(t) = f(t, x_t).
$$

DEFINITION 1. For a given point $(t_0, \varphi_0) \in G(T; S)$ a continuous function $x(t; \varphi_0, t_0)$ of t is said to be *a solution of* (1.2) *through* φ_0 *at* $t = t_0$ (or *through* (t_0, φ_0)), if there is a number $\delta > 0$ such that

- (i) for each $t, t_0 \leqq t < t_0 + \delta$, $(t, x_t(\varphi_0\, t_0))$ belongs to $G(T; S)$,
- (ii) $x_{t_0}(\varphi_0, t_0) = \varphi_0$,
- (iii) $x(t; \varphi_0, t_0)$ has the right-hand derivative for any $t, t_0 \leqq t < t_0 + \delta$, and $x(t; \varphi_0, t_0)$ satisfies (1.2) for all $t, t_0 \leqq t < t_0 + \delta$.

For functional differential equations, we can see the following proposition.

PROPOSITION 1. *If $f(t, \varphi)$ in the system* (1.2) *is continuous and if (t_0, φ_0) is an interior point of $G(T; S)$, then there exists a solution of* (1.2) *through φ_0 at $t = t_0$.*

Furthermore, if $G(T; S) = [T_0, \infty) \times C(E^p)$ for a constant $T_0 \geqq 0$ and if $f(t, \varphi)$ is bounded there, then all solutions of (1.2) *exist in the future.*

For the system (1.2) and another system

$$
(1.3) \qquad \dot{y}(t) = g(t, y_t),
$$

defined on $G(T; S)$, we shall give the following definitions: Let S_0 be a subset S.

DEFINITION 2. The systems (1. 2) and (1. 3) are said to be *asymptotically equivalent on* S_0, if the following conditions are satisfied;

(i) all solutions of both (1. 2) and (1. 3) starting from $G(T; S_0)$ exist in the future,

(ii) for any bounded closed subset S^* of S_0, there exists a $T^*(S^*) \geqq 0$ such that for any given solution of (1. 2) or (1. 3) starting from $G(T; S^*)$, we can find a solution of (1. 3) or (1. 2), respectively, starting at $t = t_0$, $t_0 \geqq T^*(S^*)$, which tends to the given solution of (1. 2) or (1. 3) as $t \to \infty$.

DEFINITION 3. The systems (1. 2) and (1. 3) are said to be *eventually asymptotically equivalent on* S_0, if for any bounded closed subset S^* of S_0, there exists a $T^*(S^*) \geqq 0$ such that

(i) all solutions of both (1. 2) and (1. 3) starting from $[T^*(S^*), \infty) \times S^*$ exist in the future

and that

(ii) for any given solution of (1. 2) or (1. 3) starting from $[T^*(S^*), \infty) \times S^*$, we can find a solution of (1. 3) or (1. 2), respectively, starting at $t = t_0$, $t_0 \geqq T^*(S^*)$, which tends to the given solution of (1. 2) or (1. 3) as $t \to \infty$.

The following proposition will show an interesting property of the eventually asymptotic equivalence.

PROPOSITION 2. *If the systems* (1. 2) *and* (1. 3) *are eventually asymptotically equivalent on S, then for any bounded solution of the system* (1. 2) (*or* (1. 3)) *we can find a solution of the system* (1. 3) (*or* (1. 2)), *which tends to the bounded solution as* $t \to \infty$.

Here, we say the solution $x(t; \varphi_0, t_0)$ *of* (1. 2) *to be bounded, if there exists a bounded closed subset* S^* *of* S *such that* $x_t(\varphi_0, t_0) \in S^*$ *for all* $t \geqq t_0$.

PROOF. Let $x(t)$ be bounded solution of (1. 2) (or (1. 3)). Then, as mentioned above, we can find a bounded closed subset S^* of S such that x_t remains in S^* in the future. For this S^* we choose $T^*(S^*) \geqq 0$ as in Definition 3. Since $x(t)$ can be assumed to start from $[T^*(S^*), \infty) \times S^*$, we can find a solution $y(t)$ of (1. 3) (or (1. 2)) such that $x(t) - y(t) \to 0$ as $t \to \infty$. Thus, the proof is completed.

Since in this paper we use Liapunov functionals, we shall state here some of their fundamental properties. Let $V(t, \varphi)$ be a continuous functional defined on $G(T; S) \subset [0, \infty) \times C(E^p)$ and satisfying a Lipschitz condition with respect to φ, that is, there exists a continuous function $L(\tau, \alpha) > 0$, monotone in α, such that if (t, φ), $(t, \varphi') \in G(T; S) \cap [0, \tau] \times \overline{C}_\alpha(E^p)$ for any $\alpha > 0$ and any $\tau > 0$, then we have

$$|V(t, \varphi) - V(t, \varphi')| \leqq L(\tau, \alpha) \|\varphi - \varphi'\| . \tag{1.4}$$

We shall denote

$$D^+_{(1.2)} V(t, \varphi) = \overline{\lim_{\delta \to +0}} \frac{1}{\delta} \{V(t + \delta, x_{t+\delta}(\varphi, t)) - V(t, \varphi)\}$$

and

$$D^-_{(1.2)} V(t, \varphi) = \underline{\lim_{\delta \to +0}} \frac{1}{\delta} \{V(t + \delta, x_{t+\delta}(\varphi, t)) - V(t, \varphi)\} ,$$

where $x(s; \varphi, t)$ is a solution of the system (1.2) through φ at $s = t$. Namely, $D^+_{(1.2)} V(t, \varphi)$ (or $D^-_{(1.2)} V(t, \varphi)$) denotes the upper (or lower) right-hand derivative of $V(t, \varphi)$ along a solution of the system (1.2). Here, we can see that $D^+_{(1.2)} V(t, \varphi)$ and $D^-_{(1.2)} V(t, \varphi)$ are determined independing of a particular solution $x(s; \varphi, t)$ in the right-hand sides of the equations above, even if the solution of (1.2) is not unique for the initial value problem. In general, for any continuous real-valued function $v(s)$ of s we shall denote the upper (or lower) right-hand derivative of $v(s)$ at $s = t$ by $D^+v(t)$ (or $D^-v(t)$). Moreover, we can verify that for a perturbed system

$$\dot{x}(t) = f(t, x_t) + X(t, x_t) \tag{1.5}$$

of the system (1.2) with a continuous perturbation term $X(t, \varphi)$, we have

$$\begin{cases} D^+v(t) \leqq D^+_{(1.2)} V(t, x_t) + L(t, \|x_t\|) |X(t, x_t)| , \\ D^-v(t) \geqq D^-_{(1.2)} V(t, x_t) - L(t, \|x_t\|) |X(t, x_t)| , \end{cases} \tag{1.6}$$

where $x(t)$ is a solution of (1.5), $v(t) = V(t, x_t)$ and $L(\tau, \alpha)$ is the one in the relation (1.4).

It should be noted that even if the system is defined on a domain $[T_0, \infty) \times C_{\alpha_0}(E^p)$, where T_0 and α_0 are positive constants, the desired Liapunov functional will not be necessarily constructed on $[T_0, \infty) \times C_{\alpha_0}(E^p)$, and, in general, the Liapunov functional will be constructed on such a domain as $G(T; S)$ (cf, see Proposition 4 in [10]).

Finally, for a convenience, a product system of the form

$$\begin{cases} \dot{x}(t) = f(t, x_t) \\ \dot{u}(t) = f(t, u_t) \end{cases}$$

will be represented by (1. 2)* corresponding to the system (1. 2).

2. A boundary value problem. In order to study behaviors of solutions in a neighborhood of a singular point of ordinary differential equations, Hukuhara has considered a special type of the boundary value problem [7]. For ordinary differential equations, Nagumo has obtained some results concerning this boundary value problem [12]. By the same arguments as used by Nagumo, we shall obtain a result for functional differential equations, which we shall use in the proofs of theorems concerning the asymptotic equivalence.

First of all, we shall state the following lemmas. Lemma 1 is the well-known Stone-Weierstrass Theorem. For the proof, refer to [6] or [11].

LEMMA 1. *Let Ω be a compact topological space, and let $\mathfrak{C}(\Omega)$ be the algebra of all continuous E^p-valued functions on Ω. Then, a subalgebra $\mathfrak{L}$ of $\mathfrak{C}(\Omega)$ is dense, if the unit function of $\mathfrak{C}(\Omega)$ belongs to $\mathfrak{L}$ and if for any pair of a, $b \in \Omega$, $a \neq b$, there exists an $f \in \mathfrak{L}$ such that $f(a) \neq f(b)$.*

We shall consider the case where Ω is a compact subset of a Banach space with a norm $\|x\|$ for an $x \in \Omega$. Let $\mathfrak{L}$ consist of all E^p-valued functions defined on Ω and satisfying a Lipschitz condition, that is,

(2. 1) for each $f \in \mathfrak{L}$, there exists a constant $L(f) \geqq 0$ such that

$$|f(x) - f(y)| \leqq L(f) \|x - y\|$$

for all x, $y \in \Omega$.

Now, we shall show that $\mathfrak{L}$ satisfies all assumptions in Lemma 1. In fact, for any f, $g \in \mathfrak{L}$ and any real α, we can assume that

$$L(f+g) = L(f) + L(g), \ L(f \cdot g) = L(f) \||g\|| + L(g) \||f\||,$$
$$L(\alpha f) = |\alpha| L(f), \ \ L(f_0) = 0,$$

where each component of $(f+g)(x)$ (or $(f \cdot g)(x)$) is the sum (or product) of the corresponding components of $f(x)$ and $g(x)$, f_0 denotes the unit function of $\mathfrak{C}(\Omega)$ and

$$|||f||| = \sup \{|f(x)| \,|\, x \in \Omega\} \ (<\infty)$$

for any $f \in \mathfrak{C}(\Omega)$. From these, it follows that f_0, $f+g$, $f \cdot g$ and αf belong to $\mathfrak{L}$, that is, $\mathfrak{L}$ is a subalgebra of $\mathfrak{C}(\Omega)$ with the unit function. Next, for arbitrary pair $a, b \in \Omega$, $a \neq b$, we shall exhibit an $f \in \mathfrak{L}$ such that $f(a) \neq f(b)$. Obviously, we can assume that $f(x)$ is a scalar function, that is, $\mathfrak{C}(\Omega)$ is the algebra of continuous real-valued functions. If we set

$$f_a(x) = \|x - a\|,$$

we can easily see that $f_a \in \mathfrak{L}$ with $L(f_a) = 1$, $f_a(a) = 0$ and $f_a(b) \neq 0$, which shows that f_a is the required.

Thus, we have the following lemma.

LEMMA 2. *Let Ω be a compact subset of a Banach space, and let $\mathfrak{L}$ be the subset of $\mathfrak{C}(\Omega)$ such that each element of $\mathfrak{L}$ satisfies a Lipschitz condition in the sense of* (2.1). *Then, $\mathfrak{L}$ is dense in $\mathfrak{C}(\Omega)$. Namely, for any $f \in \mathfrak{C}(\Omega)$ there exists a sequence $\{f_k\} \subset \mathfrak{C}(\Omega)$ such that $f_k(x)$ satisfies a Lipschitz condition for any k and that the sequence $\{f_k\}$ converges uniformly to f on Ω.*

The following lemma is an immediate consequence of a result due to Dugundji [2].

LEMMA 3. *Let Ω be a closed subset of a Banach space $\mathfrak{B}$, and let f be a continuous function mapping Ω into a compact convex subset K of E^p. Then, f has a continuous extension mapping $\mathfrak{B}$ into K.*

Now, we shall prove the following theorem.

