Tôhoku Math. Journ.
Vol. 18, No. 4, 1966

IMMERSIONS OF LENS SPACES*

FUICHI UCHIDA

(Received June 23, 1966)

Introduction. In this paper we consider the immersions of lens spaces in Euclidean spaces, where the immersion means C^∞-differentiable one. M. W. Hirsch [1] proved the following beautiful result.

THEOREM. *Let k-dimensional differentiable manifold M^k be immersible in Euclidean $(m+r)$-space R^{m+r} with a transversal r-field, then M^k is immersible in R^m, if $k < m$.*

The purpose of this note is to prove the following theorems by making use of the above result.

THEOREM A. *For any odd integer $p \geqq 3$ and any positive integer n, the $(2n+1)$-dimensional lens space $L^n(p)$ is immersible in Euclidean $2\left[\dfrac{3n+4}{2}\right]$-space, where $[x]$ means the integer part of x.*

THEOREM B. *For any odd integer $p \geqq 3$ and any integer $n \geqq 5$, $L^n(p)$ is immersible in Euclidean $(4n-4)$-space.*

When $5 \leqq n \leqq 6$, Theorem B is better than Theorem A.

1. Lens spaces. Let S^{2n+1} be the unit $(2n+1)$-sphere. A point of S^{2n+1} is represented by a sequence $(z_0, z_1, \cdots, z_n)$, where z_i $(i = 0, 1, \cdots, n)$ are complex numbers with $\sum_{i=0}^{n} |z_i|^2 = 1$. Let p be an integer greater than one and λ be the rotation of S^{2n+1} defined by

$$\lambda(z_0, z_1, \cdots, z_n) = (z_0 e^{2\pi i/p}, z_1 e^{2\pi i/p}, \cdots, z_n e^{2\pi i/p}).$$

Let Λ denote the topological transformation group of S^{2n+1} of order p generated

* This work was supported by the Sakkokai Foundation.

by λ. Then the lens space mod p is defined to be the orbit spaces:

$$L^n(p) = S^{2n+1}/\Lambda .$$

It is the compact connected orientable differentiable $(2n+1)$-manifold without boundary, and $L^n(p)$ is canonically embedded in $L^{n+1}(p)$.

As a *CW*-complex, $L^n(p)$ has one cell e^i in each dimension $(0 \leqq i \leqq 2n+1)$, and the integral cohomology groups of $L^n(p)$ are given by (cf. [7], p. 67)

$$H^i(L^n(p), Z) = \begin{cases} Z_p & i = 2, 4, \cdots, 2n , \\ Z & i = 0, 2n+1 , \\ 0 & \text{for other } i . \end{cases}$$

By making use of universal coefficient theorem, we have the cohomology groups of $L^n(p)$ over Z_k which are given by

$$H^i(L^n(p), Z_k) = \begin{cases} Z_k & i = 0, 2n+1 , \\ Z_{(p,k)} & i = 1, 2, 3, \cdots, 2n , \end{cases} \tag{1.1}$$

where (p, k) is the greatest common measure of p and k.

2. Stiefel manifolds. Let $V_{n,m}$ be the Stiefel manifold of m-frames in Euclidean n-space. Then, by normal cell decomposition ([7], p. 54), the $2k$-skeleton of $V_{n,n-k}$ is the stunted projective space P^{2k}/P^{k-1} if $n > 2k$, and the n-skeleton of $V_{n,n-k}$ is the stunted projective space P^{n-1}/P^{k-1} if $n \leqq 2k$.

For any positive integer n, the integral reduced homology groups of the projective space P^n are given by

$$\widetilde{H}_r(P^n, Z) = \begin{cases} 0 & \text{if } r \text{ is even and } 0 \leqq r \leqq n , \\ Z_2 & \text{if } r \text{ is odd and } 0 < r < n , \\ Z & \text{if } r \text{ is odd and } r = n . \end{cases}$$

So the integral reduced homology groups of the stunted projective space P^m/P^{2k} are given by

$$\widetilde{H}_r(P^m/P^{2k}, Z) = \begin{cases} Z_2 & \text{if } r \text{ is odd and } 2k < r < m , \\ Z & \text{if } r \text{ is odd and } r = m , \\ 0 & \text{for other } r. \end{cases}$$

If $k \geqq 1$, then P^m/P^{2k} is simply connected, and following the generalized Hurewicz theorem (cf. [2], p. 305), the homotopy group $\pi_r(P^m/P^{2k})$ is a finite group and has only 2-primary component if $r \leqq 2\left[\frac{m}{2}\right]$.

For example, if $4n \geqq 4k > 2n+3$, the homotopy group $\pi_r(V_{2n+3,2k})$ is a finite group and has only 2-primary component for $r \leqq 4(n-k)+6$.

3. Local coefficient. Let ξ be an n-dimensional vector bundle over an arcwise connected space X, and $\xi^{(m)}$ be the associated $V_{n,m}$-bundle. And let $\xi_r^{(m)}$ be the bundle of coefficients ([6], p. 151) associated with $\pi_r(V_{n,m})$. Then we have the following result (cf. Serre, Homologie singulière des espaces fibrés, p. 445, Prop. 3).

(3. 1) If ξ is orientable, then $\xi_r^{(m)}$ is trivial, for any m and r.