THEOREM 1. *In a system*

$$\begin{cases} \dot{x}(t) = f(t, x_t, y_t) \\ \dot{y}(t) = g(t, x_t, y_t), \end{cases} \tag{2.2}$$

where x, y are n, m-vectors, if $f(t, \varphi, \psi)$ and $g(t, \varphi, \psi)$ are continuous and bounded on $[a, b] \times C(E^n) \times C(E^m)$, then for any given $x_0 \in E^n$ and any $(\varphi_0, \psi_0) \in C(E^n) \times C(E^m)$ there exists a solution $(x(t), y(t))$ of (2.2) *which satisfies the condition*

$$x(b) = x_0, y_a = \psi_0 \text{ and } x_a - \varphi_0 \text{ is a constant.} \tag{2.3}$$

PROOF. Let A_0 be a bound of both $f(t,\varphi,\psi)$ and $g(t,\varphi,\psi)$ on $[a,b] \times C(E^n) \times C(E^m)$, let $x_0 \in E^n$ and $(\varphi_0, \psi_0) \in C(E^n) \times C(E^m)$ be given, and let

$$\alpha = \|\varphi_0\| + |x_0| + A(b-a),$$
$$A_1 = \|\varphi_0\| + \alpha + A(b-a),$$
$$A_2 = \|\psi_0\| + A(b-a),$$

where $A = A_0 + 1$. Let $\mathfrak{F}$ denote a family of such a continuous $E^n \times E^m$-valued function $(x(t), y(t))$ as is defined on $[a-h, b]$ and satisfies the condition

$$(2.4)\qquad \begin{cases} |x(t)| \leqq A_1 \text{ and } |y(t)| \leqq A_2 \text{ on } [a,b]; \\ |x(t) - x(s)|,\ |y(t)-y(s)| \leqq A|t-s| \text{ for any } t,\ s \in [a,b]; \\ x(t-a) = \varphi_0(t-a) + q \text{ and } y(t-a) = \psi_0(t-a) \text{ for all } t, \\ a-h \leqq t \leqq a, \text{ and for a } q \in S_\alpha(E^n), \end{cases}$$

where

$$S_r(E^p) = \{x \mid x \in E^p,\ |x| \leqq r\}.$$

We represent by Ω the subset of $C(E^n) \times C(E^m)$ such that

$$\Omega = \{(t, x_t, y_t) \mid (x(s), y(s)) \in \mathfrak{F} \text{ and } t \in [a,b]\}.$$

Then, it can be seen that the set Ω is a compact subset of the Banach space $E^1 \times C(E^n) \times C(E^m)$. Thus, by Lemma 2 we can find a sequence $\{(f_k(t,\varphi,\psi), g_k(t,\varphi,\psi))\}$ such that $(f_k(t,\varphi,\psi), g_k(t,\varphi,\psi))$ is defined on Ω and satisfies a Lipschitz condition there for each k and that the sequence $\{(f_k(t,\varphi,\psi), g_k(t,\varphi,\psi))\}$ converges to $(f(t,\varphi,\psi), g(t,\varphi,\psi))$ uniformly on Ω. Here, we can assume that

$$|f(t,\varphi,\psi) - f_k(t,\varphi,\psi)|,\ |g(t,\varphi,\psi) - g_k(t,\varphi,\psi)| \leqq 1$$

for all k on Ω, which implies that for all k

$$|f_k(t,\varphi,\psi)|,\ |g_k(t,\varphi,\psi)| \leqq A \quad \text{on } \Omega,$$

that is,

$$(f_k(t,\varphi,\psi),\ g_k(t,\varphi,\psi)) \in S_A(E^n) \times S_A(E^m)$$

for all $(t,\varphi,\psi) \in \Omega$ and all k. Since $K = S_A(E^n) \times S_A(E^m)$ is closed and convex,

$(f_k(t, \varphi, \psi), g_k(t, \varphi, \psi))$ has a continuous extension mapping $[a, b] \times C(E^n) \times C(E^m)$ into K by Lemma 3. We shall denote this extension by $(f_k(t, \varphi, \psi), g_k(t, \varphi, \psi))$ again.

Consider the system

$$\text{(2. 5)} \qquad \begin{cases} \dot{x}(t) = f_k(t, x_t, y_t) \\ \dot{y}(t) = g_k(t, x_t, y_t), \end{cases}$$

and let $(x(t; q), y(t; q))$ be a solution of (2. 5) through $(\varphi_0 + \langle q \rangle, \psi_0)$ at $t = a$ for a $q \in E^n$ (for the notation $\langle q \rangle$, see Section 1). By Proposition 1, $(x(t; q), y(t; q))$ exists for all $t \in [a, b]$. Since we have

$$\text{(2. 6)} \qquad \begin{cases} x(t; q) = \varphi_0(0) + q + \displaystyle\int_a^t f_k(\tau, x_\tau(q), y_\tau(q))\, d\tau \\ y(t; q) = \psi_0(0) + \displaystyle\int_a^t g_k(\tau, x_\tau(q), y_\tau(q))\, d\tau \end{cases}$$

for all $t \in [a, b]$, we can verify that if $q \in S_\alpha(E^n)$, then $(x(t; q), y(t; q))$ satisfies the condition (2. 4). Therefore, for any $q \in S_\alpha(E^n)$, $(x(t; q), y(t; q))$ belongs to the family $\mathfrak{F}$, or in other words, $(t, x_t(q), y_t(q))$ belongs to Ω for all $t \in [a, b]$, and hence, since $(f_k(t, \varphi, \psi), g_k(t, \varphi, \psi))$ satisfies a Lipschitz condition on Ω, $(x(t; q), y(t; q))$ is continuous in $q \in S_\alpha(E^n)$. On the other hand, the first equation of (2. 6) implies that we have

$$|x(b; q) - q| \leqq \|\varphi_0\| + A(b-a)$$

for all $q \in E^n$. From these, it follows that the function $F(q)$ defined by

$$F(q) = q - x(b; q) + x_0$$

is a continuous function mapping $S_\alpha(E^n)$ into itself. Thus, by Brouwer's fixed point theorem, we can find a $q_0 \in S_\alpha(E^n)$ such that $F(q_0) = q_0$, that is,

$$x(b; q_0) = x_0 .$$

This proves the existence of a solution of (2. 5) satisfying the condition (2. 3), which we shall denote by $(x^k(t), y^k(t))$.

Since $(x^k(t), y^k(t))$ belongs to the family $\mathfrak{F}$, the sequence $\{(x^k(t), y^k(t)) \mid k \geqq 1\}$ is normal, and hence, this sequence has a subsequence which converges uniformly to a function $(x(t), y(t))$. It can be easily proved that $(x(t), y(t))$ is a solution of the system (2. 2) satisfying the condition (2. 3). Thus, the theorem is completely proved.

3. The case where the perturbation terms are integrable. In this section, we shall discuss the asymptotic equivalence between a system

$$(3.1)\qquad \begin{cases} \dot{x}(t) = f(t, x_t) \\ \dot{y}(t) = g(t, x_t, y_t) \end{cases}$$

and its perturbed system

$$(3.2)\qquad \begin{cases} \dot{x}(t) = f(t, x_t) + X(t, x_t, y_t) \\ \dot{y}(t) = g(t, x_t, y_t) + Y(t, x_t, y_t), \end{cases}$$

where x, y are n, m-vectors and all functions in the right-hand sides of (3. 1) and (3. 2) are completely continuous on $\Omega(T_0, H_0)$ for some continuous and monotone functions $T_0(r) \geqq 0$ and $H_0(r) > 0$ ($H_0(r)$ may be infinite).

Throughout this section, let $H(r)$ be a given continuous and non-increasing function such that $H_0(r) > H(r) > 0$. The following assumption will be made:

(3. 3) There exists a continuous function $\lambda(t, r) > 0$ such that

$$\int^{\infty} \lambda(t, r) dt < \infty$$

and that

$$|X(t, \varphi, \psi)|,\ |Y(t, \varphi, \psi)| \leqq \lambda(t, \alpha)$$

for any $\alpha > 0$ and any $(t, \varphi, \psi) \in \bar{\Omega}_\alpha(T_0, H)$.

Here, $\lambda(t, r)$ can be assumed to be non-decreasing in r.

First of all, we shall prove the following lemmas.

LEMMA 4. *In addition to assumption* (3. 3), *we assume that there exists a continuous Liapunov functional* $W(t, \varphi, \psi)$ *defined on* $\Omega(T_0, H_0)$ *and satisfying the following conditions*:

$$(3.4)\qquad a_1(|\psi(0)|) \leqq W(t, \varphi, \psi) \leqq b_1(\|\psi\|, \alpha)$$

on $\Omega_\alpha(T_0, H_0)$ *for any* $\alpha > 0$, *where* $a_1(r)$, $b_1(r, s)$ *are continuous and non-decreasing function of* (r, s), $a_1(r) > 0$ *for* $r > 0$ *and* $b(0, s) = 0$ *for all* $s \geqq 0$.

(3.5) *For any $\alpha > 0$, on the domain $\bar{\Omega}_\alpha(T_0, H)$, we have*

$$|W(t, \varphi, \psi) - W(t, \varphi', \psi')| \leqq L_1(\alpha)\{\|\varphi-\varphi'\| + \|\psi-\psi'\|\}$$

and

$$D^+_{(3.1)} W(t, \varphi, \psi) \leqq -c(\alpha) W(t, \varphi, \psi),$$

where $L_1(r) > 0$ and $c(r) > 0$ are continuous and monotone functions of $r > 0$.

Then, there exist continuous and monotone functions $T_1(r) \geqq 0$ and $H_1(r) > 0$ of $r > 0$ as follows: For any $\alpha > 0$, let $(x(t), y(t))$ be a solution of the system (3.1) or (3.2) starting at $t = t_0$, $t_0 \geqq T_1(\alpha)$, such that $\|y_{t_0}\| \leqq H_1(\alpha)$. Then, so long as $\|x_t\| < \alpha$, the segment (x_t, y_t) remains in $\Omega(H)$ in the future and $y(t)$ tends to zero as $t \to \infty$ uniformly with respect to x_{t_0}.

PROOF. For any solution $(x(t), y(t))$ of the system (3.2) (or (3.1)), so long as $(t, x_t, y_t) \in \bar{\Omega}_\alpha(T_0, H)$, we have

$$D^+ w(t) \leqq -c(\alpha) w(t) + 2L_1(\alpha)\lambda(t, \alpha), \tag{3.6}$$

by (1.6), (3.3) and (3.5), where

$$w(t) = W(t, x_t, y_t).$$

Choose functions $H_1(r)$ and $T_1(r)$ so that

$$\begin{cases} b_1(H_1(r), r) < a_1(H(r)), \\ \displaystyle\int_{T_1(r)}^\infty \lambda(t, r)\, dt < \frac{a_1(H(r)) - b_1(H_1(r), r)}{2L_1(r)}. \end{cases} \tag{3.7}$$

Let $(x(t), y(t))$ be a solution of the system (3.2) (or (3.1)) through (t_0, φ_0, ψ_0) such that $t_0 \geqq T_1(\alpha)$ and $\|\psi_0\| \leqq H_1(\alpha)$ for a given $\alpha > 0$, and suppose that for a $\tau > 0$

$$(t, x_t, y_t) \in \Omega_\alpha(T_0, H) \quad \text{for all} \quad t, t_0 \leqq t < \tau.$$

Then, from (3.6), it follows that

$$w(\tau) \leqq b_1(\|\psi_0\|, \alpha)\, e^{-c(\alpha)(\tau - t_0)} + 2L_1(\alpha)\int_{t_0}^\tau e^{-c(\alpha)(\tau - t)}\lambda(t, \alpha)\, dt, \tag{3.8}$$

the right-hand side of which is less than $a_1(H(\alpha))$ by (3. 7), and consequently, by (3. 4), we have

$$\|y_\tau\| < H(\alpha) \leqq H(\|x_\tau\|).$$

This and the fact that $(\varphi_0, \psi_0) \in \Omega(H_1) \subset \Omega(H)$ imply that so long as $\|x_t\| < \alpha$, $(x_t, y_t) \in \Omega(H)$ in the future. Moreover, since the right-hand side of (3. 8) tends to zero as $\tau \to \infty$ and $a_1(|y(\tau)|) \leqq w(\tau)$, we can see that $y(t) \to 0$ as $t \to \infty$, so long as $\|x_t\| < \alpha$.

LEMMA 5. *Suppose that all assumptions in Lemma* 4 *are satisfied. Moreover, for the system*

$$\dot{x}(t) = f(t, x_t), \tag{3.9}$$

we assume that

(3. 10) (3. 9) *has a bounded solution* $u(t) = x(t; \varphi_0^*, t_0^*)$ *with a bound* $B_0 \geqq 0$

and that there exists a continuous Liapunov functional $V(t, \varphi_1, \varphi_2)$ *defined on* $D(T_0)$ *(see Section* 1) *and satisfying the following conditions*;

$$a_2(|\varphi_1(0) - \varphi_2(0)|) \leqq V(t, \varphi_1, \varphi_2) \leqq b_2(\|\varphi_1 - \varphi_2\|, \alpha) \tag{3.11}$$

on $\overline{D}_\alpha(T_0)$ *for any* $\alpha > 0$, *where* $a_2(r) > 0$ *and* $b_2(r, s) > 0$ *are continuous and non-decreasing functions of* $r > 0$, $s \geqq 0$ *and* $a_2(r) \to \infty$ *as* $r \to \infty$,

$$|V(t, \varphi_1, \varphi_2) - V(t, \varphi_1', \varphi_2')| \leqq L_2(\alpha)\{\|\varphi_1 - \varphi_1'\| + \|\varphi_2 - \varphi_2'\|\} \tag{3.12}$$

on $\overline{D}_\alpha(T_0)$ *for any* $\alpha > 0$, *where* $L_2(r) > 0$ *is continuous and non-decreasing*,

$$D^+_{(3.9)^*} V(t, \varphi_1, \varphi_2) \leqq 0 \tag{3.13}$$

(*for the notation* (3. 9)*, *see Section* 1).

Then, there exist continuous and monotone functions $B_1(r) > 0$, $T_2(r) \geqq 0$ *and* $H_2(r) > 0$ *of* $r > 0$ *such that for any* $\alpha > 0$, *all solutions of the systems* (3. 1) *and* (3. 2) *starting from* $\overline{\Omega}_\alpha(T_2, H_2)$ *remain in* $\Omega_{B_1(\alpha)}(H)$ *in the future and their y-components tend to zero as* $t \to \infty$.

PROOF. Let $B_1(r)$, $H_2(r)$ and $T_2(r)$ be chosen so that

$$(3.14)\quad \begin{cases} a_2(B_1(r)-B_0) > b_2(r+B_0, \max\{r, B_0\}), \\ \displaystyle\int_{T_2(r)}^{\infty} \lambda(t, B_1(r))\, dt < \frac{a_2(B_1(r)-B_0) - b_2(r+B_0, \max\{r, B_0\})}{L_2(B_1(r))}, \end{cases}$$

$$(3.15)\quad \begin{cases} H_2(r) = H_1(B_1(r)), \\ T_2(r) \geqq \max\{T_1(B_1(r)), t_0^*\}, \end{cases}$$

where $H_1(r)$ and $T_1(r)$ are those given in Lemma 4.

Let $(t_0, \varphi_0, \psi_0) \in \bar{\Omega}_\alpha(T_2, H_2)$, and let $(x(t), y(t))$ be a solution of the system (3.2) or (3.1) through (φ_0, ψ_0) at $t = t_0$. Obviously, by (3.15) we have

$$(3.16)\quad \|\psi_0\| \leqq H_2(\|\varphi_0\|) = H_1(B_1(\|\varphi_0\|)),\ t_0 \geqq T_2(\|\varphi_0\|) \geqq T_1(B_1(\|\varphi_0\|)).$$

Suppose that

$$(3.17)\quad (x_t, y_t) \in \bar{\Omega}(H) \quad \text{on} \quad [t_0, \tau]$$

for a $\tau > t_0$, and suppose that $\|x_t\| < B_1(\alpha)$ on $[t_0, \tau')$ for a τ', $\tau \geqq \tau' > t_0$. Then, for $v(t) = V(t, u_t, x_t)$, we have

$$D^+v(t) \leqq L_2(B_1(\alpha))\lambda(t, B_1(\alpha)) \quad \text{on} \quad [t_0, \tau')$$

by (1.6), (3.3), (3.12) and (3.13), which implies that

$$a_2(|x(\tau')-u(\tau')|) \leqq v(\tau') \leqq b_2(\alpha+B_0, \max\{\alpha, B_0\}) + L_2(B_1(\alpha))\int_{t_0}^{\tau'} \lambda(t, B_1(\alpha))\, dt$$

and hence by (3.14), $\|x_{\tau'}\| < B_1(\alpha)$. From this and $\|x_{t_0}\| = \|\varphi_0\| \leqq \alpha < B_1(\alpha)$, it follows that $\|x_t\| < B_1(\alpha)$ on $[t_0, \tau]$ or that

(3.18) under the assumption (3.17), we have

$$\|x_t\| < B_1(\|\varphi_0\|) \quad \text{on} \quad [t_0, \tau].$$

Now, we shall show that

$$(x_t, y_t) \in \Omega(H) \quad \text{for all} \quad t \geqq t_0.$$

If not, there exists a $\tau > t_0$ such that

$$(x_t, y_t) \in \Omega(H) \text{ for all } t \in [t_0, \tau) \text{ and } \|y_\tau\| = H(\|x_\tau\|).$$

Since we have (3. 17), by (3. 18) we have

$$\|x_t\| < B_1(\|\varphi_0\|) \quad \text{on} \quad [t_0, \tau].$$

Therefore, by using Lemma 4, this and (3. 16) imply that

$$(x_t, y_t) \in \Omega(H) \quad \text{on} \quad [t_0, \tau],$$

which contradicts with $\|y_\tau\| = H(\|x_\tau\|)$. Thus, we have that

$$(x_t, y_t) \in \Omega(H) \quad \text{for all} \quad t \geqq t_0$$

and that $y(t) \to 0$ as $t \to \infty$. On the other hand, we have

$$\|x_t\| < B_1(\|\varphi_0\|) \leqq B_1(\alpha) \quad \text{for all} \quad t \geqq t_0$$

by (3. 18). From these, it follows that

$$(x_t, y_t) \in \Omega_{B_1(\alpha)}(H) \quad \text{for all} \quad t \geqq t_0.$$

LEMMA 6. *In addition to all assumptions in Lemma* 4 *and the condition* (3. 10), *we assume that there exists a continuous Liapunov functional* $V(t, \varphi_1, \varphi_2)$ *defined on* $D(T_0)$ *and satisfying the condition* (3. 12) *and the following conditions*;

(3. 19) $$a_2(|\varphi_1(0) - \varphi_2(0)|) \leqq V(t, \varphi_1, \varphi_2) \leqq b_2(|\varphi_1(0) - \varphi_2(0)|, \alpha)$$

on $\bar{D}_\alpha(T_0)$ *for any* $\alpha > 0$, *where* $a_2(r)$ *and* $b_2(r, s)$ *satisfy the same conditions as in* (3. 11),

(3. 20) $$D^-_{(3.9)^*} V(t, \varphi_1, \varphi_2) \geqq 0$$

and that

(3. 21) *there exist functions* $\eta(r) > 0$ *and* $T'(r) \geqq 0$ *such that for any* $\alpha > 0$ *we have*

$$F(t, \alpha + \eta(\alpha))h \leqq \eta(\alpha) \quad \textit{for all} \quad t \geqq T'(\alpha),$$

where

$$F(t, r) = \sup\{|f(t, \varphi)| \mid \varphi \in \bar{C}_r(E^n)\}$$

for $f(t, \varphi)$ *in the system* (3. 1).

Then, for given $\varphi_0 \in C(E^n)$ *there exist continuous and monotone functions* $B_2(r) > 0$, $H_3(r) > 0$ *and* $T_3(r) \geqq 0$ *of* $r > 0$, *which may depend on* φ_0, *such that for any* $\alpha > 0$, *any* $(t_0, <x_0>, \psi_0) \in \bar{\Omega}_\alpha(T_3, H_3)$ *and any* $t_1 \geqq t_0$, *we can find a solution* $(x(t), y(t))$ *of the system* (3. 1) (*or* (3. 2)), *which satisfies*

$$(x_t, y_t) \in \Omega_{B_2(\alpha)}(H) \quad on \quad [t_0, t_1] \tag{3.22}$$

and

$$x(t_1) = x_0, y_{t_0} = \psi_0 \quad and \ x_{t_0} - \varphi_0 \ is \ a \ constant. \tag{3.23}$$

PROOF. First of all, let functions $B'(r) > 0$, $B''(r) > 0$ and $B(r) > 0$ be chosen so that

$$a_2(B'(r) - B_0) > b_2(r + B_0, \max\{r, B_0\}),$$

$$B''(r) > B'(r) + \|\varphi_0 - <\varphi_0(0)>\|,$$

$$B(r) = B''(r) + \eta(B''(r)),$$

where B_0 is the one given in (3. 10), and let $B_2(r)$ be a continuous and non-decreasing function such that

$$B_2(r) \geqq B(r). \tag{3.24}$$

Moreover, let $H_3(r)$ and $T_3(r)$ be such that

$$\begin{cases} H_3(r) = H_1(B_2(r)), \\ T_3(r) \geqq \max\{T_1(B_2(r)), T'(B''(r)), t_0^*\} \end{cases} \tag{3.25}$$

and

$$\begin{cases} \displaystyle\int_{T_3(r)}^{\infty} \lambda(t, B(r))\, dt < \frac{a_2(B'(r) - B_0) - b_2(r + B_0, \max\{r, B_0\})}{L_2(B(r))} \\ \displaystyle\int_{T_3(r)}^{\infty} \lambda(t, B(r))\, dt < B''(r) - B'(r) - \|\varphi_0 - <\varphi_0(0)>\|, \end{cases} \tag{3.26}$$

where t_0^* is the one given in (3. 10), $T'(r)$ is the one given in (3. 21), which can be assumed to be continuous and non-decreasing in r, and $H_1(r)$, $T_1(r)$ are those given in Lemma 4.

Now, for a given $(t_0, <x_0>, \psi_0) \in \bar{\Omega}_\alpha(T_3, H_3)$ we shall define $f^*(t, \varphi)$, $g^*(t, \varphi, \psi)$, $X^*(t, \varphi, \psi)$ and $Y^*(t, \varphi, \psi)$ by replacing (t, φ, ψ) in $f(t, \varphi)$, $g(t, \varphi, \psi)$, $X(t,$

$\varphi, \psi)$ and $Y(t, \varphi, \psi)$, respectively, with

$$(t, \varphi \min\{1, B(|x_0|)/\|\varphi\|\}, \psi \min\{1, H(\min\{\|\varphi\|, B(|x_0|)\})/\|\psi\|\}).$$

Then, we can easily see that these functions are continuous on $[t_0, \infty) \times C(E^n) \times C(E^m)$ and are bounded on $I' \times C(E^n) \times C(E^m)$ for any compact subinterval I' of $[t_0, \infty)$, because $f(t, \varphi)$ etc. are completely continuous. Moreover, we have

$$(3.27) \qquad \begin{cases} |f^*(t, \varphi)| \leqq F(t, B(|x_0|)), \\ |X^*(t, \varphi, \psi)| \leqq \lambda(t, B(|x_0|)) \end{cases}$$

on $[t_0, \infty) \times C(E^n) \times C(E^m)$.

By applying Theorem 1, for any $t_1 \geqq t_0$ there exists a solution $(x(t), y(t))$ of the system

$$(3.28) \qquad \begin{cases} \dot{x}(t) = f^*(t, x_t) + X^*(t, x_t, y_t) \\ \dot{y}(t) = g^*(t, x_t, y_t) + Y^*(t, x_t, y_t), \end{cases}$$

which satisfies the condition (3.23). Then, $(x(t), y(t))$ is a solution of the system (3.2) on $[t_0, t_1]$ satisfying the conditions (3.22) and (3.23), if $(x_t, y_t) \in \Omega_{B(|x_0|)}(H)$ for all $t, t_0 \leqq t \leqq t_1$ (note that $B(|x_0|) \leqq B_2(|x_0|) \leqq B_2(\alpha)$). Now, we shall prove that $(x_t, y_t) \in \Omega_{B(|x_0|)}(H)$ on $[t_0, t_1]$.

First of all, we shall show that

(3.29) if $|x(t)| < B'(|x_0|)$ for some $t, t_1 \geqq t \geqq t_0 + h$, then

$$\|x_t\| < B(|x_0|) - \|\varphi_0 - \langle\varphi_0(0)\rangle\|.$$

In fact, since

$$x(s) = x(t) - \int_s^t \{f^*(\tau, x_\tau) + X^*(\tau, x_\tau, y_\tau)\}\, d\tau$$

for any s, $t \geqq s \geqq t_0$, we have

$$|x(s)| \leqq |x(t)| + \int_s^t \{|f^*(\tau, x_\tau)| + |X^*(\tau, x_\tau, y_\tau)|\}\, d\tau$$

$$< B'(|x_0|) + \max_{t \geqq \tau \geqq s} F(\tau, B(|x_0|))(t-s) + \int_s^t \lambda(\tau, B(|x_0|))\, d\tau$$

by (3.27). From this, (3.21) and (3.26), it follows that

$$|x(s)| < B(|x_0|) - \|\varphi_0 - <\varphi_0(0)>\|$$

for any $s, t \geqq s \geqq t - h$, and hence

$$\|x_t\| < B(|x_0|) - \|\varphi_0 - <\varphi_0(0)>\| .$$

Suppose that

$$|x(t)| < B'(|x_0|) \quad \text{for all} \quad t, \tau < t \leqq t_1 ,$$

for a $\tau \geqq t_0 + h$, and let

$$v(t) = V(t, x_t, u_t) ,$$

where $u(t)$ is the one given in the condition (3.10). Then, by (3.29) and (1.6),

$$D^- v(t) \geqq - L_2(B(|x_0|))\lambda(t, B(|x_0|)) \quad \text{on} \quad (\tau, t_1] ,$$

and hence

$$v(\tau) \leqq v(t_1) + L_2(B(|x_0|)) \int_\tau^{t_1} \lambda(t, B(|x_0|))\, dt ,$$

which implies

$$|x(\tau)| < B'(|x_0|) .$$

Therefore, we can prove

$$|x(t)| < B'(|x_0|) \quad \text{on} \quad [t_0 + h, t_1] , \tag{3.30}$$

because $|x(t_1)| = |x_0| < B'(|x_0|)$.

Since we have $|x(t_0+h)| < B'(|x_0|)$, (3.29) implies that

$$|x(t_0)| \leqq \|x_{t_0+h}\| < B(|x_0|) - \|\varphi_0 - <\varphi_0(0)>\| . \tag{3.31}$$

By (3.23), $x_{t_0} - \varphi_0$ is a constant, and hence, we have

$$x_{t_0} = <x(t_0)> + \varphi_0 - <\varphi_0(0)> .$$

Thus, by (3.31) we have

$$\|x_{t_0}\| < B(|x_0|) .$$

From this, (3.29) and (3.30), it follows that

$$\|x_t\| < B(|x_0|) \quad \text{for all} \quad t, t_1 \geqq t \geqq t_0 . \tag{3.32}$$

Finally, since $(t_0, <x_0>, \psi_0) \in \overline{\Omega}(T_3, H_3)$, we have

$$\|\psi_0\| \leqq H_3(|x_0|) = H_1(B_2(|x_0|)) \leqq H_1(B(|x_0|)),$$
$$t_0 \geqq T_3(|x_0|) \geqq T_1(B_2(|x_0|)) \geqq T_1(B(|x_0|))$$

(see (3.25)), which implies that

$$(x_t, y_t) \in \Omega(H) \quad \text{for all} \quad t, t_1 \geqq t \geqq t_0$$

by (3.32) and Lemma 4. From this and (3.32), it follows that

$$(x_t, y_t) \in \Omega_{B(|x_0|)}(H) \qquad \text{on} \qquad [t_0, t_1].$$

Thus, we prove completely this lemma.