4. Immersions of lens spaces.

THEOREM A. *For any odd integer $p \geqq 3$ and any positive integer n, the $(2n+1)$-dimensional lens space $L^n(p)$ is immersible in Euclidean $2\left[\frac{3n+4}{2}\right]$-space.*

PROOF. Let $f: L^{n+1}(p) \rightarrow R^{4n+6}$ be an immersion. Then the normal bundle ν of this immersion is an orientable $(2n+3)$-dimensional vector bundle. We consider the existence of a cross-section of the bundle $\nu^{(2k)}$ with fibre $V_{2n+3,2k}$ associated with ν. The obstructions are contained in

$$H^{r+1}(L^{n+1}(p), \pi_r(V_{2n+3,2k})),$$

where the local coefficients are all trivial by (3. 1). And if $n \geqq k$ and $4k > 2n+3$, then $\pi_r(V_{2n+3,2k})$ is a finite group and has only 2-primary component for $r \leqq 4(n-k)+6$. See the end of the section 2.

Let $2k=n+3$ when n is odd and $2k=n+2$ when n is even, then $\pi_r(V_{2n+3,2k})$ is a finite group and has only 2-primary component for $r \leqq 2n$. And hence

$$H^{r+1}(L^{n+1}(p), \pi_r(V_{2n+3,2k})) = 0$$

for $r \leqq 2n$ by (1. 1). Hence $\nu^{(2k)}$ has a cross-section over $L^n(p)$, the $(2n+1)$-skeleton of $L^{n+1}(p)$. Thus $L^n(p)$ is immersible in Euclidean $(4n+6-2k)$-space. Therefore, $L^n(p)$ is immersible in Euclidean $2\left[\frac{3n+4}{2}\right]$-space if $n > 1$. Clearly,

$L^1(p)$ is immersible in R^6. This completes the proof.

THEOREM B. *$L^n(p)$ is immersible in R^{4n-4}, if $n \geqq 5$.*

PROOF. By Theorem A, $L^n(p)$ is immersible in R^{4n-2} if $n \geqq 5$. Then the normal bundle ν of this immersion is an orientable $(2n-3)$-dimensional vector bundle. We consider the existence of a cross-section of the bundle $\nu^{(2)}$ with fibre $V_{2n-3,2}$ associated with ν. The obstructions are contained in

$$H^{r+1}(L^n(p), \pi_r(V_{2n-3,2}))\,,$$

where the local coefficients are all trivial by (3.1). And if $n \geqq 5$, then

$$\begin{aligned} \pi_{2n-5}(V_{2n-3,2}) &= Z_2\,, & \pi_{2n-4}(V_{2n-3,2}) &= Z_2\,, \\ \pi_{2n-3}(V_{2n-3,2}) &= Z_4\,, & \pi_{2n-2}(V_{2n-3,2}) &= Z_2+Z_2\,, \\ \pi_{2n-1}(V_{2n-3,2}) &= Z_2\,, & \pi_{2n}(V_{2n-3,2}) &= 0\,, \end{aligned}$$

by Paechter [4]. Thus $H^{r+1}(L^n(p), \pi_r(V_{2n-3,2})) = 0$ for all r by (1.1). Hence $\nu^{(2)}$ has a cross-section over $L^n(p)$, and hence $L^n(p)$, is immersible in R^{4n-2} with transversal 2-field. Therefore $L^n(p)$ is immersible in R^{4n-4}.

REMARK. If k is odd and $k \geqq 15$, then $\pi_{k+13}(V_{k+2,2}) = 0$ by making use of Paechter's method (cf. [4], p. 260). When $n \geqq 13$, $L^n(p)$ is immersible in R^{4n-10} by Theorem A. Then, if $n \geqq 14$, $L^n(p)$ is immersible in R^{4n-12}, by similar argument. And this result is better than Theorem A if $n=14$.

REMARK. Recently, T. Kambe [3] has proved the following non-immersibility theorem of lens spaces: $L^n(p)$ cannot be immersed in Euclidean $2(n+L(n,p))$-space, where $L(n,p)$ is the integer defined by

$$L(n,p) = \max\left\{ i \leqq \left[\frac{n}{2}\right] \,\middle|\, \binom{n+i}{i} \not\equiv 0 \mod p^{1+\left[\frac{n-2i}{p-1}\right]} \right\}.$$

REFERENCES

[1] M. W. HIRSCH, Immersions of manifolds, Trans. Amer. Math. Soc., 93(1959), 242–276.
[2] S. T. HU, Homotopy theory, New York, Academic Press (1959).
[3] T. KAMBE, The structure of K_Λ-rings of the lens spaces and their applications, Journ. Math. Soc. Japan, 18(1966), 135–146.
[4] G. F. PAECHTER, The groups $\pi_r(V_{n,m})$ (I), Quart. Journ. Math. Oxford, 7(1956), 249–268.

[5] B. J. SANDERSON, Immersions and embeddings of projective spaces, Proc. London Math. Soc., 14(1964), 137–153.
[6] N. E. STEENROD, The topology of fibre bundles, Princeton Univ. (1951).
[7] ———, Cohomology operations, Princeton Univ. (1962).
[8] F. UCHIDA, Immersions and embeddings of tangent bundles, Tôhoku Math. Journ., 17 (1965), 161–170.

TÔHOKU UNIVERSITY.