REMARK 1. In the case where $h = 0$ or $f(t, \varphi)$ is bounded on $\Omega(T_0, H_0)$, the condition (3.21) is always satisfied.

Now, we shall prove the following theorem concerning the eventually asymptotic equivalence between the systems (3.1) and (3.2).

THEOREM 2. *In addition to all assumptions given in Lemma* 5 *and the condition* (3.21), *we suppose that there exists a continuous Liapunov functional* $V(t, \varphi_1, \varphi_2)$ *which satisfies the conditions given in Lemma* 6.

Then, there exists a function $H^*(r) > 0$ *of* $r > 0$ *such that the systems* (3.1) *and* (3.2) *are eventually asymptotically equivalent on* $\Omega(H^*)$. *More precisely, for a fixed* $\varphi_0 \in C(E^n)$ *and any* $\alpha > 0$, *there exist an* $H'(\alpha) > 0$ *and* $T^*(\alpha) \geqq 0$ *such that for any solution of* (3.1) (*or* (3.2)) *starting from* $\Omega_\alpha(H^*)$ *at* $t = t_0$, $t_0 \geqq T^*(\alpha)$, *and for any* $\psi_0 \in C_{H'(\alpha)}(E^m)$, *we can find a solution of* (3.2) (*or* (3.1), *respectively*) *which passes through* $(\varphi_0 + <x_0>, \psi_0)$ *at* $t = t_0$ *for some* $x_0 \in E^n$ *and which tends to the given solution of* (3.1) (*or* (3.2)) *as* $t \to \infty$, *where* $H'(\alpha)$ *and* $T^*(\alpha)$ *may depend on* φ_0.

PROOF. Clearly, all conditions in Lemma 6 are satisfied. Let $H^*(r) = H_2(r)$, and let $H'(\alpha) > 0$ and $T^*(\alpha) \geqq 0$ be

$$\begin{cases} H'(\alpha) = \min \{H_2(B^*(\alpha)),\ H_3(B_1(\alpha))\}, \\ T^*(\alpha) = \max \{T_2(B^*(\alpha)),\ T_3(B_1(\alpha))\} \end{cases} \tag{3.33}$$

with $B^*(\alpha) = B_2(B_1(\alpha))$, where (H_2, T_2, B_1) is the one given in Lemma 5 and

(H_3, T_3, B_2) is the one in Lemma 6 for a given $\varphi_0 \in C(E^n)$.

Now, let $(\bar{x}(t), \bar{y}(t))$ be a given solution of the system (3.1) (or (3.2)) starting from $\Omega_\alpha(H^*)$ at $t = t_0$, $t_0 \geqq T^*(\alpha)$. Then, by Lemma 5, we can see that $(\bar{x}(t), \bar{y}(t))$ exists and remains in $\Omega_{B_1(\alpha)}(H)$ in the future and $\bar{y}(t) \to 0$ as $t \to \infty$. By applying Lemma 6 under the consideration of (3.33), for a fixed $\psi_0 \in C_{H'(\alpha)}(E^m)$ and for any $s \geqq t_0$ we can find a solution $(x(t; s), y(t; s))$ of the system (3.2) (or (3.1)) such that

$$x(s; s) = \bar{x}(s),\ y_{t_0}(s) = \psi_0 \text{ and } x_{t_0}(s) - \varphi_0 \text{ is a constant}$$

and that we have

$$(t_0, x_{t_0}(s), y_{t_0}(s)) \in \Omega_{B^*(\alpha)}(T_2, H_2)\,.$$

Moreover, by Lemma 5, we have

$$(x_t(s), y_t(s)) \in \Omega_{B(\alpha)}(H) \quad \text{for all } t \geqq t_0\,,$$

where $B(\alpha) = B_1(B^*(\alpha))$. Therefore, the family $\{(x(t; s), y(t; s)) | s \geqq t_0\}$ is uniformly bounded and equi-continuous on any compact subinterval of $[t_0, \infty)$. From this, it follows that there exists a divergent sequence $\{s_k\}$, $s_k \in [t_0, \infty)$, such that the sequence $\{(x(t; s_k), y(t; s_k))\}$ converges to an $(x(t), y(t))$ uniformly on any compact subinterval of $[t_0, \infty)$. Then, obviously $(x(t), y(t))$ is a solution of the system (3.2) (or (3.1), respectively) which satisfies the condition

$$x_{t_0} - \varphi_0 \text{ is a constant and } (t_0, x_{t_0}, y_{t_0}) \in \bar{\Omega}_{B^*(\alpha)}(T_2, H_2)\,. \tag{3.34}$$

Moreover, $y(t)$ tends to zero as $t \to \infty$ by Lemma 5 and (3.34).

Let $V(t, \varphi_1, \varphi_2)$ be the Liapunov functional satisfying the conditions in Lemma 6. For any $\varepsilon > 0$, if $\tau(\varepsilon, \alpha)$ is chosen so that

$$\int_{\tau(\varepsilon,\alpha)}^{\infty} \lambda(t, B(\alpha))\, dt < \frac{a_2(\varepsilon)}{L_2(B(\alpha))}\,,$$

we have

$$|x(t; s) - \bar{x}(t)| < \varepsilon \quad \text{for all} \quad t, s \geqq t \geqq \tau(\varepsilon, \alpha)\,,$$

because we have

$$V(t, x_t(s), \bar{x}_t) \leqq L_2(B(\alpha)) \int_t^s \lambda(\tau, B(\alpha))\, d\tau$$

for all t, $t_0 \leqq t \leqq s$. Hence, we have

$$|x(t) - \bar{x}(t)| \leqq \varepsilon \quad \text{for all} \quad t \geqq \tau(\varepsilon, \alpha),$$

that is, $x(t)$ tends to $\bar{x}(t)$ as $t \to \infty$. Thus, the proof is completed.

If all functions in the right-hand sides of the systems (3. 1) and (3. 2) and the Liapunov functional $W(t, \varphi, \psi)$ in Lemma 4 are defined on $\Delta(T_0) \times C(E^m)$ for some continuous and non-decreasing function $T_0(r) \geqq 0$ of $r > 0$ and if $a_1(r)$ in (3. 4) in Lemma 4 tends to infinity with r, then corresponding to the conditions (3. 3), (3. 4) and (3. 5) we assume the following conditions:

(3. 35) There is a continuous function $\lambda(t, \alpha, \beta) > 0$ such that

$$\int^{\infty} \lambda(t, \alpha, \beta)\, dt < \infty$$

and that

$$|X(t, \varphi, \psi)|, \ |Y(t, \varphi, \psi)| \leqq \lambda(t, \alpha, \beta)$$

on $\bar{\Delta}_\alpha(T_0) \times \bar{C}_\beta(E^m)$ for any $\alpha > 0$ and $\beta > 0$.

(3. 36) $$a_1(|\psi(0)|) \leqq W(t, \varphi, \psi) \leqq b_1(\|\psi\|, \alpha)$$

on $\bar{\Delta}_\alpha(T_0) \times C(E^m)$ for any $\alpha > 0$, where $a_1(r)$, $b_1(r, s)$ are continuous and non-decreasing in (r, s), $a_1(r) \to \infty$ as $r \to \infty$ and $a_1(r) > 0$ for $r > 0$.

(3. 37) $$|W(t, \varphi, \psi) - W(t, \varphi', \psi')| \leqq L(\alpha, \beta)\, \{\|\varphi - \varphi'\| + \|\psi - \psi'\|\}.$$

and

$$D^+_{(3.1)} W(t, \varphi, \psi) \leqq -\, c(\alpha, \beta)\, W(t, \varphi, \psi)$$

on $\bar{\Delta}_\alpha(T_0) \times \bar{C}_\beta(E^m)$ for any $\alpha > 0$ and $\beta > 0$, where $L(\alpha, \beta) > 0$ and $c(\alpha, \beta) > 0$.

In this case, by choosing a $K(\alpha, \beta) > 0$ so that

$$a_1(K(\alpha, \beta)) > b_1(\beta, \alpha),$$

we can replace $H_1(\alpha)$, $H(r)$ in the proof of Lemma 4 by $\beta, K(\alpha, \beta)$, respectively, and $H_2(\alpha)$, $H(r)$ in the proof of Lemma 5 by β, $K(B_1(\alpha), \beta)$, respectively, where $B_1(\alpha)$ is determined by the first inequality in (3. 14), if T_1 etc. in Lemmas 4, 5 and 6 are chosen depending on α and β.

Thus, we can prove the following corollary.

COROLLARY. *Let all functions in the right-hand sides of the systems*

(3.1) and (3.2) be defined on $\Delta(T_0) \times C(E^m)$ for some continuous and non-decreasing function $T_0(r)$. Suppose that there exist two continuous Liapunov functionals $V(t, \varphi_1, \varphi_2)$ in Lemmas 5 and 6, and moreover, suppose that there exists a continuous Liapunov functional $W(t, \varphi, \psi)$ defined on $\Delta(T_0) \times C(E^m)$ and satisfying the conditions (3.36) and (3.37).

If the conditions (3.10), (3.21) and (3.35) hold good, then the systems (3.1) and (3.2) are eventually asymptotically equivalent on $C(E^n) \times C(E^m)$.

4. Linear system and its perturbed system. By using the same arguments as used in Section 3, we shall discuss the asymptotic equivalence between a linear system

$$\dot{z}(t) = G(z_t) \tag{4.1}$$

and its perturbed system

$$\dot{z}(t) = G(z_t) + G^*(t, z_t)\,, \tag{4.2}$$

where z is an l-vector, $G(\xi)$ is a continuous linear function mapping $C(E^l)$ into E^l and $G^*(t, \xi)$ is continuous on $[0, \infty) \times C(E^l)$.

When $G(\xi)$ has just n eigenvalues with non-negative real parts (n may be zero), we can transform the systems (4.1) and (4.2) into the systems

$$\begin{cases} \dot{x}(t) = Ax(t) \\ \dot{y}(t) = G(y_t) \end{cases} \tag{4.3}$$

and

$$\begin{cases} \dot{x}(t) = Ax(t) + X(t, x_t, y_t) \\ y_t = z_{t-s}(\psi) + \displaystyle\int_s^t Z_{t-\tau} Y(\tau, x_\tau, y_\tau)\, d\tau \quad t \geqq s, \end{cases} \tag{4.4}$$

respectively, by a suitable transformation which preserves asymptotic equivalence properties and which transforms $\xi \in C(E^l)$ into $(x, \psi) \in E^n \times \widetilde{C}$, where $(s, \psi) \in [0, \infty) \times \widetilde{C}$ is a parameter and $\widetilde{C}$ is a subspace of $C(E^l)$ such that all eigenvalues of the restriction of $G(\xi)$ to $\widetilde{C}$ have negative real parts. For the transformation, see [4] and [9]. Here, A, $X(t, \varphi, \psi)$, $Y(t, \varphi, \psi)$, $z(t; \xi)$ and $Z(t)$ satisfy the following conditions: A is a constant (n, n)-matrix whose characteristic roots are zero or the eigenvalues of $G(\xi)$ with positive real parts. $X(t, \varphi, \psi)$ and $Y(t, \varphi, \psi)$ are continuous on $[0, \infty) \times C(E^n) \times \widetilde{C}$ and, actually, are functions of $(t, \varphi(0), \psi)$. $z(t; \xi)$ is a solution of (4.1) through ξ at $t = 0$. $Z(t)$ is an (l, l)-matrix solution of (4.1) through Γ at $t = 0$, where Γ is an (l, l)-matrix, components of which are piecewise continuous functions defined on $[-h, 0]$. Moreover, $z_t(\xi)$ and Z_t denote the segments of the functions $z(s; \xi)$ and $Z(s)$, respectively, at $s = t$.

Furthermore, we recall the following lemmas (cf. [4], [9]).

LEMMA 7. *For any continuous function $Y(t)$ and any $(s, \psi) \in [0, \infty) \times \widetilde{C}$*

$$z_{t-s}(\psi) + \int_s^t Z_{t-\tau} Y(\tau)\, d\tau$$

belongs to the space $\widetilde{C}$ for all $t \geqq s$.

LEMMA 8. *Let $\widehat{C}$ be any subspace of $C(E^l)$. If all eigenvalues of the restriction of $G(\xi)$ to $\widehat{C}$ have negative real parts, then there exist two positive constants c and L such that*

$$\|z_t(\xi, t_0)\| \leqq L\|\xi\| \exp[-c(t-t_0)] \quad \textit{for all} \quad t \geqq t_0\,,$$

so long as $z_t(\xi, t_0)$ belongs to $\widehat{C}$, where $z(t; \xi, t_0)$ is a solution of (4. 1) through ξ at $t = t_0$.

LEMMA 9. *The conditions*

(4. 5) *all solutions of (4. 1) are bounded,*

(4. 6) *there exists a continuous function $\lambda^*(t, \alpha) > 0$ such that*

$$\int_0^\infty \lambda^*(t, \alpha)\, dt < \infty \qquad (\textit{for any } \alpha > 0)$$

and that

$$|G^*(t, \xi)| \leqq \lambda^*(t, \alpha)$$

on $[0, \infty) \times \overline{C}_\alpha(E^l)$ for any $\alpha > 0$

imply that we have

(4. 7) *A is the zero matrix,*

(4. 8) *there exists a continuous function $\lambda(t, \alpha, \beta) > 0$ such that*

$$\int^\infty \lambda(t, \alpha, \beta)\, dt < \infty \qquad (\textit{for any } \alpha > 0,\ \beta > 0)$$

and that

$$|X(t,\varphi,\psi)|,\ |Y(t,\varphi,\psi)| \leqq \lambda(t,\alpha,\beta)$$

$$\textit{for all } (t,\varphi,\psi) \in [0,\infty) \times \overline{C}_\alpha(E^n) \times \widetilde{C},\ \|\psi\| < \beta,$$

respectively.

Since each eigenvalue of the restriction of the function $G(\xi)$ to $\widetilde{C}$ has a negative real part, by Lemmas 7 and 8 we have that if $(t_0,\psi) \in [0,\infty) \times \widetilde{C}$, then

$$\|z_t(\psi,t_0)\| \leqq L\|\psi\| \exp[-c(t-t_0)] \quad \text{for all} \quad t \geqq t_0\,, \tag{4.9}$$

where c and L are positive constants. Furthermore, the above implies the following lemma.

LEMMA 10. *There exists a continuous Liapunov functional $W(t,\psi)$ defined on $[0,\infty) \times \widetilde{C}$ and satisfying the conditions;*

$$\|\psi\| \leqq W(t,\psi) \leqq L\|\psi\|\,,$$

$$|W(t,\psi) - W(t,\psi')| \leqq L\|\psi-\psi'\|\,,$$

$$D^+_{(4.1)} W(t,\psi) \leqq -cW(t,\psi)\,,$$

where c and L are those in (4.9).

For a special system of (4.4)

$$\begin{cases} \dot{x}(t) = X(t,x_t,y_t) \\ y_t = z_{t-s}(\psi) + \displaystyle\int_s^t Z_{t-\tau} Y(\tau,x_\tau,y_\tau)\,d\tau\,, \end{cases} \tag{4.10}$$

we can prove the following theorem by the same arguments as used in the proof of Theorem 1.

THEOREM 3. *If $X(t,\varphi,\psi)$ and $Y(t,\varphi,\psi)$ are continuous and bounded on $[a,b] \times C(E^n) \times \widetilde{C}$, then for any $(x_0,\psi_0) \in E^n \times \widetilde{C}$ there exists a solution $(x(t),y_t)$ of the system* (4.10), *with $(s,\psi) = (a,\psi_0)$, such that $x(b) = x_0$ and $y_a = \psi_0$.*

Here, for the definition of solutions of the system (4.4) or (4.10), see [9].

Thus, by the same arguments used in Section 3, we can prove the eventually asymptotic equivalence between the systems (4.3) and (4.4) under the assumptions

(4. 7) and (4. 8), which implies the eventually asymptotic equivalence between the systems (4. 1) and (4. 2) under the assumptions (4 .5) and (4. 6). However, we should recall Lemma 9, and the followings should be noted here: By the condition (4. 7), corresponding to the system (3. 9), we have

$$\dot{x}(t) = 0.$$

Therefore, both Liapunov functionals $V(t, \varphi_1, \varphi_2)$ in Lemmas 5 and 6 can be defined by

$$V(t, \varphi_1, \varphi_2) = |\varphi_1(0) - \varphi_2(0)|,$$

and moreover, clearly the condition (3. 21) is satisfied. Corresponding to the Liapunov functional $W(t, \varphi, \psi)$ in Lemma 4, we use the Liapunov functional $W(t, \psi)$ given in Lemma 10. In this case, corresponding to the relation (1. 6), we have

$$D^+ w(t) \leqq -cw(t) + L\|\Gamma\| |Y(t, x_t, y_t)| \tag{4. 11}$$

for any solution $(x(t), y_t)$ of the system (4. 4), where $w(t) = W(t, y_t)$ (note Lemma 7) and $\|\Gamma\|$ denotes a suitable norm of the matrix Γ.

Now, we shall prove the relation (4. 11). Let $(x(t), y_t)$ be a solution of the system (4. 4) through $(\varphi_0, \psi_0) \in C(E^n) \times \tilde{C}$ at $t = t_0$. $z_t(\xi)$ is linear in $\xi \in C(E^l)$, $z_{t+\delta}(\xi, t) = z_\delta(\xi)$ and $z_\delta(z_t(\xi)) = z_{t+\delta}(\xi)$ for any $t \geqq 0$, $\delta \geqq 0$, $\xi \in C(E^l)$, and hence we have

$$z_{t+\delta}(y_t, t) = z_\delta(y_t) = z_{t+\delta-t_0}(\psi_0) + \int_{t_0}^{t} Z_{t+\delta-\tau} Y(\tau, x_\tau, y_\tau)\, d\tau,$$

where $z(t; \xi, t_0)$ is a solution of the system (4. 1) through ξ at $t = t_0$. From this and

$$y_{t+\delta} = z_{t+\delta-t_0}(\psi_0) + \int_{t_0}^{t+\delta} Z_{t+\delta-\tau} Y(\tau, x_\tau, y_\tau)\, d\tau,$$

it follows that

$$y_{t+\delta} - z_{t+\delta}(y_t, t) = \int_{t}^{t+\delta} Z_{t+\delta-\tau} Y(\tau, x_\tau, y_\tau)\, d\tau,$$

which implies that

$$\overline{\lim_{\delta\to+0}} \frac{1}{\delta} \|y_{t+\delta}-z_{t+\delta}(y_t, t)\| \leqq \|\Gamma\|\cdot|Y(t, x_t, y_t)| . \tag{4.12}$$

On the other hand, Lemma 7 implies $z_{t+\delta}(y_t, t) \in \widetilde{C}$, and moreover, we have

$$D^+_{(4.1)} W(t, \psi) = D^+_{(4.3)} W(t, \psi) .$$

Therefore, since

$$D^+ w(t) = \overline{\lim_{\delta\to+0}} \frac{1}{\delta} \{W(t+\delta, y_{t+\delta}) - W(t, y_t)\}$$

$$\leqq \overline{\lim_{\delta\to+0}} \frac{1}{\delta} \{W(t+\delta, z_{t+\delta}(y_t, t)) - W(t, y_t)\}$$

$$+ \overline{\lim_{\delta\to+0}} \frac{1}{\delta} \{W(t+\delta, y_{t+\delta}) - W(t+\delta, z_{t+\delta}(y_t, t))\}$$

$$\leqq -cW(t, y_t) + L \overline{\lim_{\delta\to+0}} \frac{1}{\delta} \|y_{t+\delta} - z_{t+\delta}(y_t, t)\| ,$$

we have the relation (4.11) by (4.12).

Thus, we have the following theorem.

THEOREM 4. *Under the assumptions* (4.5) *and* (4.6), *the systems* (4.1) *and* (4.2) *are eventually asymptotically equivalent on* $C(E^l)$.

COROLLARY 1. *If, in addition to the assumptions* (4.5) *and* (4.6), *we assume that* $\lambda^*(t, \alpha)$ *in* (4.6) *satisfies the condition*

(4.13) *there exist continuous functions* $\lambda_1(t) > 0$ *and* $M(\alpha) > 0$ *such that*

$$\int^\infty \lambda_1(t)\, dt < \infty , \quad \int^\infty \frac{d\alpha}{M(\alpha)} = \infty$$

and

$$\lambda^*(t, \alpha) \leqq \lambda_1(t)\, M(\alpha) ,$$

then the systems (4.1) *and* (4.2) *are asymptotically equivalent on* $C(E^l)$.

This corollary follows immediately from Theorem 4 and Proposition 2, because under all assumptions of Corollary 1, all solutions of the system (4.2) are bounded in the future (for the proof of this, see p.331 in [9]).

COROLLARY 2. *Consider a system*

$$\dot{z}(t) = G(z_t) + p(t) \tag{4.14}$$

and its perturbed system

$$\dot{z}(t) = G(z_t) + p(t) + G^*(t, z_t), \tag{4.15}$$

where $G(\xi)$, $G^*(t,\xi)$ *are the same ones in the systems* (4. 1) *and* (4. 2) *and* $p(t)$ *is continuous on* $[0,\infty)$. *In addition to the condition* (4. 6), *if we assume that*

(4. 16) *all solutions of the system* (4. 14) *are bounded,*

then the systems (4. 14) *and* (4. 15) *are eventually asymptotically equivalent on* $C(E^l)$.

Furthermore, if $\lambda^*(t,\alpha)$ *in* (4. 6) *satisfies the condition* (4. 13), *then the systems* (4. 14) *and* (4. 15) *are asymptotically equivalent on* $C(E^l)$.

PROOF. Let $\bar{z}(t)$ be a solution of (4.14). Then, $\bar{z}(t)$ is bounded by a constant B_0. Transform $z(t)$ in the systems (4. 14) and (4. 15) into $z^*(t)$ by $z(t) = \bar{z}(t) + z^*(t)$. Then, the systems (4. 14) and (4. 15) are transformed into systems of the forms of the systems (4. 1) and (4. 2), respectively. In this case, from the condition (4. 16), we have the condition (4. 5), and the perturbation terms satisfies the condition (4. 6), where $\lambda^*(t,\alpha)$ must be replaced by $\lambda^*(t,\alpha + B_0)$. Thus, the first part of this corollary can be proved by Theorem 4, and the second part by Corollary 1 of Theorem 4.

REMARK 2. In the case where $h = 0$, that is, the systems (4. 14) and (4. 15) are systems of ordinary differential equations, Corollary 2 is an extension of Theorem 2 in [1], where we should note that if $h=0$, under all assumptions of Corollary 2, all solutions of the system (4. 15) are defined on $[0, \infty)$.

5. Systems with more general perturbations. In this section, by applying the same idea in Section 3, we shall discuss the eventually asymptotic equivalence between a system

$$\begin{cases} \dot{x}(t) = f(t, x_t) \\ \dot{y}(t) = g(t, x_t, y_t, z_t) \\ \dot{z}(t) = \omega(t, x_t, z_t) \end{cases} \tag{5.1}$$

and its perturbed system

$$(5.2)\qquad \begin{cases} \dot{x}(t) = f(t, x_t) + X_1(t, x_t, y_t, z_t) + X_2(t, x_t, y_t, z_t) \\ \dot{y}(t) = g(t, x_t, y_t, z_t) + Y_1(t, x_t, y_t, z_t) + Y_2(t, x_t, y_t, z_t) \\ \dot{z}(t) = \omega(t, x_t, z_t) + Z_1(t, x_t, y_t, z_t) + Z_2(t, x_t, y_t, z_t), \end{cases}$$

where x, y, z are n, m, l-vectors and all functions in the right-hand sides of the systems (5. 1) and (5. 2) are completely continuous on $\Omega(T_0, H_0) \times C(E^l)$ for some continuous and monotone functions $T_0(r) \geqq 0$ and $H_0(r) > 0$.

Let $H(r)$ be a given continuous and non-increasing function of $r > 0$ such that $H_0(r) > H(r) > 0$. Now, we assume the following conditions:

(5. 3) There exist continuous functions $K(r) > 0$ and $\sigma(t,r) > 0$ such that

$$\int^{\infty} \sigma(t, r)\, dt < \infty \qquad (\text{for any } r > 0)$$

and that

$$|\omega(t, \varphi, \xi) - \omega(t, \varphi', \xi')| \leqq K(\alpha)\|\varphi - \varphi'\| + \sigma(t, \alpha)\|\xi - \xi'\|$$

on $\bar{\Delta}_\alpha(T_0) \times C(E^l)$ for any $\alpha > 0$.

(5. 4) For any $\alpha > 0$ and any $\beta > 0$ there exist $M(\alpha) > 0$ and $N(\alpha, \beta) > 0$ such that $N(\alpha, \beta) \to 0$ as $\beta \to 0$ and that

$$|X_1(t, \varphi, \psi, \xi)|,\ |Z_1(t, \varphi, \psi, \xi)| \leqq M(\alpha)\|\psi\|,$$

$$|Y_1(t, \varphi, \psi, \xi)| \leqq N(\alpha, \beta)\|\psi\|$$

for any $(t, \varphi, \psi, \xi) \in \bar{\Omega}_\alpha(T_0, H) \times C(E^l)$, $\|\psi\| \leqq \beta$.

(5. 5) There exists a continuous function $\lambda(t, r) > 0$ such that

$$\int^{\infty} \int_t^{\infty} \lambda(\tau, r)\, d\tau\, dt < \infty \qquad (\text{for any } r > 0)$$

and that $X_2(t, \varphi, \psi, \xi)$, $Y_2(t, \varphi, \psi, \xi)$ and $Z_2(t, \varphi, \psi, \xi)$ are bounded by $\lambda(t, \alpha)$ on $\bar{\Omega}_\alpha(T_0, H) \times C(E^l)$ for any $\alpha > 0$.

Here, we can assume that $K(r)$, $\sigma(t, r)$, $M(r)$, $N(r, s)$ and $\lambda(t, r)$ are continuous and non-decreasing in (r, s).

The following lemma corresponds to Lemma 4.

LEMMA 11. *In addition to the conditions* (5. 3), (5. 4) *and* (5. 5), *we* *assume that there exists a continuous Liapunov functional* $W(t,\varphi,\psi,\xi)$ *defined on* $\Omega(T_0,H_0)\times C(E^l)$ *and satisfying the following conditions;*

$$|\psi(0)|\leqq W(t,\varphi,\psi,\xi)\leqq b(\|\psi\|,\alpha) \tag{5.6}$$

on $\Omega_\alpha(T_0,H_0)\times C(E^l)$ *for any* $\alpha>0$,

$$\begin{aligned}&|W(t,\varphi,\psi,\xi)-W(t,\varphi',\psi',\xi')|\\ &\quad\leqq P(\alpha)\|\psi-\psi'\|+Q(\alpha,\beta)\{\|\varphi-\varphi'\|+\|\xi-\xi'\|\}\end{aligned} \tag{5.7}$$

for all (t,φ,ψ,ξ), $(t,\varphi',\psi',\xi')\in\bar{\Omega}_\alpha(T_0,H)\times C(E^l)$, $\psi,\psi'\in\bar{C}_\beta(E^m)$, *and for any* $\alpha>0$, $\beta>0$,

$$D^+_{(5.1)}W(t,\varphi,\psi,\xi)\leqq -c(\alpha)W(t,\varphi,\psi,\xi) \tag{5.8}$$

on $\bar{\Omega}_\alpha(T_0,H)\times C(E^l)$ *for any* $\alpha>0$,

where $b(s,r)$, $P(r)$, $Q(r,s)$ *and* $c(r)$ *are positive, continuous and monotone in* $r>0$, $s>0$ *and* $b(0,r)=0$, $Q(r,0)=0$ *for all* $r>0$.

Then, for any $\alpha>0$ *there exist an* $H_1(\alpha)>0$ *and a* $T_1(\alpha)\geqq 0$ *and, moreover, for any* $\beta>0$, $\beta\leqq H_1(\alpha)$, *and any* $\gamma>0$ *there exist an* $A_1(s,t,\alpha,\beta,\gamma)>0$ *and a* $C_1(s,t,\alpha,\beta)>0$ *such that if* $(x(t),y(t),z(t))$ *is a solution of the system* (5.1) *or* (5.2) *starting from* $C(E^n)\times C_\beta(E^m)\times C_\gamma(E^l)$ *at* $t=t_0$, $t_0\geqq T_1(\alpha)$, *then we have*

$\|y_t\|<C_1(t_0,t,\alpha,\beta)$ *and* $\|z_t\|<A_1(t_0,t,\alpha,\beta,\gamma)$ *for all* $t\geqq t_0$, *so long as* $\|x_t\|\leqq\alpha$. *Here, we can assume that* $A_1(s,t,\alpha,\beta,\gamma)$ *and* $C_1(s,t,\alpha,\beta)$ *satisfy the following conditions: They are continuous in all their arguments and monotone in* (t,α,β,γ). $C_1(s,t,\alpha,\beta)\to 0$ *as* $t\to\infty$,

$$\int^\infty\int_t^\infty C_1(s,\tau,\alpha,\beta)\,d\tau\,dt<\infty$$

and $C_1(s,t,\alpha,\beta)<H(\alpha)$ *for any* $\alpha>0$, *any* $\beta>0$ $(\beta\leqq H_1(\alpha))$, *any* $s\geqq T_1(\alpha)$ *and any* $t\geqq s$. *Furthermore, for any* $\alpha>0$ *and any* $\varepsilon>0$ *we can find an* $H(\varepsilon,\alpha)>0$ *and a* $T(\varepsilon,\alpha)\geqq 0$ *such that for any* $\beta>0$, $\beta\leqq H(\varepsilon,\alpha)$, *and any* $s\geqq T(\varepsilon,\alpha)$ *we have*

$$\int_s^\infty C_1(s,t,\alpha,\beta)\,dt<\varepsilon \quad\text{and}\quad \int_s^\infty\int_t^\infty C_1(s,\tau,\alpha,\beta)\,d\tau\,dt<\varepsilon.$$

PROOF. Let $(x(t),y(t),z(t))$ be a solution of the system (5.1) or (5.2), and let

$$w(t) = W(t, x_t, y_t, z_t).$$

Then, by (1.6), (5.4), (5.5), (5.6), (5.7) and (5.8), we have

$$D^+w(t) \leqq - \{c(\alpha) - P(\alpha)N(\alpha, \beta) - 2Q(\alpha, \beta)M(\alpha)\}\, w(t) + L^*(\alpha, \beta)\lambda(t, \alpha),$$

so long as

(5.9) $$(t, x_t, y_t, z_t) \in \bar{\Omega}_\alpha(T_0, H) \times C(E^l) \quad \text{and} \quad \|y_t\| \leqq \beta$$

for any $\alpha > 0$ and any $\beta > 0$, where

$$L^*(\alpha, \beta) = P(\alpha) + 2Q(\alpha, \beta).$$

Therefore, if we choose $H^*(\alpha) > 0$, $H_1(\alpha) > 0$ and $T_1(\alpha) \geqq 0$ so that

$$P(\alpha)\, N(\alpha, H^*(\alpha)) + 2Q(\alpha, H^*(\alpha))\, M(\alpha) \leqq c(\alpha)/2,$$

$$H^*(\alpha) \leqq H(\alpha),$$

$$b(H_1(\alpha), \alpha) < H^*(\alpha)/R \quad \text{for some constant} \quad R > 1,$$

$$\int_{T_1(\alpha)}^{\infty} \lambda(t, \alpha)\, dt < \frac{H^*(\alpha) - Rb(H_1(\alpha), \alpha)}{L^*(\alpha, H^*(\alpha))},$$

$$T_1(\alpha) \geqq T_0(\alpha),$$

and if $\|y_{t_0}\| \leqq \beta \leqq H_1(\alpha)$ for a $t_0 \geqq T_1(\alpha)$, then by the same arguments as used in the proof of Lemma 4, we have that

$$|y(t)| \leqq w(t) < H^*(\alpha)$$

and

(5.10) $$|y(t)| \leqq w(t) < C_1(t_0, t, \alpha, \beta) \quad \text{for all} \quad t \geqq t_0,$$

so long as $(x_t, z_t) \in C_\alpha(E^n) \times C(E^l)$,

because

$$D^+w(t) \leqq - \frac{c(\alpha)}{2} w(t) + L^*(\alpha, H^*(\alpha))\lambda(t, \alpha),$$

if (5.9) holds good with $\beta = H^*(\alpha)$, where

$$C_1(s, t, \alpha, \beta) = Rb(\beta, \alpha) \exp\left[- \frac{c(\alpha)}{2}(t - s)\right]$$

$$+ L^*(\alpha, H^*(\alpha)) \int_s^t \lambda(\tau, \alpha) \exp\left[-\frac{c(\alpha)}{2}(t-\tau)\right] d\tau .$$

Easily, it can be verified that $C_1(s, t, \alpha, \beta)$ satisfies all conditions required in this lemma.

Now, we shall show that if $\|y_{t_0}\| \leqq \beta \leqq H_1(\alpha)$ and $t_0 \geqq T_1(\alpha)$, then $z(t)$ exists in the future, so long as $\|x_t\| \leqq \alpha$. Let $\bar{z}(t)$ be a solution of the system

$$\dot{z}(t) = \omega(t, <0>, z_t)$$

through $<0>$ at $t = t_0$ (for the notation $<0>$, see Section 1). By the condition (5. 3), we have

$$|\bar{z}(s)| \leqq \int_{t_0}^t |\omega(\tau, <0>, <0>)| \, d\tau + \int_{t_0}^t \sigma(\tau, 0)\|\bar{z}_\tau\| \, d\tau$$

for any $s, t \geqq s \geqq t_0$, that is,

$$\|\bar{z}_t\| \leqq \int_{t_0}^t |\omega(\tau, <0>, <0>)| \, d\tau + \int_{t_0}^t \sigma(\tau, 0)\|\bar{z}_\tau\| \, d\tau$$

for any $t \geqq t_0$, which implies that

$$\|\bar{z}_t\| \leqq \gamma_0(t_0, t) \quad \text{for all} \quad t \geqq t_0,$$

where

$$\gamma_0(s, t) = \left\{\int_s^t |\omega(\tau, <0>, <0>)| \, d\tau\right\} \exp\left[\int_s^t \sigma(\tau, 0) \, d\tau\right],$$

and hence, $\bar{z}(t)$ exists in the future. Therefore, by the conditions (5. 3), (5. 4), (5. 5) and the assertion (5. 10), we have

$$\|z_t - \bar{z}_t\| \leqq \|z_{t_0}\| + K(\alpha)\,\alpha(t-t_0) + \int_{t_0}^t \sigma(\tau, \alpha)\|z_\tau - \bar{z}_\tau\| \, d\tau$$

$$+ M(\alpha)\int_{t_0}^t C_1(t_0, \tau, \alpha, \beta) \, d\tau + \int_{t_0}^t \lambda(\tau, \alpha) \, d\tau ,$$

which implies that if $\|z_{t_0}\| < \gamma$,

$$\|z_t\| < A_1(t_0, t, \alpha, \beta, \gamma) \quad \text{for all} \quad t \geqq t_0 ,$$

so long as $\|x_t\| \leqq \alpha$, where

$$A_1(s, t, \alpha, \beta, \gamma) = \left\{\gamma + K(\alpha)\,\alpha(t-s) + M(\alpha)\int_s^t C_1(s, \tau, \alpha, \beta)d\tau + \int_s^t \lambda(\tau, \alpha)\,d\tau\right\} \exp\left[\int_s^t \sigma(\tau, \alpha)\,d\tau\right] + \gamma_0(s, t),$$

and hence, $z(t)$ exists in the future, so long as $\|x_t\| \leqq \alpha$. Thus, the proof of this lemma is completed.

LEMMA 12. *In addition to all assumptions of Lemma* 11, *we assume that there exists a continuous Liapunov functional* $V(t, \varphi_1, \varphi_2)$ *in Lemma* 5 *and that the condition* (3.10) *holds good, where* $f(t, \varphi)$ *in the system* (3.9) *mentioned in* (3.10) *is the one in the system* (5.1).

Then, there exists a continuous and non-increasing function $H_2(r) > 0$, *and for any* $\alpha > 0$ *and any* $\gamma > 0$ *there exist* $B_1(\alpha) > 0$, $T_2(\alpha) \geqq 0$, $C_2(s, t, \alpha) > 0$ *and* $A_2(s, t, \alpha, \gamma) > 0$ *such that if* $(x(t), y(t), z(t))$ *is a solution of the system* (5.1) *or* (5.2) *starting from* $\Omega_\alpha(H_2) \times C_\gamma(E^l)$ *at* $t = t_0$, $t_0 \geqq T_2(\alpha)$, *then we have*

$$\|x_t\| < B_1(\alpha), \ \|y_t\| < C_2(t_0, t, \alpha), \ \|z_t\| < A_2(t_0, t, \alpha, \gamma)$$

for all $t \geqq t_0$, *where* $C_2(s, t, \alpha)$ *and* $A_2(s, t, \alpha, \gamma)$ *are continuous in* (s, t, α, γ) *and monotone in* (t, α, γ). *Moreover,* $C_2(s, t, \alpha)$ *satisfies the following conditions:* $C_2(s, t, \alpha)$ *tends to zero as* $t \to \infty$ *and*

$$\int^\infty \int_t^\infty C_2(s, \tau, \alpha)\,d\tau\,dt < \infty\,.$$

For any $s \geqq T_2(\alpha)$ *and any* $t \geqq s$, *we have*

$$C_2(s, t, \alpha) \leqq H(B_1(\alpha))\,.$$

This lemma can be proved by Lemma 11 and by using the same arguments as used in the proof of Lemma 5 and by choosing H_2, B_1, T_2, C_2 and A_2 as follows: Let $H_1(r)$, $T_1(\alpha)$, $C_1(s, t, \alpha, \beta)$, $A_1(s, t, \alpha, \beta, \gamma)$, $H(\varepsilon, \alpha)$ and $T(\varepsilon, \alpha)$ are those given in Lemma 11, and let $t_0{}^*$ and B_0 are those given in the condition (3.10). $B_1(r)$ and $H_2(r)$ are chosen so that

$$a_2(B_1(r) - B_0) > b_2(r + B_0, \max\{r, B_0\}) \tag{5.11}$$

and

$$H_2(r) \leqq \min\{H_1(B_1(r)), H(\varepsilon(r), B_1(r))\}$$

with

$$\varepsilon(r) = \frac{a_2(B_1(r)-B_0) - b_2(r+B_0, \max\{r, B_0\})}{2L_2(B_1(r))\,M(B_1(r))}.$$

$T_2(\alpha)$ are determined so as to satisfy

$$\int_{T_2(\alpha)}^{\infty} \lambda(t, B_1(\alpha))\,dt < \varepsilon(\alpha)\,M(B_1(\alpha)),$$

$$T_2(\alpha) \geqq \max\{t_0^*, T_1(B_1(\alpha)), T(\varepsilon(\alpha), B_1(\alpha))\}.$$

Finally, $C_2(s, t, \alpha)$ and $A_2(s, t, \alpha, \gamma)$ are defined by

$$C_2(s, t, \alpha) = C_1(s, t, B_1(\alpha), H_2(\alpha)),$$

$$A_2(s, t, \alpha, \gamma) = A_1(s, t, B_1(\alpha), H_2(\alpha), \gamma).$$

LEMMA 13. *In addition to all assumptions given in Lemma* 11, *we assume that* $f(t, \varphi)$ *in the system* (5.1) *satisfies the condition* (3.21) *and that there exists a continuous Liapunov functional* $V(t, \varphi_1, \varphi_2)$ *in Lemma* 6 *with* $a_2(r) = r$ *and* $b_2(0, s) = 0$.

Let $(\bar{x}(t), \bar{y}(t), \bar{z}(t))$ *be a given solution of the system* (5.1) (*or* (5.2)) *such that* $\|\bar{x}_t\| \leqq \alpha$ *for all* $t \geqq t_0'$, *where* $t_0' \geqq 0$ *is a constant. In the case where* $(\bar{x}(t), \bar{y}(t), \bar{z}(t))$ *is a solution of the system* (5.2), *we add the assumption that we have*

$$\|\bar{y}_t\| < C^*(t, \alpha) \quad \textit{for all} \quad t \geqq t_0',$$

where $C^*(t, \alpha) > 0$ *is continuous in* (t, α), *non-decreasing in* α, $C^*(t, \alpha) \leqq H(\alpha)$ *for all* $t \geqq t_0'$ *and*

$$\int^{\infty}\int_t^{\infty} C^*(\tau, \alpha)\,d\tau\,dt < \infty.$$

Then, for a given $(\varphi_0, \xi_0) \in C(E^n) \times C(E^l)$ *and for any* $\alpha > 0$ *there exist a* $T_3(\alpha) \geqq 0$, *an* $H_3(\alpha) > 0$, *a* $B_2(\alpha) > 0$ *and an* $A(\alpha) > 0$ *such that for any* $(t_0, \psi_0) \in [T_3(\alpha), \infty) \times C_{H_3(\alpha)}(E^m)$ *and any* $t_1 \geqq t_0$, *we can find a solution* $(x(t), y(t), z(t))$ *of the system* (5.2) (*or* (5.1), *respectively*) *defined on* $[t_0, t_1]$ *and which satisfies the conditions*;

$$(5.12) \qquad \begin{cases} x(t_1) = \bar{x}(t_1),\ y_{t_0} = \psi_0,\ z(t_1) = \bar{z}(t_1), \\ x_{t_0} - \varphi_0 \ \textit{and}\ z_{t_0} - \xi_0 \ \textit{are constants}, \end{cases}$$

(5. 13) $\quad \|x_t\| < B_2(\alpha)$ *and* $\|z_t - \bar{z}_t\| < A(\alpha)$ *for all* $t,\ t_0 \leqq t \leqq t_1$.

Moreover, we can find continuous functions $B^*(t, \alpha) > 0$ *and* $A^*(t, \alpha) > 0$ *such that*

$$|x(t) - \bar{x}(t)| \leqq B^*(t, \alpha) \text{ and } |z(t) - \bar{z}(t)| \leqq A^*(t, \alpha)$$

for all $t,\ t_0 + h \leqq t \leqq t_1$, *and that* $B^*(t, \alpha)$ *and* $A^*(t, \alpha)$ *are non-decreasing in* α *and tend to zero as* $t \to \infty$.

PROOF. We choose $B'(\alpha) > 0$, $B''(\alpha) > 0$, $B_2(\alpha) > 0$, $A'(\alpha) > 0$ and $A(\alpha) > 0$ so that

$$\left\{\begin{aligned} & B'(\alpha) > \alpha,\ B''(\alpha) > B'(\alpha) + \|\varphi_0 - <\varphi_0(0)>\|, \\ & B_2(\alpha) = B''(\alpha) + \eta(B''(\alpha)) \end{aligned}\right. \tag{5. 14}$$

and

$$A'(\alpha) > K(B_2(\alpha))\{B_2(\alpha) + \alpha\}\, h,$$
$$A(\alpha) = A'(\alpha) + \|\xi_0 - <\xi_0(0)>\|.$$

Letting

$$\bar{A}(\alpha) = A'(\alpha) - K(B_2(\alpha))\{B_2(\alpha) + \alpha\}\, h,$$
$$\bar{B}(\alpha) = B''(\alpha) - B'(\alpha) - \|\varphi_0 - <\varphi_0(0)>\|$$

and

$$\varepsilon(\alpha) = \min\left\{\frac{B'(\alpha) - \alpha}{2L_2(B_2(\alpha))}, \frac{\bar{B}(\alpha)}{2}, \frac{\bar{A}(\alpha)}{5}, \frac{\bar{A}(\alpha)\, M(B_2(\alpha))}{5K(B_2(\alpha))L_2(B_2(\alpha))}\right\} \times \frac{1}{M(B_2(\alpha))},$$

we choose $H_3(\alpha)$ so that

$$H_3(\alpha) \leqq \min\{H_1(B_2(\alpha)), H(\varepsilon(\alpha), B_2(\alpha))\}.$$

Finaly, let $T_3(\alpha)$ be so large that

$$\int_{T_3(\alpha)}^{\infty} \lambda(t, B_2(\alpha))\, dt, \int_{T_3(\alpha)}^{\infty} \int_t^{\infty} \lambda(\tau, B_2(\alpha))\, d\tau\, dt < \varepsilon(\alpha)\, M(B_2(\alpha)),$$

$$\int_{T_3(\alpha)}^{\infty} \sigma(t, B_2(\alpha))\, dt < \frac{\bar{A}(\alpha)}{5A(\alpha)},$$

$$\int_{T_3(\alpha)}^{\infty} C^*(t, \alpha)\, dt, \int_{T_3(\alpha)}^{\infty}\int_t^{\infty} C^*(\tau, \alpha)\, d\tau\, dt < \varepsilon(\alpha)$$

and that

$$T_3(\alpha) \geqq \max\{t_0', T'(B''(\alpha)), T_1(B_2(\alpha)), T(\varepsilon(\alpha), B_2(\alpha))\}.$$

Here, $T_1(\alpha)$, $H_1(\alpha)$, $H(\varepsilon, \alpha)$, $T(\varepsilon, \alpha)$ are those given in Lemma 11 and $\eta(\alpha)$, $T'(\alpha)$ are those given in the condition (3.21).

Replacing φ, ψ and ξ in $f(t, \varphi)$, $g(t, \varphi, \psi, \xi)$, $\omega(t, \varphi, \xi)$, $X_i(t, \varphi, \psi, \xi)$, $Y_i(t, \varphi, \psi, \xi)$, $Z_i(t, \varphi, \psi, \xi)$ $(i = 1, 2)$ by

$$\varphi \min\{1, B_2(\alpha)/\|\varphi\|\}, \ \psi \min\{1, C(t, \alpha)/\|\psi\|\}$$

and

$$\bar{z}_t + (\xi - \bar{z}_t)\min\{1, A(\alpha)/\|\xi - \bar{z}_t\|\},$$

we denote them by $f^*(t, \varphi)$, $g^*(t, \varphi, \psi, \xi)$, $\omega^*(t, \varphi, \xi)$, $X_i^*(t, \varphi, \psi, \xi)$, $Y_i^*(t, \varphi, \psi, \xi)$, $Z_i^*(t, \varphi, \psi, \xi)$ $(i = 1, 2)$, respectively. Here, for a fixed $t_0 \geqq T_3(\alpha)$

$$C(t, \alpha) = C_1(t_0, t, B_2(\alpha), H_3(\alpha)),$$

where $C_1(s, t, \alpha, \beta)$ is the one in the Lemma 11.

Consider the systems

$$\text{(5.15)} \qquad \begin{cases} \dot{x}(t) = f^*(t, x_t) \\ \dot{y}(t) = g^*(t, x_t, y_t, z_t) \\ \dot{z}(t) = \omega^*(t, x_t, z_t) \end{cases}$$

and

$$\text{(5.16)} \qquad \begin{cases} \dot{x}(t) = f^*(t, x_t) + X_1^*(t, x_t, y_t, z_t) + X_2^*(t, x_t, y_t, z_t) \\ \dot{y}(t) = g^*(t, x_t, y_t, z_t) + Y_1^*(t, x_t, y_t, z_t) + Y_2^*(t, x_t, y_t, z_t) \\ \dot{z}(t) = \omega^*(t, x_t, z_t) + Z_1^*(t, x_t, y_t, z_t) + Z_2^*(t, x_t, y_t, z_t). \end{cases}$$

Obviously, all functions in the right-hand sides of the systems (5.15) and (5.16) are continuous and bounded on $I \times C(E^n) \times C(E^m) \times C(E^l)$ for any compact subinterval I of $[t_0, \infty)$. Moreover, especially we have

$$(5.17)\qquad \begin{cases} |f^*(t,\varphi)| \leqq F(t, B_2(\alpha)), \\ |X_1^*(t,\varphi,\psi,\xi)|,\ |Z_1^*(t,\varphi,\psi,\xi)| \leqq M(B_2(\alpha))\,C(t,\alpha), \\ |X_2^*(t,\varphi,\psi,\xi)|,\ |Z_2^*(t,\varphi,\psi,\xi)| \leqq \lambda(t, B_2(\alpha)) \end{cases}$$

on $[t_0,\infty)\times C(E^n)\times C(E^m)\times C(E^l)$ and

$$(5.18)\qquad |\omega^*(t,\varphi,\xi)-\omega(t,\varphi,\bar{z}_t)| \leqq \sigma(t, B_2(\alpha))\,A(\alpha)$$

on $[t_0,\infty)\times C_{B_2(\alpha)}(E^n)\times C(E^l)$.

Therefore, for any $(t_1,\psi_0)\in[t_0,\infty)\times C(E^m)$ there exists a solution $(x(t), y(t), z(t))$ of (5.16) (or (5.15), respectively) satisfying the condition (5.12) by Theorem 1.

Now, we shall show that if $\|\psi_0\|<H_3(\alpha)$, then

$$\|x_t\|<B_2(\alpha),\ \|y_t\|<C(t,\alpha),\ \|z_t-\bar{z}_t\|<A(\alpha)$$

for all t, $t_0\leqq t\leqq t_1$, which implies that $(x(t), y(t), z(t))$ is a solution of the system (5.2) (or (5.1), respectively) on $[t_0,t_1]$. This will be proved by the same idea as in the proof of Lemma 6.

Since we have the condition (5.17) and $t_0\geqq T_3(\alpha)$, by the same manner as in Section 3, we can prove the assertion (3.29), where $B(|x_0|)$ and $B(|x_0|)$ in (3.29) are replaced by $B'(\alpha)$ and $B_2(\alpha)$ given in (5.14). Therefore, considering the function

$$v(t)=V(t,x_t,\bar{x}_t),$$

we can see that

$$(5.19)\qquad \|x_t\|<B_2(\alpha)\quad \text{for all}\quad t, t_1\geqq t\geqq t_0,$$

and that

$$|x(t)-\bar{x}(t)|\leqq B^*(t,\alpha)\quad \text{for all}\quad t, t_1\geqq t\geqq t_0,$$

where

$$B^*(t,\alpha)=L_2(B_2(\alpha))\,M(B_2(\alpha))\int_t^\infty \bar{C}(\tau,\alpha)\,d\tau+L_2(B_2(\alpha))\int_t^\infty \lambda(\tau, B_2(\alpha))\,d\tau.$$

Here, if $(\bar{x}(t),\bar{y}(t),\bar{z}(t))$ is a solution of the system (5.1), then $\bar{C}(t,\alpha)=C(t,\alpha)$, and if $(\bar{x}(t),\bar{y}(t),\bar{z}(t))$ is a solution of the system (5.2), then $\bar{C}(t,\alpha)=\max\{C(t,\alpha), C^*(t,\alpha)\}$. Obviously, $B^*(t,\alpha)$ tends to zero as $t\to\infty$ and we have

$$(5.20)\qquad \begin{cases} \displaystyle\int_{T_3(\alpha)}^{\infty} B^*(t,\alpha)\,dt < 2\varepsilon(\alpha)\,L_2(B_2(\alpha))\,, \\ \displaystyle\int_{t_0}^{t_1} \|x_t-\bar{x}_t\|\,dt \leqq (B_2(\alpha)+\alpha)h + \int_{t_0}^{t_1-h} B^*(t,\alpha)\,dt\,, \end{cases}$$

$$(5.21)\qquad \int_s^{t_1} \|x_t-\bar{x}_t\|\,dt \leqq \int_s^{t_1} B^*(t-h,\alpha)\,dt \quad \text{for all} \quad s,t_1 \geqq s \geqq t_0+h\,.$$

Next, we shall show that

$$\|z_t-\bar{z}_t\| < A(\alpha) \quad \text{for all} \quad t,t_1 \geqq t \geqq t_0\,.$$

Since for any $t, t_1 \geqq t \geqq t_0$,

$$z(t)-\bar{z}(t) = \int_t^{t_1} \Big\{ \omega^*(s,x_s,z_s) - \omega(s,\bar{x}_s,\bar{z}_s) - Z_1(s,\bar{x}_s,\bar{y}_s,\bar{z}_s) - Z_2(s,\bar{x}_s,\bar{y}_s,\bar{z}_s) \Big\}\,ds$$

or

$$z(t)-\bar{z}(t) = \int_t^{t_1} \Big\{ \omega^*(s,x_s,z_s) - \omega(s,\bar{x}_s,\bar{z}_s) + Z_1^*(s,x_s,y_s,z_s) + Z_2^*(s,x_s,y_s,z_s) \Big\}\,ds$$

according as $(x(t),y(t),z(t))$ is a solution of the system (5. 15) or (5. 16), we have

$$(5.22)\qquad |z(t)-\bar{z}(t)| \leqq \int_t^{t_1} \Big\{ K(B_2(\alpha))\|x_s-\bar{x}_s\| + \sigma(s,B_2(\alpha))\,A(\alpha) + M(B_2(\alpha))\,\bar{C}(s,\alpha) + \lambda(s,B_2(\alpha)) \Big\}\,ds$$

for all $t, t_1 \geqq t \geqq t_0$, by the conditions (5. 17), (5. 18) and (5. 19). By (5. 20), we have

$$\|z(t)-\bar{z}(t)\| < A(\alpha) - \|\xi_0 - \langle\xi_0(0)\rangle\|\,,$$

which implies that

$$\|z_t - \bar{z}_t\| < A(\alpha) \quad \text{for all} \quad t, t_1 \geqq t \geqq t_0 .$$

Moreover, from (5. 21) and (5. 22), it follows that

$$|z(t) - \bar{z}(t)| \leqq A^*(t, \alpha) \quad \text{for all} \quad t, t_1 \geqq t \geqq t_0 + h,$$

where

$$A^*(t, \alpha) = K(B_2(\alpha)) \int_{t-h}^{\infty} B^*(\tau, \alpha)\, d\tau + A(\alpha) \int_t^{\infty} \sigma(\tau, B_2))\, d\tau$$

$$+ M(B_2(\alpha)) \int_t^{\infty} \bar{C}(\tau, \alpha)\, d\tau + \int_t^{\infty} \lambda(\tau, B_2(\alpha))\, d\tau,$$

and clearly, $A^*(t, \alpha)$ tends to zero as $t \to \infty$.

Finally, we shall prove that if $\|\psi_0\| < H_3(\alpha)$ and $t_0 \geqq T_3(\alpha)$, then

$$\|y_t\| < C(t, \alpha) \quad \text{for all} \quad t, t_1 \geqq t \geqq t_0 .$$

Suppose that there exists a $\tau, t_0 < \tau \leqq t_1$, such that

$$\|y_\tau\| = C(\tau, \alpha) \text{ and } \|y_t\| < C(t, \alpha) \text{ on } [t_0, \tau).$$

As was seen above, $\|x_t\| < B_2(\alpha)$ and $\|z_t - \bar{z}_t\| < A(\alpha)$ on $[t_0, \tau]$, and hence $(x(t), y(t), z(t))$ is a solution of the system (5. 2) (or (5. 1)) on $[t_0, \tau]$. Therefore, Lemma 11 implies

$$\|y_t\| < C(t, \alpha) \quad \text{on} \quad [t_0, \tau],$$

which contradicts $\|y_\tau\| = C(\tau, \alpha)$. Thus, we prove completely this lemma.

Corresponding to Theorem 2, by applying Lemmas 12 and 13 instead of Lemmas 5 and 6, we can prove the following theorem.

THEOREM 5. *Suppose that the condition* (3. 21) *and all assumptions in Lemma* 12 *are satisfied and that there exists a continuous Liapunov functional* $V(t, \varphi_1, \varphi_2)$ *in Lemma* 13.

Then, there exists a continuous function $H^*(r) > 0$ *such that the systems* (5. 1) *and* (5. 2) *are eventually asymptotically equivalent on* $\Omega(H^*) \times C(E^l)$. *More precisely, for a given* $(\varphi_0, \xi_0) \in C(E^n) \times C(E^l)$ *and any* $\alpha > 0$ *there exist an* $H(\alpha) > 0$ *and a* $T^*(\alpha) \geqq 0$ *such that for any solution of the system* (5. 1) (*or* (5. 2)) *starting from* $\Omega_\alpha(H^*) \times C(E^l)$ *at* $t = t_0, t_0 \geqq T^*(\alpha)$, *and for* *any* $\psi_0 \in C_{B'(\alpha)}(E^m)$, *we can find a solution of the system* (5.2) (*or* (5.1)) *through* $(\varphi_0 + <x_0>, \psi_0, \xi_0 + <z_0>)$ *at* $t = t_0$, *for some* $(x_0, z_0) \in E^m \times E^l$, *which tends to the given solution of* (5.1) (*or* (5.2)) *as* $t \to \infty$.

Actually, we can put

$$H^*(r) = H_2(r),$$

$$T^*(\alpha) = \max \{T_2(B_2(B_1(\alpha)), T_3(B_1(\alpha))\},$$

$$H'(\alpha) = \min \{H_2(B_2(B_1(\alpha)), H_3(B_1(\alpha))\},$$

where (H_2, T_2, B_1) is the one given in Lemma 12 and (H_3, T_3, B_2) is the one in Lemma 13 for a given $(\varphi_0, \xi_0) \in C(E^n) \times C(E^l)$.

6. Asymptotic behaviors of solutions near an integral manifold. Consider a system of ordinary differential equations

$$\dot{w} = G(w) \tag{6.1}$$

and its perturbed system

$$\dot{\ }(t) = G(w(t)) + G^*(t, w_t), \tag{6.2}$$

where w is an $(n+m+1)$-vector, $G(w)$ is twice continuously differentiable on E^{n+m+1} and $G^*(t, \zeta)$ is a continuous function on $[0, \infty) \times C(E^{n+m+1})$. Suppose that (6.1) has a continuous $(n+1)$-parameter family of periodic solutions

$$w = w_0(\bar{\omega}(x)t + s, \mathrm{x}) \tag{6.3}$$

with real parameter $(s, x) \in E^1 \times U$, where U is an open set in E^n. In the (t, w)-space, such a family of periodic solutions (6.3) defines an $(n+1)$-dimensional integral manifold $\mathfrak{M}$.

Furthermore, we suppose the following conditions:

(6.4) $\bar{\omega}(x)$ is scalar and is continuous function of $x \in U$, and $w_0(z+\pi, x) = w_0(z, x)$.

(6.5) m of the characteristic exponents of the linear variational equations of (6.3) have negative real parts for each $(s, x) \in E^1 \times U$.

$$\text{(6. 6)} \qquad \operatorname{rank}\left[\frac{\partial w_0(z, x)}{\partial z}, \frac{\partial w_0(z, x)}{\partial x}\right] = n + 1$$

for all $(z, x) \in E^1 \times U$.

Here, by restricting our considerations in a neighborhood of a given point $x_0 \in U$ and by replacing x in (6. 3) with $x + x_0$, we can assume that U is an open sphere with the center at the origin of E^n.

Under the assumptions (6. 4), (6. 5) and (6. 6), Hale and Stokes [5] have introduced local coordinates (x, y, z), where x, y, z, are n, m, 1-vectors, in a neighborhood of the manifold $\mathfrak{M}$ in such a way that $\mathfrak{M}$ is given by

$$\text{(6. 7)} \qquad x = \text{constant}, \ y = 0, \ z = \bar{\omega}(x)t + s .$$

Systems (6. 1) and (6. 2) in the new variables have the forms given by

$$\text{(6. 8)} \qquad \begin{cases} \dot{x}(t) = X_1(x_t, y_t, z_t) \\ \dot{y}(t) = g(x_t, y_t) + Y_1(x_t, y_t, z_t) \\ \dot{z}(t) = \omega(x_t) + Z_1(x_t, y_t, z_t) \end{cases}$$

and

$$\text{(6. 9)} \qquad \begin{cases} \dot{x}(t) = X_1(x_t, y_t, z_t) + X_2(t, x_t, y_t, z_t) \\ \dot{y}(t) = g(x_t, y_t) + Y_1(x_t, y_t, z_t) + Y_2(t, x_t, y_t, z_t) \\ \dot{z}(t) = \omega(x_t) + Z_1(x_t, y_t, z_t) + Z_2(t, x_t, y_t, z_t), \end{cases}$$

respectively, where $\omega(\varphi) = \bar{\omega}(\varphi(0))$, $g(\varphi, \psi)$ is linear in ψ and, actually, $g(\varphi, \psi)$, $X_1(\varphi, \psi, \xi)$, $Y_1(\varphi, \psi, \xi)$ and $Z_1(\varphi, \psi, \xi)$ are functions of $(\varphi(0), \psi(0), \xi(0))$. Here, it is noted that (6. 7) is a solution of

$$\text{(6. 10)} \qquad \begin{cases} \dot{x}(t) = 0 \\ \dot{y}(t) = g(x_t, y_t) \\ \dot{z}(t) = \omega(x_t) . \end{cases}$$

By observing the systems (6. 8) and (6. 9), Hale has shown the following: Under the assumptions (6. 4), (6. 5), (6. 6) and

(6. 11) there exist continuous functions $\lambda(t) > 0$, $M(r) > 0$ such that

$$\int^{\infty}\int^{\infty}\lambda(\tau)\,d\tau\,dt<\infty$$

and that

$$|G^*(t,\zeta)|\leqq\lambda(t)\,M(\alpha)$$

on $[0,\infty)\times\overline{C}_\alpha(E^{n+m+1})$ for any $\alpha>0$,

if $(t_0, w(t_0))$ is sufficiently closed to the manifold $\mathfrak{M}$ and if t_0 is sufficiently large, then for any solution $w(t)$ of the system (6. 1) or (6. 2) through $(t_0, w(t_0))$ there exists a $(s_0, x_0)\in E^1\times U$ such that

$$w(t)-w_0(\bar{\omega}(x_0)t+s_0,x_0)\rightarrow 0\quad\text{as}\quad t\rightarrow\infty\,,$$

that is, the manifold $\mathfrak{M}$ is stable with an asymptotic amplitude and an asymptotic phase (see [3]).

Now, by the same arguments as used in the previous section, we can prove the following theorem, which is a converse of Hale's result in some sense.

THEOREM 6. *Under the assumptions* (6. 4), (6. 5) *and* (6. 6), *if* $G^*(t,\zeta)$ *in the system* (6. 2) *satisfies the condition* (6. 11) *or*

(6. 12) *there exists a continuous function* $\lambda^*(t,r)>0$ *such that*

$$\int^{\infty}\int_t^{\infty}\lambda^*(\tau,r)\,d\tau\,dt<\infty$$

and that

$$|G^*(t,\zeta)|\leqq\lambda^*(t,\alpha)$$

on $[0,\infty)\times\overline{C}_\alpha(E^{n+m+1})$ *for any* $\alpha>0$,

then for any $(s_0,x_0)\in E^1\times U$ *there exists a family of solutions of the systems* (6. 1) *and* (6. 2), *which tend to* $w_0(\bar{\omega}(x_0)t+s_0,x_0)$ *as* $t\rightarrow\infty$, *that is, there exists a family of solutions of the systems* (6. 1) *and* (6. 2) *with a given asymptotic amplitude and a given asymptotic phase.*

PROOF. By applying Hale and Stokes' transformation, we consider the systems (6. 10) and (6. 9), which are of special forms of the systems (5. 1) and (5. 2), respectively.

First of all, it should be noted that by the properties of Hale and Stokes' transformation, we can see the followings: Letting $\alpha_0 > 0$ be the radius of the open sphere U, $\omega(\varphi)$, $(X_1(\varphi, \psi, \xi)$, $Y_1(\varphi, \psi, \xi)$ $Z_1(\varphi, \psi, \xi))$ and $(X_2(t, \varphi, \psi, \xi)$, $Y_2(t, \varphi, \psi, \xi)$, $Z_2(t, \varphi, \psi, \xi))$ are defined on $[0, \infty) \times \Omega_{\alpha_0}(H_0) \times C(E^1)$ (for some continuous function $H_0(r) > 0$) and satisfy the conditions (5. 3), (5. 4) and (5. 5), respectively, where α in these conditions is restrained in $[0, \alpha_0)$. Moreover, there exists a continuous Liapunov functional $W(t, \varphi, \psi, \xi)$ defined on $[0, \infty) \times \Omega_{\alpha_0}(H_0) \times C(E^1)$ and satisfying all conditions in Lemma 11 for the system (6. 10). For the above, see [3] and [9].

Now, let $V(t, \varphi_1, \varphi_2)$ be defined by

$$V(t, \varphi_1, \varphi_2) = |\varphi_1(0) - \varphi_2(0)| .$$

Then, $V(t, \varphi_1, \varphi_2)$ satisfies all conditions given in Lemmas 12 and 13 for the system

$$\dot{x}(t) = 0, \tag{6.13}$$

which corresponds to the system (3. 9). Obviously, the condition (3. 21) holds good with $\eta(\alpha) = 0$, and the system (6. 13) has a bounded solution $x(t) = 0$.

Therefore, we can see that all assumptions in Theorem 5 are satisfied on the domain $[0, \infty) \times \Omega_{\alpha_0}(H_0) \times C(E^1)$. Moreover, if φ_0 in Lemma 13 is given so that $\|\varphi_0 - <\varphi_0(0)>\|$ is suitably small, then $B_1(\alpha)$ and $B_2(\alpha)$ in Lemmas 12 and 13 can be chosen to be less than α_0 for any $\alpha > 0$, $\alpha < \alpha_0$, because B_1 and B_2 were chosen so that the conditions (5. 11) and (5. 14) are satisfied. Thus, the eventually asymptotic equivalence between the systems (6. 8) (or (6. 9)) and (6. 10) on $\Omega_{\alpha_0}(H^*) \times C(E^1)$ (for an $H^*(r) > 0$) follows immediately from Theorem 5. Since Hale and Stokes' transformation preserves the asymptotic equivalence properties, we can complete the proof.

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